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AN ARROW–HURWICZ-TYPE ITERATION FOR THE THERMALLY COUPLED INCOMPRESSIBLE MAGNETOHYDRODYNAMICS MODEL WITH GRAD-DIV STABILIZATION*

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Abstract. This paper shows that, for the stationary thermally coupled incompressible magnetohydrodynamics problem, an application of the grad-div stabilization technique and a modification of the Arrow–Hurwicz iteration can improve the convergence rate of the Arrow–Hurwicz algorithm and remove restrictions on the relaxation parameter α for this algorithm. Based on the grad-div stabilization method, we design an Arrow–Hurwicz-type iterative finite element algorithm for solving this problem. A convergence analysis as well as numerical tests show that the proposed iteration performs better for the considered problem compared to the standard Arrow–Hurwicz iteration.

Key words. Arrow–Hurwicz-type iteration, thermally coupled magnetohydrodynamics, grad-div stabilization, convergence analysis

AMS subject classifications. 65N12, 65N30

1. Introduction.

We consider an Arrow–Hurwicz-type iterative finite element algorithm with grad-div stabilization for approximating solutions of the stationary thermally coupled incompressible magnetohydrodynamics (STCIMHD) model [28, 29]. On a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$, the non-dimensional form of the STCIMHD equations reads as

$$
\begin{aligned}
-\mathrm{Re}^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + s\,\mathbf{H}\times\mathrm{curl}\mathbf{H} &= \mathbf{f} + \beta T\mathbf{j}, && \text{in } \Omega,\\
s\,\mathrm{Rm}^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{H}) - s\,\mathrm{curl}(\mathbf{u}\times\mathbf{H}) &= \mathbf{g}, && \text{in } \Omega,\\
-\kappa\Delta T + \mathbf{u}\cdot\nabla T &= \gamma, && \text{in } \Omega,\\
\mathrm{div}\,\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{H} &= 0, && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

with boundary conditions

$$
\begin{aligned}
\mathbf{u}|_{\partial\Omega} &= 0, && \text{(no-slip condition)},\\
\mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0 \ \text{ and } \ \mathbf{n}\times\mathrm{curl}\mathbf{H}|_{\partial\Omega} &= 0, && \text{(perfectly conducting wall)},\\
T|_{\Gamma_D} = 0 \ \text{ and } \ \nabla T\cdot\mathbf{n}|_{\Gamma_N} &= 0, && \text{(insulated wall)},
\end{aligned}
\tag{1.2}
$$

where $\mathbf{u}$ is the velocity field, $\mathbf{H}$ is the magnetic field, p is the pressure, and T is the temperature. The parameters Re,Rm,κ, s, and β are positive and denote the hydrodynamic Reynolds number, the magnetic Reynolds number, the thermal conductivity, the coupling number, and the thermal expansion coefficient, respectively. Besides, $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$, $\mathbf{j}$ denotes a unit vector in the direction opposite to gravity, $\Gamma_D = \partial\Omega \setminus \Gamma_N$, where Γ_N is a regular open subset of $\partial\Omega$, $\mathbf{g}$ represents the known applied current with $\mathrm{div}\,\mathbf{g} = 0$, $\mathbf{f}$ is a force term for the magnetic induction, and γ is a given heat source.

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Although the multi-physical field coupling of the STCIMHD model makes the numerical simulation challenging, an investigation of its numerical properties is important due to the broad applications of the model. There is a large amount of literature on numerical investigations of the STCIMHD equations in recent years. For example, the existence and uniqueness of a weak solution and the convergence analysis of finite element approximations for the STCIMHD system were discussed in [28, 29]. In [30], a general approach to stationary, electromagnetically and thermally driven liquid-metal flows was studied by Meir and Schmidt. Bermúdez et al. [3] proposed and analyzed some existence results of weak solutions to the stationary magnetohydrodynamics systems of equations including Joule heating. Moreover, Yang and Zhang [37] proposed three iterative methods with a finite element discretization in space for the STCIMHD equations. For more extensive investigations we refer to [2, 5, 7, 17, 24, 25, 27, 33, 34, 38, 39] and their references.

However, due to the existence of incompressibility constraints, many numerical methods for solving the STCIMHD equations need to solve saddle point systems at each iteration. Hence, the Arrow–Hurwicz iterative finite element method, which is an inexact version of the Uzawa method, is proposed to approach this problem [18]. It is an efficient method for dealing with saddle point problems, and it is applied, for instance, for the steady incompressible Navier-Stokes equations [8], the stationary magnetohydrodynamics flow [40], the Smagorinsky model [19], etc. Although the Arrow–Hurwicz algorithm avoids solving saddle point problems exactly, there exists certain restrictions on the parameter α (see [18, Remark 3.2]), where $\alpha > 0$ is a relaxation parameter for the pressure which plays an important role in the convergence of the Arrow–Hurwicz algorithm. Hence, it is natural to ask whether this restriction can be removed.

As a common and powerful tool for improving the solution quality, the grad-div stabilization has been widely studied both analytically and computationally [1, 4, 9, 15, 20, 21, 22, 23, 26, 31, 32, 36, 41]. It adds a penalty term with respect to the continuity equation to the momentum equation and can mitigate the lack of mass conservation and improve the solution accuracy by reducing the effect of the pressure on the velocity error [16]. A grad-div enhanced Arrow–Hurwicz iterative finite element algorithm was recently investigated in [11]. It was shown that applying the idea of the grad-div stabilization to the Arrow–Hurwicz iteration can improve convergence. Besides, Takhirov et al. [35] have proposed an improved Arrow–Hurwicz-type method for approximating the steady-state Navier-Stokes equations, which is inspired from the artificial compressibility method.

Inspired by [11] and [35], in this paper we propose an Arrow–Hurwicz-type iterative finite element algorithm with grad-div stabilization for approximating the solution of the STCIMHD equations. This iterative algorithm improves the convergence rate of the Arrow–Hurwicz algorithm and eliminates the restriction for the relaxation parameter α of the algorithm. The remainder of this article is structured as follows. In Section 2, we introduce some basic notations and results for the problem (1.1)–(1.2) and also recall the finite element method and its stability. In Section 3, we propose an Arrow–Hurwicz-type iterative algorithm for the considered problem. Then, the iterative error is analyzed for the presented algorithm. In Section 4, the relationship between algorithm parameters and iterative linear solutions is investigated. Some numerical experiments are performed to validate the efficiency of the proposed algorithm in the final section.

2. Preliminaries. Throughout this paper, $(\cdot, \cdot)$ and $\|\cdot\|_0$ denote the inner product and norm on $L^2(\Omega)$, respectively. Besides, we write $H^1(\Omega)$ for the usual Sobolev space $W^{1,2}(\Omega)$. Next, in order to write the variational form of (1.1)–(1.2), we define the following function spaces:

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$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v}|_{\partial\Omega} = 0\}, \qquad \mathbf{W} = \{\mathbf{B} \in H^1(\Omega)^2 : \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$Q = \{S \in H^1(\Omega) : S|_{\Gamma_D} = 0\}, \qquad M = \{q \in L^2(\Omega) : (q, 1) = 0\}.$$

Additionally, we introduce the product space $\mathbf{D} = \mathbf{X} \times \mathbf{W}$, equipped with the norm $\|\nabla(\mathbf{w}, \mathbf{\Phi})\|_0^2 = \|\nabla\mathbf{w}\|_0^2 + \|\nabla\mathbf{\Phi}\|_0^2$, for all $(\mathbf{w}, \mathbf{\Phi}) \in \mathbf{D}$.

Moreover, we define three continuous bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$, and $a_2(\cdot,\cdot)$ on $Q \times Q$, $\mathbf{X} \times \mathbf{X}$, and $\mathbf{W} \times \mathbf{W}$, respectively, by

$$\begin{aligned}
a_0(T, S) &= \kappa(\nabla T, \nabla S),\\
a_1(\mathbf{u}, \mathbf{v}) &= \mathrm{Re}^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}),\\
a_2(\mathbf{H}, \mathbf{B}) &= s\,\mathrm{Rm}^{-1}\big((\mathrm{curl}\mathbf{H}, \mathrm{curl}\mathbf{B}) + (\mathrm{div}\,\mathbf{H}, \mathrm{div}\,\mathbf{B})\big),
\end{aligned}$$

and three trilinear forms $b_0(\cdot,\cdot,\cdot)$, $b_1(\cdot,\cdot,\cdot)$, and $b_2(\cdot,\cdot,\cdot)$ on $\mathbf{X} \times Q \times Q$, $\mathbf{X} \times \mathbf{X} \times \mathbf{X}$, and $\mathbf{W} \times \mathbf{W} \times \mathbf{X}$, respectively, by

$$\begin{aligned}
b_0(\mathbf{u}, T, S) =& (\mathbf{u}\cdot\nabla T, S) + 0.5((\mathrm{div}\,\mathbf{u})T, S) = 0.5(\mathbf{u}\cdot\nabla T, S) - 0.5(\mathbf{u}\cdot\nabla S, T),\\
b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) =& ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) + 0.5((\mathrm{div}\,\mathbf{u})\mathbf{w}, \mathbf{v}) = 0.5((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) - 0.5((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}),\\
b_2(\mathbf{H}, \mathbf{B}, \mathbf{v}) =& s(\mathbf{H} \times \mathrm{curl}\mathbf{B}, \mathbf{v}).
\end{aligned}$$

These forms satisfy the following properties [10, 12]:

$$\begin{aligned}
|b_0(\mathbf{u}, T, S)| &\le N_0\|\nabla\mathbf{u}\|_0\|\nabla T\|_0\|\nabla S\|_0,\\
|b_1(\mathbf{u}, \mathbf{w}, \mathbf{v})| &\le N_1\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{w}\|_0\|\nabla\mathbf{v}\|_0,\\
|b_2(\mathbf{H}, \mathbf{B}, \mathbf{v})| &\le sN_2\|\nabla\mathbf{H}\|_0\|\nabla\mathbf{B}\|_0\|\nabla\mathbf{v}\|_0,
\end{aligned} \tag{2.1}$$

for all $T, S \in Q$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$, and $\mathbf{H}, \mathbf{B} \in \mathbf{W}$, where $N_i > 0$, $i = 0, 1, 2$, are constants depending on Ω.

