

## A FOOTNOTE ON QUATERNION BLOCK-TRIDIAGONAL SYSTEMS\*

CECÍLIA COSTA<sup>†</sup> AND ROGÉRIO SERÓDIO<sup>‡</sup>

**Abstract.** Two direct methods, for solving a system of linear equations having matrices of quaternion entries as coefficients and independent terms, are proposed. Second-kind Chebyshev polynomials, in a matrix argument, are used as a tool to do certain calculations.

**Key words.** block tridiagonal matrices, quaternions, Chebyshev polynomials.

**AMS subject classifications.** 65F05.

**1. Commutative case.** Consider the linear system  $\mathbb{A}X = B$ , written as follows:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

where  $\mathbb{A}$  is a  $n \times n$  matrix, partitioned into pairwise commuting blocks  $A_{ij}$ ,  $i, j = 1, \dots, n$ ;  $X = [X_1 \ X_2 \ \cdots \ X_n]^T$  and  $B = [B_1 \ B_2 \ \cdots \ B_n]^T$ .

In general, we can solve the given system applying a generalized Cramer rule.

- When each  $B_j$ , for  $j = 1, \dots, n$ , commutes with each  $A_{ij}$ ,  $i, j = 1, \dots, n$ , the solution of the system is given by  $X$ , with blocks  $X_i = \Delta^{-1} \Delta_i$ ,  $i = 1, \dots, n$ , where  $\Delta$  is the formal determinant of the coefficient matrix and  $\Delta_i$  are the formal determinants obtained by replacing the  $i^{\text{th}}$  column in  $\Delta$ , with the block-independent terms [10].

- When some  $B_j$ ,  $j = 1, \dots, n$ , do not commute with some  $A_{ij}$ ,  $i, j = 1, \dots, n$ ,  $\Delta_i$  must be replaced by a symbolic determinant —  $\Delta_i^*$  — formally analogous to  $\Delta_i$ ; however, the  $B_j$ ,  $j = 1, \dots, n$ , should appear on the right in the computation of  $\Delta_i^*$  [9].

If  $\mathbb{A}$  is a full matrix, then  $\Delta_i^*$  (and also  $\Delta_i$ ) is too difficult to compute. We have to choose another way of finding the solution. We may apply this to the inverse of  $\mathbb{A}$ , computing the solution by  $X = \mathbb{A}^{-1}B$ .

**2. Noncommutative case.** When passing from a commutative situation to a non-commutative, the use of the Cramer rule brings many difficulties. The references [7, 11] are just a sample of the extensive work done, mainly by Chinese mathematicians, on quaternion matrices. In this note, we use a very special matrix form, which is the reason why we could make the following extension.

In the special case of  $\mathbb{A}$  being a block-tridiagonal block-symmetric matrix (of order  $n$ ) of the form  $\mathbb{A} = [(-I) \ (A) \ (-I)]$ , with  $I$  denoting the identity matrix of order  $p$  and

---

\* Received November 1, 1998. Accepted for publication December 1, 1999. Recommended by F. Marcellán. Work done at the Grupo de Teoria do Controlo, Instituto de Sistemas e Robótica, Pólo de Coimbra, Portugal and Centro de Matemática da UBI, Covilhã, Portugal.

<sup>†</sup> Secção de Matemática, Universidade de Trás-os-Montes e Alto Douro, 5000 Vila Real, Portugal (nop62466@mail.telepac.pt).

<sup>‡</sup>Departamento de Matemática, Universidade da Beira Interior, 6200 Covilhã, Portugal (rserodio@noe.ubi.pt).

$A \in \mathcal{M}_{p \times p}(\mathbb{H})$  and  $B_j \in \mathcal{M}_{p \times q}(\mathbb{H})$ ,  $j = 1, \dots, n$ , it is possible to apply Chebyshev polynomials to compute  $\Delta$ ,  $\Delta_i^*$  and each block of  $\mathbb{A}^{-1}$  (denoted by  $(\mathbb{A}^{-1})_{ij}$ ,  $i, j = 1, \dots, n$ ).

(a) Using Onana and Mbala's method [4], we have

$$\Delta^{-1} = J_n^{-1}$$

$$\Delta_i^* = J_{n-i} \left( \sum_{k=1}^{i-1} J_{k-1} B_k (-1)^{i+k} \right) + J_{i-1} \left( \sum_{k=1}^n J_{n-k} B_k (-1)^{i+k} \right)$$

where  $J_n$ ,  $n \in \mathbb{N}$ , denotes the modified  $n^{\text{th}}$ -order second-kind Chebyshev polynomials, with matrix argument  $A$ .

(b) Using Kershaw's [2] or Rózsa and Romani's [6] method, we have

$$(\mathbb{A}^{-1})_{rs} = U_n^{-1} U_{r-1} U_{n-s}, \quad 1 \leq r \leq s \leq n, \quad (\mathbb{A}^{-1})_{rs} = (\mathbb{A}^{-1})_{sr}$$

where  $U_n$ ,  $n \in \mathbb{N}$ , denotes the  $n^{\text{th}}$ -order second-kind Chebyshev polynomials, with matrix argument  $\frac{1}{2}A$ .

**3. Remarks.** a) In the scalar case, the determinant of tridiagonal matrices is a certain kind of continuant [5]. To compute a continuant, a three term recurrence is usually used [5, 1, 8, 3]. Some special kind of continuants were first associated, as far as we know, with the expressions of Chebyshev Polynomials, in 1900, by Pascal [5], as referred to in [6].

