A FOOTNOTE ON QUATERNION BLOCK-TRIDIAGONAL SYSTEMS

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Abstract. Two direct methods, for solving a system of linear equations having matrices of quaternion entries as coefficients and independent terms, are proposed. Second-kind Chebyshev polynomials, in a matrix argument, are used as a tool to do certain calculations.

Key words. block tridiagonal matrices, quaternions, Chebyshev polynomials.

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1. Commutative case. Consider the linear system \( AX = B \), written as follows:

\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix}
\]

where \( A \) is an \( n \times n \) matrix, partitioned into pairwise commuting blocks \( A_{ij}, i,j = 1, \ldots, n \); \( X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^T \) and \( B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}^T \).

In general, we can solve the given system applying a generalized Cramer rule.

- When each \( B_j \), for \( j = 1, \ldots, n \), commutes with each \( A_{ij} \), \( i,j = 1, \ldots, n \), the solution of the system is given by \( X_i = \Delta_i^{-1} \Delta_i \), \( i = 1, \ldots, n \), where \( \Delta \) is the formal determinant of the coefficient matrix and \( \Delta_i \) are the formal determinants obtained by replacing the \( i^{th} \) column in \( \Delta \), with the block-independent terms [10].

- When some \( B_j \), \( j = 1, \ldots, n \), do not commute with some \( A_{ij} \), \( i,j = 1, \ldots, n \), \( \Delta_i \) must be replaced by a symbolic determinant — \( \Delta_i^* \) — formally analogous to \( \Delta_i \); however, the \( B_j \), \( j = 1, \ldots, n \), should appear on the right in the computation of \( \Delta_i^* \) [9].

If \( A \) is a full matrix, then \( \Delta_i^* \) (and also \( \Delta_i \)) is too difficult to compute. We have to choose another way of finding the solution. We may apply this to the inverse of \( A \), computing the solution by \( X = A^{-1} B \).

2. Noncommutative case. When passing from a commutative situation to a noncommutative, the use of the Cramer rule brings many difficulties. The references [7, 11] are just a sample of the extensive work done, mainly by Chinese mathematicians, on quaternion matrices. In this note, we use a very special matrix form, which is the reason why we could make the following extension.

In the special case of \( A \) being a block–tridiagonal block–symmetric matrix (of order \( n \)) of the form \( A = \begin{bmatrix} (-I)(A)(-I) \end{bmatrix} \), with \( I \) denoting the identity matrix of order \( p \) and

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A footnote on quaternion block-tridiagonal systems

$A \in \mathcal{M}_{p \times q}(\mathbb{H})$ and $B_j \in \mathcal{M}_{p \times q}(\mathbb{H})$, $j = 1, \ldots, n$, it is possible to apply Chebyshev polynomials to compute $\Delta$, $\Delta_i^*$ and each block of $A^{-1}$ (denoted by $(A^{-1})_{ij}$, $i, j = 1, \ldots, n$).

(a) Using Onana and Mbala’s method [4], we have

$$\Delta_i^{-1} = J_n^{-1}$$

$$\Delta_i^* = J_n^i - j \left( \sum_{k=1}^{i-1} J_{k-1} B_k (-1)^{i+k} \right) + J_{i-1} \left( \sum_{k=1}^{n} J_{n-k} B_k (-1)^{i+k} \right)$$

where $J_n$, $n \in \mathbb{N}$, denotes the modified $n^{th}$-order second-kind Chebyshev polynomials, with matrix argument $A$.

(b) Using Kershaw’s [2] or Rózsa and Romani’s [6] method, we have

$$(A^{-1})_{rs} = U_{n-r}^{-1} U_{n-s}^{-1}, \quad 1 \leq r \leq s \leq n, \quad (A^{-1})_{rs} = (A^{-1})_{sr}$$

where $U_n$, $n \in \mathbb{N}$, denotes the $n^{th}$-order second-kind Chebyshev polynomials, with matrix argument $\frac{1}{2}A$.

3. Remarks. a) In the scalar case, the determinant of tridiagonal matrices is a certain kind of continuant [5]. To compute a continuant, a three term recurrence is usually used [5, 1, 8, 3]. Some special kind of continuants were first associated, as far as we know, with the expressions of Chebyshev Polynomials, in 1900, by Pascal [5], as referred to in [6].

We followed the generalization to the matrix case of such association, given in [2, 6].

b) We underline that the quaternionic skew-field $\mathbb{H}$ is noncommutative. The present extension is possible, because the only matrices involved are powers of $A$, and thus, in this particular case, we have commutativity between blocks. We have block symmetry, as well, but scalar symmetry is lost.

c) Another remark is due. In both methods, it is necessary to compute the inverse of quaternionic matrices ($J_n^{-1}$ and $U_n^{-1}$).

Following [12], we can guarantee the unicity of the inverse of a quaternionic matrix, when it exists, and we can compute it.

The process to obtain the inverse is, briefly, the following: we transform the quaternionic matrix $W$ into a complex one — denoted by $W_c$ —; then, we invert $W_c$ (in classical sense), obtaining the complex matrix $W_c^{-1}$; finally, transforming $W_c^{-1}$ into a quaternionic matrix, we obtain $W^{-1}$. The main result involved in this process is the existence of the isomorphism $\varphi$, defined [12] as follows:

$$\varphi : M_{n \times n}(\mathbb{H}) \longrightarrow M_{2n \times 2n}(\mathbb{C})$$

$$M = M_1 + M_2 j \longmapsto \varphi(M) = \begin{bmatrix} M_1 & M_2 \\ -M_2 & M_1 \end{bmatrix}.$$

As an illustrative example of this process, let us compute the inverse of the matrix

$$W = \begin{bmatrix} j & -3k \\ -k & 1 + j + k \end{bmatrix} \in M_{2 \times 2}(\mathbb{H}).$$

We decompose $W$ as follows

$$W = W_1 + W_2 j = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3i \\ -i & 1 + i \end{bmatrix} j.$$
Applying $\varphi$ to $W$, we obtain
\[
\varphi(W) \equiv W_c = \begin{bmatrix}
0 & 0 & 1 & -3i \\
0 & 1 & -i & 1 + i \\
-1 & -3i & 0 & 0 \\
-i & -1 + i & 0 & 1
\end{bmatrix} \in M_{4\times4}(\mathbb{C}).
\]

The inverse of $W_c$ is the complex matrix $W_c^{-1} = \begin{bmatrix}
T_1 & T_2 \\
-T_2 & T_1
\end{bmatrix}$, with
\[
T_1 = \begin{bmatrix}
\frac{1}{p} & -\frac{1}{p}i \\
\frac{1}{p}i & \frac{1}{p}
\end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix}
\frac{1}{3p} + \frac{1}{q}i & -\frac{1}{3p} + 2i \\
-\frac{1}{3p} + \frac{1}{q}i & \frac{1}{q} + \frac{2}{3}i
\end{bmatrix}.
\]

Finally, the inverse of $W$ is $\varphi^{-1}(W_c^{-1})$, which is the quaternion matrix
\[
W^{-1} = \begin{bmatrix}
\frac{1}{p} - \frac{1}{2}j + \frac{2}{3}k & -\frac{1}{3p} - \frac{1}{2}j + \frac{2}{3}k \\
\frac{1}{3p} - \frac{1}{2}j + \frac{2}{3}k & \frac{1}{q} + \frac{2}{3}j - \frac{1}{2}k
\end{bmatrix} \in M_{2\times2}(\mathbb{H}).
\]

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