Then, the STCIMHD equations (1.1)–(1.2) can be rewritten as follows:

Find $((\mathbf{u}, \mathbf{H}), T, p) \in \mathbf{D} \times Q \times M$ such that for all $((\mathbf{v}, \mathbf{B}), S, q) \in \mathbf{D} \times Q \times M$

$$a_0(T, S) + b_0(\mathbf{u}, T, S) = (\gamma, S), \tag{2.2}$$

$$\begin{aligned}
A_0\left((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})\right) + A_1((\mathbf{u}, \mathbf{H}), (\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p)&\\
= (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T, (\mathbf{v}, \mathbf{B})),&
\end{aligned} \tag{2.3}$$

$$d((\mathbf{u}, \mathbf{H}), q) = 0, \tag{2.4}$$

where

$$\begin{aligned}
A_1\left((\mathbf{u}, \mathbf{H}), (\mathbf{w}, \mathbf{\Phi}), (\mathbf{v}, \mathbf{B})\right) &= b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b_2(\mathbf{H}, \mathbf{\Phi}, \mathbf{v}) - b_2(\mathbf{H}, \mathbf{B}, \mathbf{w}),\\
d((\mathbf{v}, \mathbf{B}), p) &= (\mathrm{div}\,\mathbf{v}, p),\\
A_0\left((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})\right) &= a_1(\mathbf{u}, \mathbf{v}) + a_2(\mathbf{H}, \mathbf{B}),\\
(\mathbf{F}, (\mathbf{v}, \mathbf{B})) &= (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{B}), \qquad \text{and}\\
G(T, (\mathbf{v}, \mathbf{B})) &= \beta(T\mathbf{j}, \mathbf{v}).
\end{aligned}$$

In order to discuss the well-posedness of the variational formulation (2.2)–(2.4), we verify the coercivity and continuity property of $A_0\left((\cdot,\cdot), (\cdot,\cdot)\right)$ and the continuity property of $A_1\left((\cdot,\cdot), (\cdot,\cdot), (\cdot,\cdot)\right)$.

LEMMA 2.1 ([18]). *For all* $(\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}),(\mathbf{v},\mathbf{B})\in\mathbf{D}$, *there holds*

$$
\begin{aligned}
A_0\left((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})\right) &\le c_A\|(\mathbf{u},\mathbf{H})\|_0\|(\mathbf{v},\mathbf{B})\|_0,\\
A_0\left((\mathbf{u},\mathbf{H}),(\mathbf{u},\mathbf{H})\right) &\ge \nu_A\|(\mathbf{u},\mathbf{H})\|_0^2,\\
A_1\left((\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}),(\mathbf{v},\mathbf{B})\right) &\le N\|(\mathbf{u},\mathbf{H})\|_0\|(\mathbf{w},\mathbf{\Phi})\|_0\|(\mathbf{v},\mathbf{B})\|_0,
\end{aligned}
$$

where

$$c_A=\max\{\mathrm{Re}^{-1},4s\mathrm{Rm}^{-1}\},\ \nu_A=\min\{\mathrm{Re}^{-1},s\mathrm{Rm}^{-1}c_1\},\ N=\sqrt{2}\max\{N_1,N_2s\}.$$

Now, we establish the following existence and uniqueness results for the problem (1.1)–(1.2):

THEOREM 2.2 ([28, 29, 37]). *Let* $\gamma\in Q'$, $\mathbf{F}\in\mathbf{D}'$, κ, β, *and* ν_A *satisfy the following uniqueness condition:*

$$0<\delta<1,\qquad \textit{where}\quad \delta=\delta_1+\delta_2\quad \textit{with}$$

$$\delta_1:=\nu_A^{-2}N(\|\mathbf{F}\|_{-1}+\kappa^{-1}\beta\|\gamma\|_{-1}),\qquad \delta_2:=\nu_A^{-1}\kappa^{-2}\beta N_0\|\gamma\|_{-1},$$

$$\|\gamma\|_{-1}=\sup_{T\in Q,T\neq 0}\frac{(\gamma,T)}{\|\nabla T\|_0}\qquad\qquad \|\mathbf{F}\|_{-1}=\sup_{(\mathbf{u},\mathbf{H})\in\mathbf{D},(\mathbf{u},\mathbf{H})\neq\mathbf{0}}\frac{(\mathbf{F},(\mathbf{u},\mathbf{H}))}{\|\nabla(\mathbf{u},\mathbf{H})\|_0}.$$

Then the problem (1.1)–(1.2) *admits a unique solution* $((\mathbf{u},\mathbf{H}),p,T)\in\mathbf{D}\times M\times Q$ *such that*

$$\nu_A\|\nabla(\mathbf{u},\mathbf{H})\|_0\le\|\mathbf{F}\|_{-1}+\beta\kappa^{-1}\|\gamma\|_{-1},\qquad \kappa\|\nabla T\|_0\le\|\gamma\|_{-1}.$$

From now on, let h be a real positive parameter. The conforming finite element subspaces $(\mathbf{X}_h,\mathbf{W}_h,Q_h,M_h)$ of $(\mathbf{X},\mathbf{W},Q,M)$ are characterized by $K_h=K_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles K, assumed to be uniformly regular as $h\to 0$. Next, we define the product space $\mathbf{D}_h=\mathbf{X}_h\times\mathbf{W}_h$. Further, we assume that $\mathbf{D}_h\times M_h$ admits the following discrete inf-sup condition: For each $q_h\in M_h$, there exists $(\mathbf{v}_h,\mathbf{B}_h)\in\mathbf{D}_h$ such that [29]

$$\sup_{(\mathbf{v}_h,\mathbf{B}_h)\in\boldsymbol{D}_h,(\mathbf{v}_h,\mathbf{B}_h)\neq(\mathbf{0},\mathbf{0})}\frac{|d((\mathbf{v}_h,\mathbf{B}_h),q_h)|}{\|(\mathbf{v}_h,\mathbf{B}_h)\|_0}\ge\widetilde{\beta}\|q_h\|_0,\tag{2.5}$$

where $\widetilde{\beta}>0$ is a constant depending on Ω.

Then, according to the above definition of the finite element subspaces, the finite element approximation for (2.2)–(2.4) is to find $((\mathbf{u}_h,\mathbf{H}_h),T_h,p_h)\in\mathbf{D}_h\times Q_h\times M_h$ such that for all $((\mathbf{v},\mathbf{B}),S,q)\in\mathbf{D}_h\times Q_h\times M_h$

$$a_0(T_h,S)+b_0(\mathbf{u}_h,T_h,S)=(\gamma,S),\tag{2.6}$$

$$
\begin{aligned}
A_0\left((\mathbf{u}_h,\mathbf{H}_h),(\mathbf{v},\mathbf{B})\right)+A_1\left((\mathbf{u}_h,\ \mathbf{H}_h),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{v},\mathbf{B})\right)-d((\mathbf{v},\mathbf{B}),p_h)&\\
=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_h,(\mathbf{v},\mathbf{B})),&\qquad(2.7)
\end{aligned}
$$

$$d((\mathbf{u}_h,\mathbf{H}_h),q)=0.\tag{2.8}$$

The following results can be found in [29, 37], which describe the stability of the numerical solutions obtained by (2.6)–(2.8).

THEOREM 2.3. *Let* $((\mathbf{u}_h,\mathbf{H}_h),T_h,p_h)\in\mathbf{D}_h\times Q_h\times M_h$ *be a solution of the finite element discretization* (2.6)–(2.8). *Then, under the assumptions of Theorem* 2.2, *there holds*

$$\nu_A\|\nabla(\mathbf{u}_h,\mathbf{H}_h)\|_0\le\|\mathbf{F}\|_{-1}+\beta\kappa^{-1}\|\gamma\|_{-1},\qquad \kappa\|\nabla T_h\|_0\le\|\gamma\|_{-1}.$$

3. An Arrow–Hurwicz-type iterative algorithm. In this section, we present an Arrow–Hurwicz-type iterative finite element algorithm with grad-div stabilization for the STCIMHD model and then show an iterative error estimate of the proposed algorithm.

ALGORITHM 3.1 (Arrow–Hurwicz-type iterative algorithm).
Let $\rho>0$ and $\alpha>0$ be two relaxation parameters. Then, we find

$$\left((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}\right),T_h^{n+1},p_h^{n+1})\in\mathbf{D}_h\times Q_h\times M_h$$

by the following two steps:

Step I: Choose an initial function pair $\left((\mathbf{u}_h^0,\mathbf{H}_h^0),T_h^0,p_h^0\right)\in\mathbf{D}_h\times Q_h\times M_h$ defined by solving the following equations:

$$\begin{aligned}
a_0(T_h^0,S)&=(\gamma,S), && \forall S\in Q_h,\\
A_0\left((\mathbf{u}_h^0,\mathbf{H}_h^0),(\mathbf{v},\mathbf{B})\right)-d((\mathbf{v},\mathbf{B}),p_h^0)&=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_h^0,(\mathbf{v},\mathbf{B})), && \forall(\mathbf{v},\mathbf{B})\in\mathbf{D}_h,\\
d\left((\mathbf{u}_h^0,\mathbf{H}_h^0),q\right)&=0, && \forall q\in M_h.
\end{aligned}$$

Step II: For $n=1,2,\dots$ and given $((\mathbf{u}_h^n,\mathbf{H}_h^n),T_h^n,p_h^n)\in\mathbf{D_h}\times Q_h\times M_h$, find the update $\left((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),T_h^{n+1},p_h^{n+1}\right)\in\mathbf{D}_h\times Q_h\times M_h$ as solution of

$$\rho^{-1}\mathrm{Re}\kappa\left(\nabla T_h^{n+1}-\nabla T_h^n,\nabla S\right)+a_0\left(T_h^{n+1},S\right)+b_0\left(\mathbf{u}_h^n,T_h^{n+1},S\right)=(\gamma,S),\tag{3.1}$$

$$\begin{aligned}
&\rho^{-1}\left(\nabla\mathbf{u}_h^{n+1}-\nabla\mathbf{u}_h^n,\nabla\mathbf{v}\right)+\rho^{-1}\mathrm{Re}\,a_2\left(\mathbf{H}_h^{n+1}-\mathbf{H}_h^n,\mathbf{B}\right)-d\left((\mathbf{v},\mathbf{B}),p_h^n\right)\\
&\qquad+A_0\left((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),(\mathbf{v},\mathbf{B})\right)\\
&\qquad+A_1\left((\mathbf{u}_h^n,\mathbf{H}_h^n),(\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),(\mathbf{v},\mathbf{B})\right)\\
&\qquad+\frac{\rho}{\alpha}\left(\operatorname{div}\mathbf{u}_h^{n+1},\operatorname{div}\mathbf{v}\right)\\
&\qquad\qquad=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_h^{n+1},(\mathbf{v},\mathbf{B})),
\end{aligned}\tag{3.2}$$

$$\alpha\left(p_h^{n+1}-p_h^n,q\right)+\rho d\left((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),q\right)=0,\tag{3.3}$$

for all $(\mathbf{v},\mathbf{B})\in\mathbf{D}_h$, $S\in Q_h$, and $q\in M_h$.