We followed the generalization to the matrix case of such association, given in [2, 6].

b) We underline that the quaternionic skew-field  $\mathbb{H}$  is noncommutative. The present extension is possible, because the only matrices involved are powers of  $A$ , and thus, in this particular case, we have commutativity between blocks. We have block symmetry, as well, but scalar symmetry is lost.

c) Another remark is due. In both methods, it is necessary to compute the inverse of quaternionic matrices ( $J_n^{-1}$  and  $U_n^{-1}$ ).

Following [12], we can guarantee the unicity of the inverse of a quaternionic matrix, when it exists, and we can compute it.

The process to obtain the inverse is, briefly, the following: we transform the quaternionic matrix  $W$  into a complex one — denoted by  $W_c$  —; then, we invert  $W_c$  (in classical sense), obtaining the complex matrix  $W_c^{-1}$ ; finally, transforming  $W_c^{-1}$  into a quaternionic matrix, we obtain  $W^{-1}$ . The main result involved in this process is the existence of the isomorphism  $\varphi$ , defined [12] as follows:

$$\varphi : M_{n \times n}(\mathbb{H}) \longrightarrow M_{2n \times 2n}(\mathbb{C})$$

$$M = M_1 + M_2 j \longmapsto \varphi(M) = \begin{bmatrix} M_1 & M_2 \\ -M_2 & M_1 \end{bmatrix}.$$

As an illustrative example of this process, let us compute the inverse of the matrix

$$W = \begin{bmatrix} j & -3k \\ -k & 1 + j + k \end{bmatrix} \in M_{2 \times 2}(\mathbb{H}).$$

We decompose  $W$  as follows

$$W = W_1 + W_2 j = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3i \\ -i & 1 + i \end{bmatrix} j.$$

Applying  $\varphi$  to  $W$ , we obtain

$$\varphi(W) \equiv W_c = \begin{bmatrix} 0 & 0 & 1 & -3i \\ 0 & 1 & -i & 1+i \\ -1 & -3i & 0 & 0 \\ -i & -1+i & 0 & 1 \end{bmatrix} \in M_{4 \times 4}(\mathbb{C}).$$

The inverse of  $W_c$  is the complex matrix  $W_c^{-1} = \begin{bmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{bmatrix}$ , with

$$T_1 = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6}i \\ \frac{1}{18}i & \frac{1}{18} \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} -\frac{1}{3} + \frac{1}{6}i & -\frac{1}{6} + \frac{2}{3}i \\ -\frac{1}{18} + \frac{2}{9}i & -\frac{2}{9} - \frac{1}{18}i \end{bmatrix}.$$

Finally, the inverse of  $W$  is  $\varphi^{-1}(W_c^{-1})$ , which is the quaternion matrix

$$W^{-1} = \begin{bmatrix} \frac{1}{6} - \frac{1}{3}j + \frac{1}{6}k & -\frac{1}{6}i - \frac{1}{6}j + \frac{2}{3}k \\ \frac{1}{18}i - \frac{1}{18}j + \frac{2}{9}k & \frac{1}{18} - \frac{2}{9}j - \frac{1}{18}k \end{bmatrix} \in M_{2 \times 2}(\mathbb{H}).$$

**Acknowledgments.** Thanks are due to Professor José Vitória for helpful discussions on the subject, during the preparation of an early draft of this paper. We are also grateful to an anonymous referee for providing the Rózsa and Romani's reference and for calling our attention to some work by Chinese mathematicians.

#### REFERENCES

- [1] P. B. FISCHER, *Determinanten*, Sammlung Götschen, Leipzig, pp 114, 1921.
- [2] D. KERSHAW, *The explicit inverses of two commonly occurring matrices*, Math. Comp, 23 (1969), pp. 189–191.
- [3] T. MUIR, *A Treatise on the Theory of Determinants*, Dover Publications, New York, pp. 521, 1960.
- [4] A. ONANA AND H. A. MBALA, *Direct method for a class of symmetric linear systems*, Comm. Numer. Methods Engng., 9 (1993), pp. 721–727 [MR 94f:65020].
- [5] E. PASCAL, *Die Determinanten*, Teubner, Leipzig, 1900, pp. 155–157.
- [6] P. RÓZSA AND F. ROMANI, *A reduction theorem for the characteristic polynomial of periodic block tridiagonal matrices*, in Numer. Methods, D. Greenspan and P. Rózsa, eds., North-Holland, Amsterdam, pp. 37–47, 1991.
- [7] C. J. TU, *Cramer rules for weighted system over the quaternion field*, J. Zhangzhou Teachers College (Natural Science Editions), 9(2) (1995), pp. 17–22.
- [8] J. VICENTE GONÇALVES, *Curso de Álgebra Superior*, Tipografia Atlântica, Coimbra, pp. 304, 1933.
- [9] J. VITÓRIA, *Matrices partitionnées en blocs commutatifs*, Gaz. Math., 117–120 (1970), pp. 49–57 [MR 49 # 328].
- [10] P. M. XIAN, *Block eigenvalues of block compound matrices*, Acta Math. Sinica, 34 (1991), pp. 48–55 (in Chinese).
- [11] C. L. XUAN, *Definition of determinant and Cramer solutions over the quaternion field*, Acta Math. Sinica (NS), 7(2)(1991), pp. 172–180 [MR 92f:15011].
- [12] F. ZHANG, *Quaternions and matrices of quaternions*, Linear Algebra Appl., 251 (1997), pp. 21–57 (1997) [MR 97h:15020].