REMARK 3.2. The usual Arrow–Hurwicz algorithm is a generalization of the Uzawa algorithm, where the terms $a_0(T^{n+1},S)$ and $A_0\left((u_h^{n+1},H_h^{n+1}),(\mathbf{v},\mathbf{B})\right)$ in Algorithm 3.1 read $a_0(T^n,S)$ and $A_0\left((u_h^n,H_h^n),(\mathbf{v},\mathbf{B})\right)$. Although the Arrow–Hurwicz algorithm avoids solving saddle point problems exactly, there exists a restriction on the parameter α (see [18, Remark 3.2]).

For convenience, we set

$$\mathbf{e}_h^n=\mathbf{u}_h-\mathbf{u}_h^n,\quad \boldsymbol{\xi}_h^n=\mathbf{H}_h-\mathbf{H}_h^n,\quad \eta_h^n=p_h-p_h^n,\quad\text{and}\quad \theta_h^n=T_h-T_h^n,\quad n\ge0.$$

Firstly, we recall the iterative error estimates of the initial guess in Step 1.

LEMMA 3.3 ([18]). *Let* $\left((\mathbf{u}_h^0, \mathbf{H}_h^0), p_h^0, T_h^0\right) \in \mathbf{D}_h \times Q_h \times M_h$ *be the solution of Step 1. Then, under the assumptions of Theorem 2.3, we have the following results:*

$$\begin{aligned}
\|\nabla \theta_h^0\|_0 &\leq \beta^{-1}\delta_2 \left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right),\\
\|\nabla\left(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0\right)\|_0 &\leq \nu_A^{-1}\delta\left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right),\\
\|\eta_h^0\|_0 &\leq \widetilde{\beta}^{-1}\delta\left(c_A\nu_A^{-1} + 1\right)\left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right).
\end{aligned}$$

Secondly, we are going to show that the solution from Algorithm 3.1 is bounded.

THEOREM 3.4. *Assume that* $\{(\mathbf{u}_h^n, \mathbf{H}_h^n), p_h^n, T_h^n\}$ *is the function sequence from Algorithm 3.1. If* $\operatorname{Re}\nu_A > \beta$ *and the parameter* ρ *satisfies*

$$\rho \leq \min\left\{\nu_A^{-1}\kappa\delta_1, \frac{\operatorname{Re}\nu_A - \beta}{\operatorname{Re}c_A + \nu_A\delta_1 + \kappa^{-1}N_0\|\gamma\|_{-1} - \nu_A\delta_2}\right\}, \tag{3.4}$$

then $\{(\mathbf{u}_h^n, \mathbf{H}_h^n), p_h^n, T_h^n\}$ *is uniformly bounded with respect to* h *and* n.

Proof. Subtracting (2.6) from (3.1), we have

$$\begin{aligned}
a_0(\theta_h^{n+1}, S) + b_0(\mathbf{e}_h^n, T_h, S) + b_0(\mathbf{u}_h^n, \theta_h^{n+1}, S)&\\
+ \rho^{-1}\operatorname{Re}\kappa\left(\nabla(\theta_h^{n+1} - \theta_h^n), \nabla S\right) &= 0.
\end{aligned}\tag{3.5}$$

Choosing $S = \theta_h^{n+1}$ in (3.5) and combining (2.1) with the fact that $b_0\left(\mathbf{u}_h^n, \theta_h^{n+1}, \theta_h^{n+1}\right) = 0$, we get

$$\left(\rho^{-1}\operatorname{Re}\kappa + \kappa\right)\|\nabla\theta_h^{n+1}\|_0^2 \leq \rho^{-1}\operatorname{Re}\kappa\|\nabla\theta_h^n\|_0\|\nabla\theta_h^{n+1}\|_0 + N_0\|\nabla\mathbf{e}_h^n\|_0\|\nabla T_h\|_0\|\nabla\theta_h^{n+1}\|_0,$$

which, together with Theorem 2.3 and the definition of the norm of the product space $\mathbf{D}$, yields

$$\left(\rho^{-1}\operatorname{Re}\kappa + \kappa\right)\|\nabla\theta_h^{n+1}\|_0 \leq \rho^{-1}\operatorname{Re}\kappa\|\nabla\theta_h^n\|_0 + \kappa^{-1}N_0\|\gamma\|_{-1}\|\nabla\left(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n\right)\|_0. \tag{3.6}$$

Then, subtracting (2.7) from (3.2), we obtain

$$\begin{aligned}
\rho^{-1}&\left(\nabla\left(\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\right), \nabla\mathbf{v}\right) + \rho^{-1}\operatorname{Re}a_2\left(\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^n, \mathbf{B}\right)\\
&+ A_0\left(\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right), (\mathbf{v}, \mathbf{B})\right)\\
&+ A_1\left(\left(\mathbf{u}_h^n, \mathbf{H}_h^n\right), \left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right), (\mathbf{v}, \mathbf{B})\right)\\
&+ A_1\left(\left(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n\right), \left(\mathbf{u}_h, \mathbf{H}_h\right), (\mathbf{v}, \mathbf{B})\right)\\
&- d\left((\mathbf{v}, \mathbf{B}), \eta_h^n\right) + \frac{\rho}{\alpha}\left(\operatorname{div}\mathbf{e}_h^{n+1}, \operatorname{div}\mathbf{v}\right)\\
=&\ G\left(\theta_h^{n+1}, (\mathbf{v}, \mathbf{B})\right).
\end{aligned}\tag{3.7}$$

Setting $(\mathbf{v}, \mathbf{B}) = (\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})$ in (3.7) and using Lemma 2.1 and Young's inequality yields

$$\begin{aligned}
&\left(\rho^{-1}\operatorname{Re}\nu_A + \nu_A\right)\|\nabla\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right)\|_0^2\\
&\qquad - d\left(\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right), \eta_h^n\right) + \frac{\rho}{\alpha}\left(\nabla\cdot\mathbf{e}_h^{n+1}, \nabla\cdot\mathbf{e}_h^{n+1}\right)\\
&\leq \operatorname{Re}c_A\|\nabla\left(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n\right)\|_0\|\nabla\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right)\|_0 + \beta\|\theta_h^{n+1}\|_{-1}\|\nabla\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right)\|_0\\
&\qquad + N\|\nabla\left(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n\right)\|_0\|\nabla\left(\mathbf{u}_h, \mathbf{H}_h\right)\|_0\|\nabla\left(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}\right)\|_0.
\end{aligned}\tag{3.8}$$

Note that $A_1\left(\left(\mathbf{u}_h^n, \mathbf{H}_h^n\right), (\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}), (\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\right) = 0$.

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Moreover, combining (2.8) with (3.2) gives

$$\frac{\alpha}{\rho}\left(\eta_h^{n+1}-\eta_h^n,q\right)+d\left((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),q\right)=0. \tag{3.9}$$

Let $q=\eta_h^n$ in (3.9). Then with help of the polarization identity $2ab=(a+b)^2-a^2-b^2$, we have

$$\frac{\alpha}{2\rho}\left(\|\eta_h^{n+1}\|_0^2-\|\eta_h^n\|_0^2-\|\eta_h^{n+1}-\eta_h^n\|_0^2\right)+d\left((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),\eta_h^n\right)=0. \tag{3.10}$$

Next, in order to estimate the term $\|\eta_h^{n+1}-\eta_h^n\|_0$, we select $q=\eta_h^{n+1}-\eta_h^n$ in (3.9) and use the Cauchy–Schwarz inequality to obtain

$$\|\eta_h^{n+1}-\eta_h^n\|_0\le\rho\alpha^{-1}\|\operatorname{div}\mathbf{e}_h^{n+1}\|_0. \tag{3.11}$$

Hence, using (3.10), (3.11), and Theorem 2.3, we rewrite (3.8) as

$$\begin{aligned}(\rho^{-1}\mathrm{Re}\,\nu_A+\nu_A)&\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2\\&\le\frac{\alpha}{2\rho}\|\eta_h^n\|_0^2+(\mathrm{Re}\,c_A+\nu_A\delta_1)\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0\\&\qquad+\beta\|\theta_h^{n+1}\|_{-1}\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0,\end{aligned}$$

which is combined with (3.6) and Young's inequality to give

$$\begin{aligned}&\left(\rho^{-1}\mathrm{Re}\,\nu_A+\nu_A\right)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2\\&\qquad+\frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2+\beta\nu_A^{-1}(\rho^{-1}\mathrm{Re}\,\kappa+\kappa)\|\nabla\theta_h^{n+1}\|_0^2\\&\le\frac{\alpha}{2\rho}\|\eta_h^n\|_0^2+\left(\mathrm{Re}\,c_A+\nu_A\delta_1\right)\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0\\&\qquad+\beta\|\theta_h^{n+1}\|_{-1}\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0+\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^n\|_0\|\nabla\theta_h^{n+1}\|_0\\&\qquad+\beta\nu_A^{-1}\kappa^{-1}N_0\|\gamma\|_{-1}\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0\|\nabla\theta_h^{n+1}\|_0\\&\le\frac{\alpha}{2\rho}\|\eta_h^n\|_0^2+\left(\mathrm{Re}\,c_A+\nu_A\delta_1\right)\left(\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2\right)\\&\qquad+\beta\left(\rho\|\nabla\theta_h^{n+1}\|_0^2+\rho^{-1}\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2\right)\\&\qquad+\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\left(\|\nabla\theta_h^{n+1}\|_0^2+\|\nabla\theta_h^n\|_0^2\right)\\&\quad+\kappa^{-1}N_0\|\gamma\|_{-1}\left(\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2+\nu_A^{-2}\beta^2\|\nabla\theta_h^{n+1}\|_0^2\right).\end{aligned}$$

Then, it is easy to obtain that

$$\begin{aligned}&\left(\rho^{-1}\mathrm{Re}\,\nu_A+\nu_A\delta_2-\mathrm{Re}\,c_A-\rho^{-1}\beta\right)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2\\&\qquad+\frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2+\left(\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+\beta\nu_A^{-1}\kappa\delta_1-\rho\beta\right)\|\nabla\theta_h^{n+1}\|_0^2\\&\quad\le\frac{\alpha}{2\rho}\|\eta_h^n\|_0^2+\left(\mathrm{Re}\,c_A+\nu_A\delta_1+\kappa^{-1}N_0\|\gamma\|_{-1}\right)\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2\\&\qquad\quad+\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^n\|_0^2.\end{aligned} \tag{3.12}$$

Under the assumption (3.4) it holds that

$$\rho^{-1}\mathrm{Re}\,\nu_A+\nu_A\delta_2-2\mathrm{Re}\,c_A-\rho^{-1}\beta-\nu_A\delta_1-\kappa^{-1}N_0\|\gamma\|_{-1}\ge0$$

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and $\beta\nu_A^{-1}\kappa\delta_1 - \rho\beta \geq 0$. Thus, dropping the nonnegative term in (3.12) gives

$$\begin{aligned}
&\left(\mathrm{Re}\, c_A + \nu_A\delta_1 + \kappa^{-1}N_0\|\gamma\|_{-1}\right)\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 \\
&\qquad + \frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2 + \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^{n+1}\|_0^2 \\
&\leq \left(\mathrm{Re}\, c_A + \nu_A\delta_1 + \kappa^{-1}N_0\|\gamma\|_{-1}\right)\|\nabla(\mathbf{e}_h^{n}, \boldsymbol{\xi}_h^{n})\|_0^2 + \frac{\alpha}{2\rho}\|\eta_h^{n}\|_0^2 \\
&\qquad + \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^{n}\|_0^2,
\end{aligned}$$

which combined with Lemma 3.3 results in

$$\begin{aligned}
&\left(\mathrm{Re}\, c_A + \nu_A\delta_1 + \kappa^{-1}N_0\|\gamma\|_{-1}\right)\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 \\
&\qquad + \frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2 + \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^{n+1}\|_0^2 \\
&\leq \left(\mathrm{Re}\, c_A + \nu_A\delta_1 + \kappa^{-1}N_0\|\gamma\|_{-1}\right)\nu_A^{-2}\delta^2\left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right)^2 \\
&\qquad + \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\delta_2^2\beta^{-1}\left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right)^2 \\
&\qquad + \frac{\alpha}{2\rho}\widetilde{\beta}^{-2}\delta^2\left(c_A\nu_A^{-1} + 1\right)^2\left(\|\mathbf{F}\|_{-1} + \beta\kappa^{-1}\|\gamma\|_{-1}\right)^2.
\end{aligned}$$

So, $\{(\mathbf{u}_h^n, \mathbf{H}_h^n), p_h^n, T_h^n\}$ is uniformly bounded, independent of the iterative number n and the mesh size h. ∎

Now, we are going to develop our convergence rate analysis for the Arrow–Hurwicz-type iterative algorithm.

THEOREM 3.5. *Under assumptions of Theorem 3.4, the following estimate holds:*

$$\begin{aligned}
&\varrho_1\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2 + \varrho_2\|\theta_h^{n+1}\|_0^2 \\
&\qquad\qquad \leq \varpi\left(\varrho_1\|\nabla(\mathbf{e}_h^{n}, \boldsymbol{\xi}_h^{n})\|_0^2 + \frac{\alpha}{2\rho}\|\eta_h^{n}\|_0^2 + \varrho_2\|\theta_h^{n}\|_0^2\right),
\end{aligned}$$

where

$$0 < \varrho_1 < \rho^{-1}\mathrm{Re}\,\nu_A + \nu_A\delta_2, \quad 0 < \varrho_2 < \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa + \beta\nu_A^{-1}\kappa\delta_1, \quad \textit{and} \quad 0 < \varpi < 1$$

are three generic constants independent of n and h.

Proof. In fact, according to Lemma 3.3 and Theorems 3.4, 2.3, there exists a positive constant D_1 independent of n and h such that

$$\|\nabla(\mathbf{u}_h^n, \mathbf{H}_h^n)\|_0 \leq D_1. \tag{3.13}$$

Then, rewrite (3.7) to find

$$
\begin{aligned}
d((\mathbf{v},\mathbf{B}),\eta_h^n) = {} & \left(\rho^{-1}\mathrm{Re}\, + 1\right) A_0\left((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\right),(\mathbf{v},\mathbf{B})) - G\left(\theta_h^{n+1},(\mathbf{v},\mathbf{B})\right) \\
& - \rho^{-1}\mathrm{Re}\, A_0\left((\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n}),(\mathbf{v},\mathbf{B})\right) \\
& + A_1\left((\mathbf{u}_h^n,\mathbf{H}_h^n),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),(\mathbf{v},\mathbf{B})\right) \\
& + A_1\left((\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n}),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{v},\mathbf{B})\right) \\
& + \frac{\rho}{\alpha}\left(\mathrm{div}\,\mathbf{e}_h^{n+1},\mathrm{div}\,\mathbf{v}\right).
\end{aligned}
$$

Applying the discrete inf-sup condition (2.5), Lemma 2.1, (3.13), and Theorem 2.3, we obtain

$$
\begin{aligned}
\widetilde{\beta}\|\eta_h^n\|_0 \le {} & \left(\rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0 - \beta\|\nabla\theta_h^{n+1}\|_0 \\
& + (\rho^{-1}\mathrm{Re}\, c_A + \delta_1\nu_A)\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0 \\
\le {} & \left(\rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)(\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0 + \|\nabla\theta_h^{n+1}\|_0) \\
& + (\rho^{-1}\mathrm{Re}\, c_A + \delta_1\nu_A)\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0 \\
& - \left(\beta + \rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)\|\nabla\theta_h^{n+1}\|_0,
\end{aligned}
$$

where we notice that $\|\nabla\cdot\mathbf{e}_h^m\|_0 \le \|\nabla\mathbf{e}_h^m\|_0$. Then, by the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we get

$$
\begin{aligned}
\widetilde{\beta}^2\|\eta_h^n\|_0^2 \le {} & 2\left(\rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)^2(\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0 + \|\nabla\theta_h^{n+1}\|_0)^2 \\
& + 2\Big((\rho^{-1}\mathrm{Re}\, c_A + \nu_A\delta_1)\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0 \\
& \qquad - \left(\beta + \rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)\|\nabla\theta_h^{n+1}\|_0\Big)^2 \\
\le {} & 4(\rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1})^2(\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\nabla\theta_h^{n+1}\|_0^2) \\
& + 4\left(\rho^{-1}\mathrm{Re}\, c_A + \nu_A\delta_1\right)^2\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0^2 \\
& + 4\left(\beta + \rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)^2\|\nabla\theta_h^{n+1}\|_0^2.
\end{aligned} \tag{3.14}
$$

Furthermore, using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ again, inequality (3.6) becomes

$$
\begin{aligned}
& \left(\rho^{-1}\mathrm{Re}\,\kappa + \kappa\right)^2\|\nabla\theta_h^{n+1}\|_0^2 \\
& \qquad \le 2\rho^{-2}\mathrm{Re}^2\kappa^2\|\nabla\theta_h^n\|_0^2 + 2\kappa^{-2}N_0^2\|\gamma\|_{-1}^2\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0^2.
\end{aligned} \tag{3.15}
$$

Next, combining (3.14) and (3.15), we have

$$
\begin{aligned}
\widetilde{\beta}^2\|\eta_h^n\|_0^2 \le {} & 4\left(\rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)^2(\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\nabla\theta_h^{n+1}\|_0^2) \\
& + 8\Big(\left(\rho^{-1}\mathrm{Re}\, c_A + \nu_A\delta_1\right)^2 \\
& \qquad + (\beta + \rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1})^2 \\
& \qquad\qquad \times (\rho^{-1}\mathrm{Re}\,\kappa + \kappa)^{-2}\kappa^{-2}N_0^2\|\gamma\|_{-1}^2\Big)\|\nabla(\mathbf{e}_h^{n},\boldsymbol{\xi}_h^{n})\|_0^2 \\
& + 8\Big(\left(\beta + \rho^{-1}\mathrm{Re}\, c_A + c_A + ND_1 + \rho\alpha^{-1}\right)^2 \\
& \qquad \times \left(\rho^{-1}\mathrm{Re}\,\kappa + \kappa\right)^{-2}\rho^{-2}\mathrm{Re}^2\kappa^2\Big)\|\nabla\theta_h^n\|_0^2.
\end{aligned}
$$

Now, arrange the above inequality to get

$$\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\|\nabla\theta_h^{n+1}\|_0^2\geq B_1\|\eta_h^n\|_0^2-B_2\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2-B_3\|\nabla\theta_h^n\|_0^2, \tag{3.16}$$

where

$$\begin{aligned}
Q_1&:=\left(\rho^{-1}\mathrm{Re}\,c_A+c_A+ND_1+\rho\alpha^{-1}\right)^2,\\
Q_2&:=\left(\rho^{-1}\mathrm{Re}\,c_A+\nu_A\delta_1\right)^2\\
&\qquad+\left(\beta+\rho^{-1}\mathrm{Re}\,c_A+c_A+ND_1+\rho\alpha^{-1}\right)^2\left(\rho^{-1}\mathrm{Re}\,\kappa+\kappa\right)^{-2}\kappa^{-2}N_0^2\|\gamma\|_{-1}^2,\\
Q_3&:=\left(\beta+\rho^{-1}\mathrm{Re}\,c_A+c_A+ND_1+\rho\alpha^{-1}\right)^2\left(\rho^{-1}\mathrm{Re}\,\kappa+\kappa\right)^{-2}\rho^{-2}\mathrm{Re}^2\kappa^2,\\
B_1&:=\frac{\widetilde{\beta}^2}{4Q_1^2},\quad B_2:=\frac{2Q_2}{Q_1^2},\quad\text{and}\quad B_3:=\frac{2Q_3}{Q_1^2}.
\end{aligned}$$

Further, define

$$\begin{aligned}
D_2&:=\rho^{-1}\mathrm{Re}\,\nu_A+\nu_A\delta_2-2\mathrm{Re}\,c_A-\rho^{-1}\beta-\nu_A\delta_1-\kappa^{-1}N_0\|\gamma\|_{-1},\\
D_3&:=\beta\nu_A^{-1}\kappa\delta_1-\rho\beta,\qquad\text{and}\qquad Q_4:=\mathrm{Re}\,c_A+\nu_A\delta_1+\kappa^{-1}N_0\|\gamma\|_{-1}.
\end{aligned}$$

Then (3.12) becomes

$$\begin{aligned}
&(Q_4+D_2)\,\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2\\
&\qquad+\left(\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+D_3\right)\|\nabla\theta_h^{n+1}\|_0^2\\
&\leq Q_4\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2+\frac{\alpha}{2\rho}\|\eta_h^n\|_0^2+\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa\|\nabla\theta_h^n\|_0^2.
\end{aligned}\tag{3.17}$$

Adding and subtracting the terms $\sigma\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2$, $\sigma\|\nabla\theta_h^{n+1}\|_0^2$ to (3.17) and using (3.16) give

$$\begin{aligned}
&\big(Q_4+D_2-\sigma\big)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2\\
&\qquad+\frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2+\left(\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+D_3-\sigma\right)\|\nabla\theta_h^{n+1}\|_0^2\\
&\leq(Q_4+\sigma B_2)\,\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2+\left(\frac{\alpha}{2\rho}-\sigma B_1\right)\|\eta_h^n\|_0^2\\
&\qquad+\left(\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+\sigma B_3\right)\|\nabla\theta_h^n\|_0^2,
\end{aligned}\tag{3.18}$$

where $\sigma>0$ is to be determined. Suppose that the conditions

$$Q_4+D_2-\sigma>0,\quad\frac{\alpha}{2\rho}-\sigma B_1>0,\quad\text{and}\quad\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+D_3-\sigma>0,$$

hold. Then one can calculate the parameter $\sigma>0$ such that

$$\frac{Q_4+\sigma B_2}{Q_4+D_2-\sigma}=\frac{\frac{\alpha}{2\rho}-\sigma B_1}{\frac{\alpha}{2\rho}}=\frac{\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+\sigma B_3}{\frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa+D_3-\sigma},$$

which leads to

$$\bar{a}_1\sigma^2-\bar{b}_1\sigma+\bar{c}_1=0,\qquad\bar{a}_1\sigma^2-\bar{b}_2\sigma+\bar{c}_2=0, \tag{3.19}$$

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where

$$
\begin{aligned}
\overline{a}_1 &= B_1, \qquad\qquad \overline{b}_1 = \frac{\alpha}{2\rho} + D_2 B_1 + B_2\frac{\alpha}{2\rho} + Q_4 B_1,\\
\overline{b}_2 &= \frac{\alpha}{2\rho} + D_3 B_1 + B_3\frac{\alpha}{2\rho} + \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa B_1,\\
\overline{c}_1 &= \frac{\alpha}{2\rho}D_2, \qquad \text{and} \quad \overline{c}_2 = \frac{\alpha}{2\rho}D_3.
\end{aligned}
$$

It is easy to verify that $\overline{b}_1 > \frac{\alpha}{2\rho} + D_2B_1$ and $\overline{b}_2 > \frac{\alpha}{2\rho} + D_3B_1$. According to (3.19), we get

$$
\overline{a}\sigma^2 - \overline{b}\sigma + \overline{c} = 0, \tag{3.20}
$$

where $\overline{a} = 2\overline{a}_1$, $\overline{b} = \overline{b}_1 + \overline{b}_2 > \frac{\alpha}{\rho} + \overline{a}_1(D_2 + D_3)$, and $\overline{c} = \overline{c}_1 + \overline{c}_2 = \frac{\alpha}{2\rho}(D_2 + D_3)$, which leads to

$$
\overline{b}^2 - 4\overline{a}\,\overline{c} > \left(\frac{\alpha}{\rho} + \overline{a}_1\left(D_2 + D_3\right)\right)^2 - \frac{4a_1\alpha}{\rho}\left(D_2 + D_3\right) = \left(\frac{\alpha}{\rho} - \overline{a}_1(D_2 + D_3)\right)^2 \geq 0.
$$

Hence, (3.20) has two real roots $\sigma_{1,2} = \frac{b\pm\sqrt{b^2-4ac}}{2a}$, and here we choose $\sigma = \frac{b-\sqrt{b^2-4ac}}{2a}$.

With the parameter σ chosen as above, it follows from (3.18) that

$$
\begin{aligned}
&\varrho_1\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \frac{\alpha}{2\rho}\|\eta_h^{n+1}\|_0^2 + \varrho_2\|\theta_h^{n+1}\|_0^2\\
&\qquad \leq \varpi\left(\varrho_1\|\nabla(\mathbf{e}_h^{n}, \boldsymbol{\xi}_h^{n})\|_0^2 + \frac{\alpha}{2\rho}\|\eta_h^{n}\|_0^2 + \varrho_2\|\theta_h^{n}\|_0^2\right),
\end{aligned}
$$

where $\varpi = 1 - \frac{2\rho\sigma B_1}{\alpha} \in (0, 1)$ and

$$
\begin{aligned}
\varrho_1 &= Q_4 + D_2 - \sigma < \rho^{-1}\mathrm{Re}\,\nu_A + \nu_A\delta_2,\\
\varrho_2 &= \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa + D_3 - \sigma < \frac{1}{2}\beta\nu_A^{-1}\rho^{-1}\mathrm{Re}\,\kappa + \beta\nu_A^{-1}\kappa\delta_1.
\end{aligned}
$$

This finishes the proof. □

REMARK 3.6. From [18, Remark 3.2], we know that if $\overline{Q}_5 < \rho < \overline{Q}_6$ and

$$
\alpha > \begin{cases} \frac{\rho^2}{2\mathrm{Re}\,(\nu_A - 2c_A) - \rho(2\overline{Q}_4 - 4c_A)}, & \overline{Q}_5 < \rho \leq \mathrm{Re}\,,\\ \frac{\rho^2}{2\mathrm{Re}\,(\nu_A + 2c_A) - \rho(2\overline{Q}_4 + 4c_A)}, & \mathrm{Re} < \rho < \overline{Q}_6 \end{cases}
$$

holds, where

$$
\begin{aligned}
\overline{Q}_4 &= 2\nu_A\delta_1 + N_0^2\kappa^{-2}\|\gamma\|_{-1}^2 + \beta^2, \qquad \overline{Q}_5 = \max\left\{0, \frac{\kappa\mathrm{Re}}{2\kappa - 2}\right\},\\
\overline{Q}_6 &= \min\left\{\frac{\mathrm{Re}\,(\nu_A + 2c_A)}{2c_A + \overline{Q}_4}, \frac{3\kappa\mathrm{Re}}{2(\kappa + 1)}\right\},
\end{aligned}
$$

then the Arrow–Hurwicz algorithm is convergent for the STCIMHD problem. However, from Theorem 3.4, we find that Algorithm 3.1 has no such restriction for the parameter α as the Arrow–Hurwicz algorithm.

4. Parameter choice. The development of effective solvers for the STCIMHD problem is an important problem in the study of incompressible fluids. A successful solver needs to have two fundamental properties: the nonlinear iteration scheme must converge in a small number of iterations and the linear systems that arise at each iteration must be efficiently solvable. The Newton and Picard iterations typically converge in a small number of nonlinear iterations [12] but create nonsymmetric saddle point linear systems that can be difficult to solve. For the considered problem, other types of nonlinear iteration schemes that lead to easier linear systems exist, such as iterated penalty (with a small penalty) or Arrow–Hurwicz methods [13, 18, 40], but they may require a large number of nonlinear iterations to converge.

The grad-div stabilization [32], which was initially studied in [9], is a simple, useful, and popular technique for incompressible flow problems. The grad-div stabilized schemes are constructed to mitigate the lack of mass conservation and can improve the numerical accuracy of the solution, and they help in reducing spurious oscillations for convection-dominated flows. Hence, this tool has been widely studied for incompressible flows over the past decade. In particular, for the Oseen equations, de Frutos et al. [6] have proved that adding a grad-div stabilization term to the Galerkin approximation has a stabilizing effect for small viscosity. Additionally, the grad-div stabilization is also known to aid in preconditioning the Schur complement that arises in the associated linear systems, although there is a trade-off because it makes the solution of the velocity block harder. We notice that it is always a challenge to find a preconditioner that performs equally well for different mesh sizes and parameter ranges. In fact, the grad-div stabilization can be seen as a different discretization of the augmented Lagrangian term, and it shares algebraic properties with an augmented Lagrangian-type term, which motivated Heister and Rapin [14] to construct a preconditioner with grad-div stabilization.

However, the parameters of the proposed algorithm affect the performance of both the linear and nonlinear solvers. Hence, it is important to investigate the relationship between algorithmic parameters and iterative linear solutions. Next, we consider the matrix representation of (3.1)–(3.3). Given the nodal basis functions of $\mathbf{D}_h \times Q_h \times M_h$, let the symbols $\overrightarrow{U}^n, \overrightarrow{H}^n, \overrightarrow{T}^n$, and $\overrightarrow{P}^n$ denote the vector representations of the discrete velocity field, the discrete magnetic field, the discrete temperature field, and the discrete pressure field, respectively. Then, by some simple calculations, we can reformulate the problem (3.1)–(3.3) in matrix form as follows:

$$\begin{bmatrix} (\rho^{-1}\mathrm{Re}+1)S_1+S_2 & 0 & 0 & 0 \\ -S_3 & \mathcal{Q} & S_5 & 0 \\ 0 & S_5^T & (\rho^{-1}\mathrm{Re}+1)S_6 & 0 \\ 0 & -\frac{\rho}{\alpha}S_7 & 0 & S_8 \end{bmatrix} \begin{bmatrix} \overrightarrow{T}^{n+1} \\ \overrightarrow{U}^{n+1} \\ \overrightarrow{H}^{n+1} \\ \overrightarrow{P}^{n+1} \end{bmatrix}$$

$$(4.1) \qquad = \begin{bmatrix} \rho^{-1}\mathrm{Re} & 0 & 0 & 0 \\ 0 & \rho^{-1}\mathrm{Re} & 0 & 1 \\ 0 & 0 & \rho^{-1}\mathrm{Re} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_1\overrightarrow{T}^n \\ S_4\overrightarrow{U}^n \\ S_6\overrightarrow{H}^n \\ S_8\overrightarrow{P}^n \end{bmatrix} + \begin{bmatrix} M_1 \\ F \\ G \\ 0 \end{bmatrix},$$

where $\mathcal{Q} := \left(\mathrm{Re}\rho^{-1}+1\right)S_4 + S_9 + \frac{\rho}{\alpha}S_{10}$, F, M_1, and G are vector representations of the body forces $\mathbf{f}, \gamma$, and $\mathbf{g}$, and where S_1–S_{10} denote matrix representations of the various discrete terms as given in the following table:

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Symbol	denotes matrix representation of
S_1	$a_0(\cdot,\cdot)$
S_2	$b_0(\cdot,\cdot,\cdot)$
S_3	$G(\cdot,(\cdot,\cdot))$
S_4	$a_1(\cdot,\cdot)$
S_5	$b_2(\cdot,\cdot,\cdot)$
S_6	$a_2(\cdot,\cdot)$
S_7	$(\nabla\cdot\mathbf{u}_h,\mathbf{v})$
S_8	mass matrix related to p_h
S_9	$b_1(\cdot,\cdot,\cdot)$
S_{10}	$(\nabla\cdot\mathbf{u}_h,\nabla\cdot\mathbf{v})$

For the linear system (4.1), the proper choice of the algorithmic parameters ρ and α may improve the condition number of the coefficient matrix, and then it can accelerate the convergence rate of the numerical solvers used. Hence, ρ and α play the role of a preconditioner. We now turn to investigate the important issue of the relationship between α, ρ, and the iterative linear solutions. Here the GMRES solver is applied as in the work [14].

Let the computational domain $\Omega=[0,1]\times[0,1]$ and the right-hand side functions $\mathbf{f},\mathbf{g}$, and γ be selected such that the exact solutions are given by

$$\begin{aligned} u_1(x,y) &= 0.5x^2(x-1)^2y(y-1)(2y-1), & u_2(x,y) &= -0.5y^2(y-1)^2x(x-1)(2x-1),\\ H_1(x,y) &= 0.5\sin(\pi x)\cos(\pi y), & H_2(x,y) &= -0.5\cos(\pi x)\sin(\pi y),\\ p(x,y) &= 0.5\cos(\pi x)\cos(\pi y), & T(x,y) &= u_1(x,y)+u_2(x,y). \end{aligned}$$

Here, we set the parameters $s=\mathrm{Rm}=\kappa=\beta=1$. We use the MINI-element to approximate the velocity, pressure, temperature and magnetic field, respectively.

Now, we consider test cases in which the linear systems are solved using the GMRES solver. If the GMRES solver fails in an iteration, then the result is denoted by an "F". In Table 4.1 and 4.2, we list the iteration number for each parameter value of ρ and α, respectively. From these tables, we find that if the value of α decreases or the value of ρ increases, then the computational time decreases. In addition, the computational time, the average number of GMRES iterations, as well as the necessary number of nonlinear iterations for different values of Re are reported in Table 4.3. From this table, we find that the GMRES solver fails to converge at large values of Re.

TABLE 4.1
Analysis of the solver performance and the influence of the stabilization on the error with respect to the parameter choice of ρ with $\alpha=1$, $h=1/64$, and $\mathrm{Re}=1$.

ρ	0.001	0.01	0.1	1	2	3	4	5
Outer iterations	F	1491	241	31	15	12	13	F
Inner iterations	F	116	131	154	140	141	132	F
CPU time	F	6205.451	1189.801	210.457	123.570	105.772	110.381	F

5. Numerical tests. The grad-div stabilization can compensate for the lack of mass conservation and improve the numerical accuracy of the solution. In this section, we mainly consider this property of the grad-div stabilization and provide some numerical examples to test the performance of the proposed algorithm. We assess the numerical performance of Algorithm 3.1 for the STCIMHD equations. We use the MINI-element to approximate the velocity, pressure, temperature, and magnetic field, respectively. Here, we take the fixed tolerance $1.0\text{e-}6$ in these tests.

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TABLE 4.2
Analysis of the solver performance and the influence of the stabilization on the error with respect to the parameter choice of α with $\rho = 1$, $h = 1/64$, and $\mathrm{Re} = 1$.

α	0.0001	0.001	0.01	0.1	1	10	100	1000
Outer iterations	F	19	19	19	31	231	1375	F
Inner iterations	F	130	133	145	154	129	116	F
CPU time	F	142.601	157.771	214.179	210.457	957.610	4524.961	F

TABLE 4.3
The CPU time, the number of nonlinear iterations and average number of GMRES iterations with different Reynolds numbers ($\rho = 3$, $\alpha = 1$, and $1/h = 64$).

Re	0.01	0.1	1	10	100	1000
CPU time	375.947	379.571	105.772	359.831	1960.997	F
Nonlinear iterations	63	61	12	49	396	F
Average number of GMRES	199	337	388	303	245	F

5.1. Convergence test. Consider the exact solution in the previous section. In Table 5.1, we list the number of iterations of Algorithm 3.1 and the Arrow–Hurwicz algorithm with different ρ. From this table, we find that both algorithms converge with the parameter $\rho = 1$. If the parameter ρ increases, then the Arrow–Hurwicz algorithm diverges while Algorithm 3.1 still works well.

TABLE 5.1
Number of iterations with different ρ ($\alpha = 1$, $h = 1/64$, $\mathrm{Re} = 1$). "—" means that the number is larger than 1000.

ρ	1	2	3	4	4.5
Algorithm 3.1	31	15	12	12	13
Arrow–Hurwicz algorithm [18]	34	—	—	—	—

TABLE 5.2
Number of iterations with different α ($\rho = 1$, $h = 1/64$, $\mathrm{Re} = 1$). "—" means that the iterative number is larger than 1000.

α	1	0.1	0.01
Algorithm 3.1	31	19	19
Arrow–Hurwicz algorithm [18]	34	—	—

Moreover, we list the number of iterations of Algorithm 3.1 and the Arrow–Hurwicz algorithm for different α in Table 5.2. From this table, we find that the Arrow–Hurwicz algorithm is divergent with $\alpha = 0.1$. However, Algorithm 3.1 is convergent for even smaller $\alpha = 0.01$.

In Table 5.3, we collect the norms of the divergence of $\mathbf{u}_h$. From this table, we find that Algorithm 3.1 and the Arrow–Hurwicz algorithm run well and keep the optimal convergence rate with $\mathrm{Re} = 1$. However, for $\mathrm{Re} = 10$, the Arrow–Hurwicz algorithm is not convergent, but Algorithm 3.1 can still be implemented and optimal convergence order for the divergent error is obtained.

5.2. Thermal driven cavity problem. In this experiment, we test Algorithm 3.1 for the thermal driven cavity problem, which is investigated in [37]. The computational domain

TABLE 5.3
Norm of the divergence and convergence orders by different algorithms ($\rho = \alpha = 1$).

	Arrow–Hurwicz [18] ($\mathrm{Re} = 1$)		Arrow–Hurwicz [18] ($\mathrm{Re} = 10$)	Algorithm 3.1 ($\mathrm{Re} = 1$)		Algorithm 3.1 ($\mathrm{Re} = 10$)	
$1/h$	$\Vert \operatorname{div} \mathbf{u}_h^{n+1} \Vert_0$	Order	$\Vert \operatorname{div} \mathbf{u}_h^{n+1} \Vert_0$	$\Vert \operatorname{div} \mathbf{u}_h^{n+1} \Vert_0$	Order	$\Vert \operatorname{div} \mathbf{u}_h^{n+1} \Vert_0$	Order
32	8.115e-4	—	Inf.	5.248e-4	—	6.706e-3	—
64	4.217e-4	0.94	Inf.	3.157e-4	0.73	3.423e-3	0.97
128	2.183e-4	0.94	Inf.	1.674e-4	0.91	1.715e-3	0.99
256	1.156e-4	0.91	Inf.	8.498e-5	0.98	8.569e-4	1.00

consists of a square cavity with differentially heated vertical walls, where the left and right walls are kept at $T = 1$ and $T = 0$, respectively. The remaining walls are insulated, and there is no heat transfer through them. No-slip boundary conditions are imposed for the velocity at all walls. For the magnetic field, we set $H_1 = 1$, $\frac{\partial H_2}{\partial \mathbf{n}} = 0$ at the horizontal walls, and $H_2 = 0$, $\frac{\partial H_1}{\partial \mathbf{n}} = 0$ at the vertical walls.

In the numerical example, the computations are performed on the uniform grid 64×64. Here, we set the model parameters $s = \mathrm{Re} = \mathrm{Rm} = 1$ and take $\mathbf{f} = \mathbf{g} = \mathbf{0}$ and $\gamma = 0$. In Table 5.4, we provide the maximum velocity for different thermal expansion coefficients β at $y = 0.5$. From this table, we can see that Algorithm 3.1 requires less CPU time than the Arrow–Hurwicz algorithm to obtain nearly the same maximum velocity value. In particular, when the value of the thermal expansion coefficient reaches $\beta = 1000$, Algorithm 3.1 still works well.

TABLE 5.4
Comparisons of the maximum velocity values obtained by different algorithms. "—" means that the iterative number is larger than 1000.

	$\beta = 1$	$\beta = 10$	$\beta = 100$	$\beta = 1000$	CPU time($\beta = 100$)
Arrow–Hurwicz algorithm [18]	0.188	0.224	0.576	—	481.867
Algorithm 3.1	0.190	0.224	0.577	3.865	512.812

Furthermore, in Figures 5.1–5.3 below, the numerical velocity streamlines, the magnetic field, and the isotherms for the considered problem obtained by Algorithm 3.1 and the Arrow–Hurwicz algorithm are displayed for different thermal expansion coefficients. These figures show that Algorithm 3.1 can capture the solutions of the large thermal expansion coefficient problem.

REFERENCES

[1] M. AKBAS, A. LINKE, L. G. REBHOLZ, AND P. W. SCHROEDER, *The analogue of grad-div stabilization in DG methods for incompressible flows: limiting behavior and extension to tensor-product meshes*, Comput. Methods Appl. Mech. Engrg., 341 (2018), pp. 917–938.
[2] S. BADIA, A. F. MARTÍN, AND R. PLANAS, *Block recursive LU preconditioners for the thermally coupled incompressible inductionless MHD problem*, J. Comput. Phys., 274 (2014), pp. 562–591.
[3] A. BERMÚDEZ, R. MUÑOZ SOLA, AND R. VÁZQUEZ, *Analysis of two stationary magnetohydrodynamics systems of equations including Joule heating*, J. Math. Anal. Appl., 368 (2010), pp. 444–468.
[4] M. A. CASE, V. J. ERVIN, A. LINKE, AND L. G. REBHOLZ, *A connection between Scott-Vogelius and grad-div stabilized Taylor-Hood FE approximations of the Navier-Stokes equations*, SIAM J. Numer. Anal., 49 (2011), pp. 1461–1481.
[5] R. CODINA AND N. HERNÁNDEZ, *Approximation of the thermally coupled MHD problem using a stabilized finite element method*, J. Comput. Phys., 230 (2011), pp. 1281–1303.
[6] J. DE FRUTOS, B. GARCÍA-ARCHILLA, V. JOHN, AND J. NOVO, *Grad-div stabilization for the evolutionary*

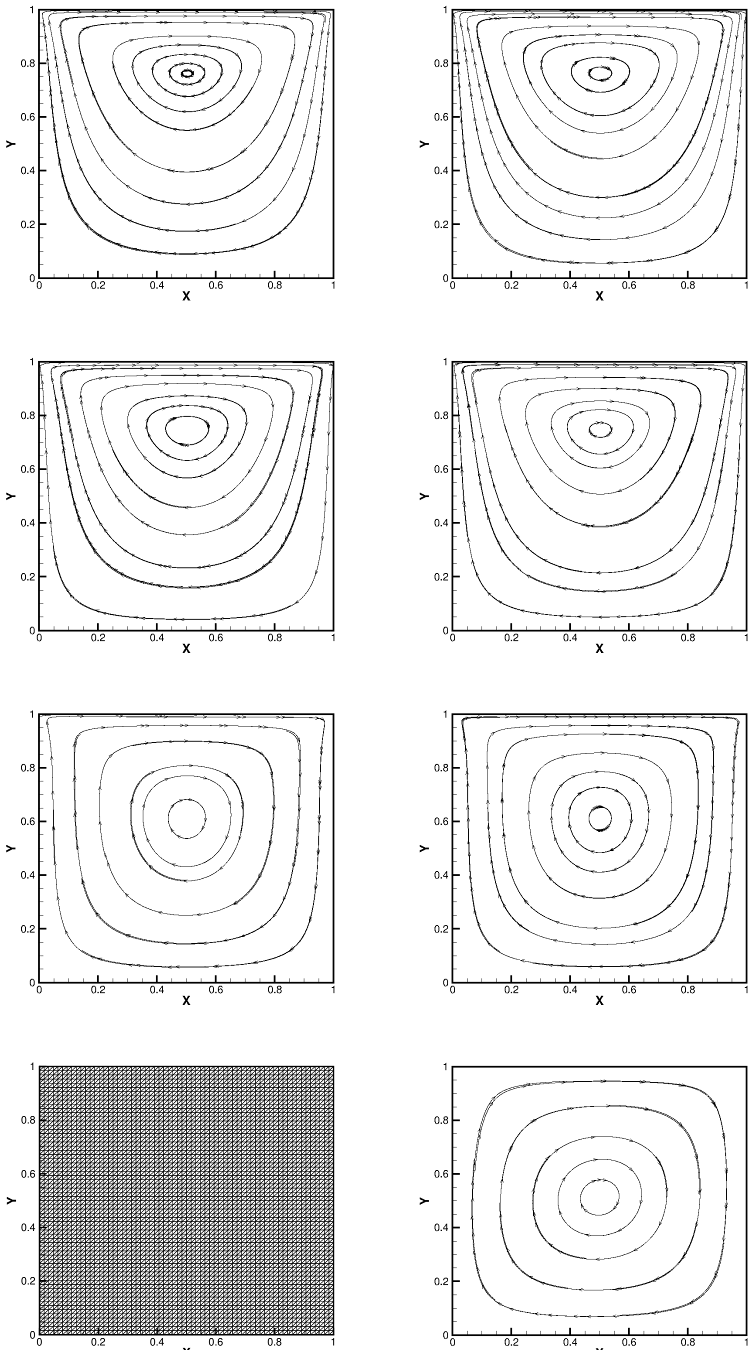

FIG. 5.1. *The velocity streamlines obtained by the Arrow–Hurwicz algorithm (the first column) and Algorithm 3.1 (the second column) at* $\beta = 1,\ 10,\ 100,$ *and* 1000 *from top to bottom.*

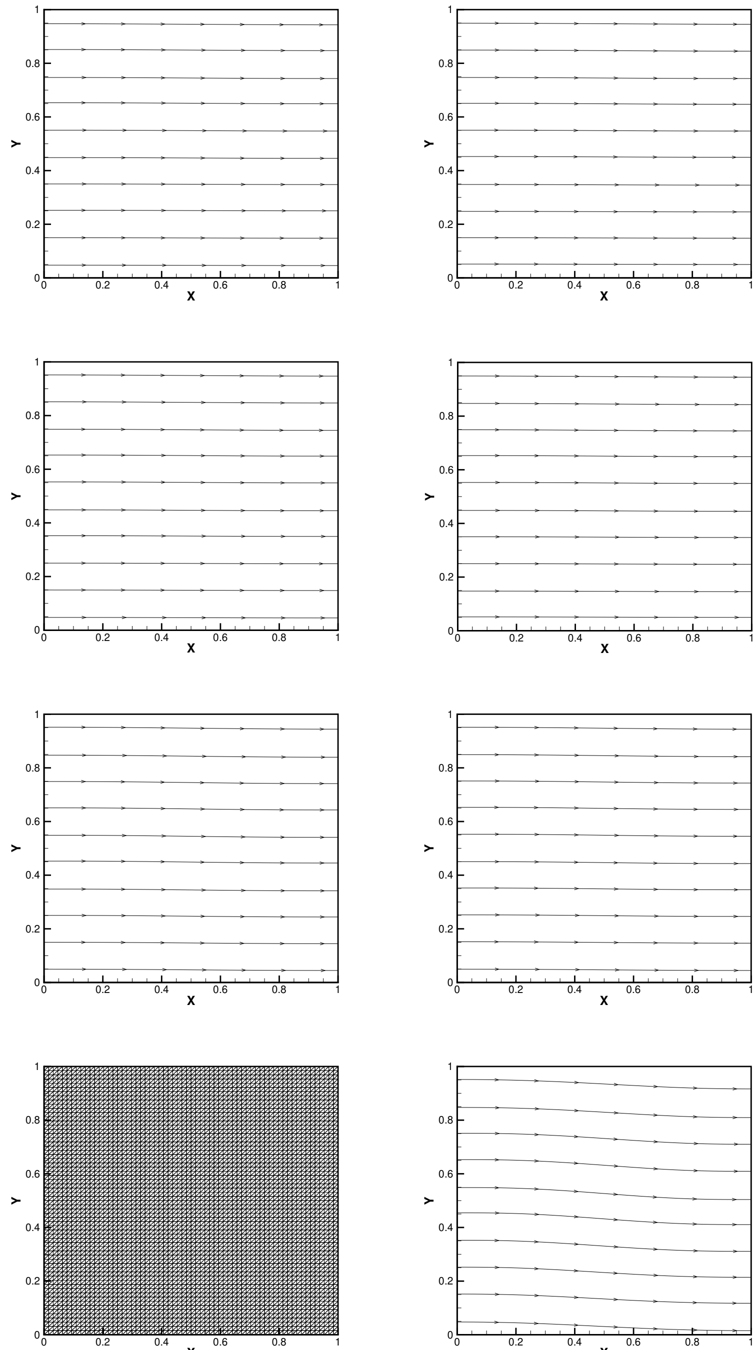

FIG. 5.2. *The magnetic fields obtained by the Arrow–Hurwicz algorithm (the first column) and Algorithm 3.1 (the second column) at* $\beta = 1$, 10, 100, *and* 1000 *from top to bottom.*

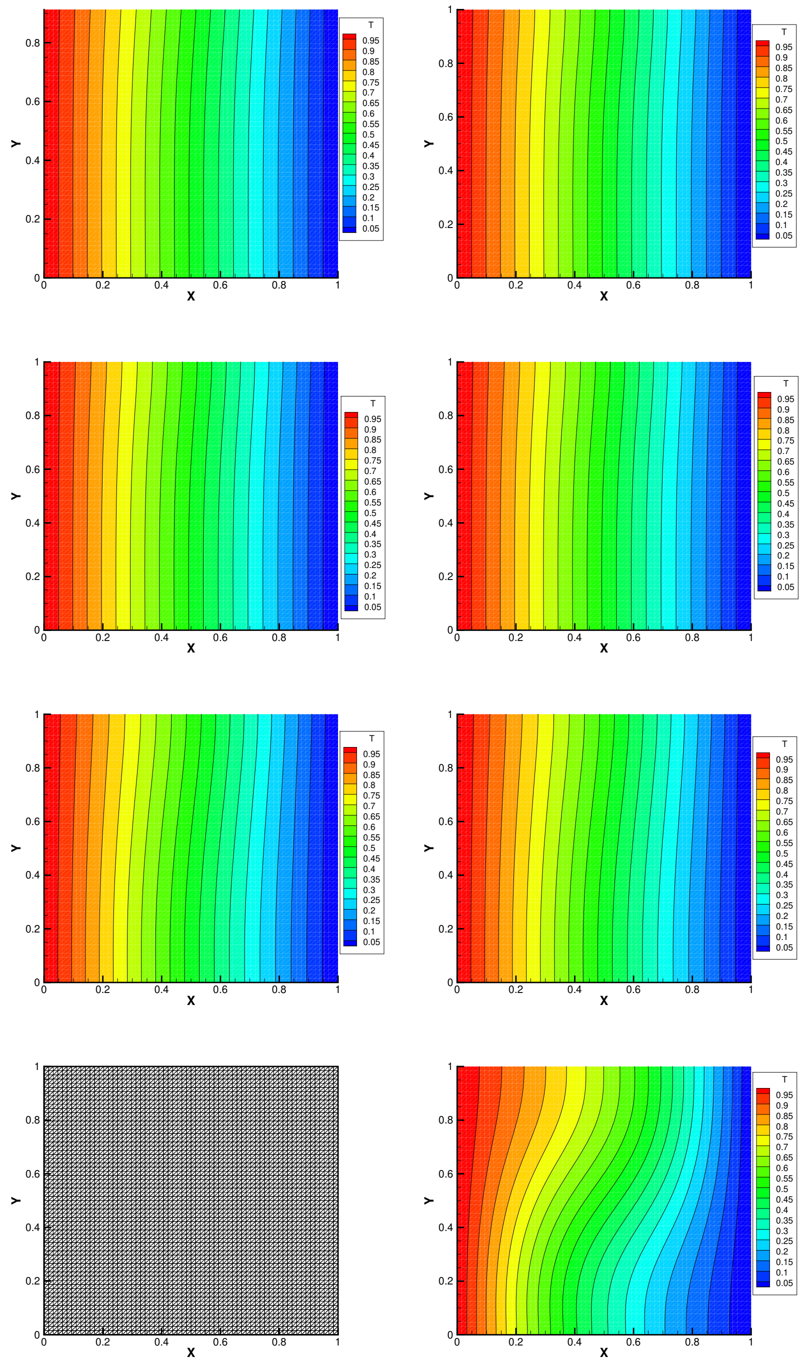

FIG. 5.3. *The isotherms obtained by the Arrow–Hurwicz algorithm (the first column) and Algorithm 3.1 (the second column) at* $\beta = 1$, 10, 100, *and* 1000 *from top to bottom.*

Oseen problem with inf-sup stable finite elements, J. Sci. Comput., 66 (2016), pp. 991–1024.

[7] Q. Ding, X. Long, and S. Mao, *Convergence analysis of Crank-Nicolson extrapolated fully discrete scheme for thermally coupled incompressible magnetohydrodynamic system*, Appl. Numer. Math., 157 (2020), pp. 522–543.

[8] B. Du, J. Huang, and H. Zheng, *Two-grid Arrow-Hurwicz methods for the steady incompressible Navier-Stokes equations*, J. Sci. Comput., 89 (2021), Paper No. 24, 24 pages.

[9] L. P. Franca and T. J. R. Hughes, *Two classes of mixed finite element methods*, Comput. Methods Appl. Mech. Engrg., 69 (1988), pp. 89–129.

[10] J.-F. Gerbeau, C. Le Bris, and T. Lelièvre, *Mathematical Methods for the Magnetohydrodynamics of Liquid Metals*, Oxford University Press, Oxford, 2006.

[11] P. G. Geredeli, L. G. Rebholz, D. Vargun, and A. Zytoon, *Improved convergence of the Arrow-Hurwicz iteration for the Navier-Stokes equation via grad-div stabilization and Anderson acceleration*, J. Comput. Appl. Math., 422 (2023), Paper No. 114920, 16 pages.

[12] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[13] M. D. Gunzburger, *Iterated penalty methods for the Stokes and Navier-Stokes equations*, in Finite Element Analysis in Fluids, T. J. Chung and G. R. Karr, eds., UAH Press, Huntsville, 1989, pp. 1040–1045.

[14] T. Heister and G. Rapin, *Efficient augmented Lagrangian-type preconditioning for the Oseen problem using grad-div stabilization*, Internat. J. Numer. Methods Fluids, 71 (2013), pp. 118–134.

[15] E. W. Jenkins, V. John, A. Linke, and L. G. Rebholz, *On the parameter choice in grad-div stabilization for the Stokes equations*, Adv. Comput. Math., 40 (2014), pp. 491–516.

[16] V. John, A. Linke, C. Merdon, M. Neilan, and L. G. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*, SIAM Rev., 59 (2017), pp. 492–544.

[17] A. Keram and P. Huang, *An iterative method for the thermally coupled incompressible magneto-hydrodynamics equations at high parameter*, Math. Methods Appl. Sci., Published online, 2023, https://doi.org/10.1002/mma.9420.

[18] ———, *The Arrow-Hurwicz iterative finite element method for the stationary thermally coupled incompressible magnetohydrodynamics flow*, J. Sci. Comput., 92 (2022), Paper No. 11, 25 pages.

[19] D. Lai, P. Huang, and Y. He, *One- and two-level Arrow–Hurwicz-type iterative algorithms for the stationary Smagorinsky model*, Commun. Nonlinear Sci. Numer. Simul., 134 (2024), Paper No. 108001, 19 pages.

[20] W. Layton, C. C. Manica, M. Neda, M. Olshanskii, and L. G. Rebholz, *On the accuracy of the rotation form in simulations of the Navier-Stokes equations*, J. Comput. Phys., 228 (2009), pp. 3433–3447.

[21] W. Li, P. Huang, and Y. He, *Grad-div stabilized finite element schemes for the fluid-fluid interaction model*, Commun. Comput. Phys., 30 (2021), pp. 536–566.

[22] A. Linke, L. G. Rebholz, and N. E. Wilson, *On the convergence rate of grad-div stabilized Taylor-Hood to Scott-Vogelius solutions for incompressible flow problems*, J. Math. Anal. Appl., 381 (2011), pp. 612–626.

[23] S. Liu and P. Huang, *A sparse grad-div stabilized algorithm for the incompressible magnetohydrodynamics equations*, Comput. Math. Appl., 138 (2023), pp. 106–119.

[24] ———, *A grad-div stabilized method using the Jacobi iteration for the thermally coupled incompressible magnetohydrodynamic system*, ZAMM Z. Angew. Math. Mech., 103 (2023), Paper No. e202200362, 22 pages.

[25] S. Liu, P. Huang, and Y. He, *Error analysis of BDF-Galerkin FEMs for thermally coupled incompressible MHD with temperature dependent parameters*, East Asian J. Appl. Math., 14 (2024), pp. 731–768.

[26] X. Lu and P. Huang, *A modular grad-div stabilization for the 2D/3D nonstationary incompressible magnetohydrodynamic equations*, J. Sci. Comput., 82 (2020), Paper No. 3, 24 pages.

[27] H. Ma and P. Huang, *A fully discrete decoupled finite element method for the thermally coupled incompressible magnetohydrodynamic problem*, J. Sci. Comput., 95 (2023), Paper No. 14, 39 pages.

[28] A. Meir, *Thermally coupled magnetohydrodynamics flow*, Appl. Math. Comput., 65 (1994), pp. 79–94.

[29] ———, *Thermally coupled, stationary, incompressible MHD flow; existence, uniqueness, and finite element approximation*, Numer. Methods Partial Differential Equations, 11 (1995), pp. 311–337.

[30] A. Meir and P. G. Schmidt, *On electromagnetically and thermally driven liquid-metal flows*, Nonlinear Anal., 47 (2001), pp. 3281–3294.

[31] M. Olshanskii, G. Lube, T. Heister, and J. Löwe, *Grad-div stabilization and subgrid pressure models for the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 3975–3988.

[32] M. A. Olshanskii and A. Reusken, *Grad-div stabilization for Stokes equations*, Math. Comp., 73 (2004), pp. 1699–1718.

[33] S. S. Ravindran, *Partitioned time-stepping scheme for an MHD system with temperature-dependent coefficients*, IMA J. Numer. Anal., 39 (2019), pp. 1860–1887.

[34] ———, *Partitioned time-stepping scheme for an MHD system with temperature-dependent coefficients*, IMA J. Numer. Anal., 39 (2019), pp. 1860–1887.

[35] A. TAKHIROV, A. ÇIBIK, F. G. EROGLU, AND S. KAYA, *An improved Arrow-Hurwicz method for the steady-state Navier-Stokes equations*, J. Sci. Comput., 96 (2023), Paper No. 52, 21 pages.

[36] Q. WANG, P. HUANG, AND Y. HE, *A generalized scalar auxiliary variable approach for the Navier–Stokes-ω/Navier–Stokes-ω equations based on the grad-div stabilization*, Commun. Nonlinear Sci. Numer. Simul., 140 (2025), Paper No. 108329, 9 pages.

[37] J. YANG AND T. ZHANG, *Stability and convergence of iterative finite element methods for the thermally coupled incompressible MHD flow*, Int. J. Numer. Methods Heat Fluid Flow, 30 (2020), pp. 5103–5141.

[38] X. YANG, P. HUANG, AND Y. HE, *A Voigt-regularization of the thermally coupled inviscid, resistive magneto-hydrodynamic*, Int. J. Numer. Anal. Model., 21 (2024), pp. 476–503.

[39] ———, *A Voigt regularization of the thermally coupled magnetohydrodynamic flow*, Z. Angew. Math. Phys., 75 (2024), Paper No. 115, 20 pages.

[40] Y.-B. YANG, Y.-L. JIANG, AND Q.-X. KONG, *The Arrow-Hurwicz iterative finite element method for the stationary magnetohydrodynamics flow*, Appl. Math. Comput., 356 (2019), pp. 347–361.

[41] Y. ZENG AND P. HUANG, *A second-order modular grad-div stabilized scheme for the Darcy–Brinkman model*, Numerical Heat Transfer, Part B, 83 (2023), pp. 205–226.