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# A GENERAL CLASS OF ITERATIVE SPLITTING METHODS FOR SOLVING LINEAR SYSTEMS*

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**Abstract.** Recently Ahmadi et al. [IEEE Trans. Parallel Distrib. Syst., 32 (2021), pp. 1452–1464] and Tagliaferro [Research Square (2022)] proposed some iterative methods for the numerical solution of linear systems which, under the classical hypothesis of strict diagonal dominance, typically converge faster than the Jacobi method but slower than the forward/backward Gauss–Seidel one. In this paper we introduce a general class of iterative methods, based on suitable splittings of the matrix that defines the system, which include all of the methods mentioned above and have the same cost per iteration in a sequential computation environment. We also introduce a partial order relation in the set of splittings and, partly theoretically and partly on the basis of a number of examples, we show that such partial order is typically connected to the speed of convergence of the corresponding methods. We pay particular attention to the case of linear systems for which the Jacobi iteration matrix is nonnegative, in which case we give a rigorous proof of the correspondence between the partial order relation and the magnitude of the spectral radius of the iteration matrices. Within the considered general class, some new specific promising methods are proposed as well.

**Key words.** linear systems, iterative methods, matrix splitting

**AMS subject classifications.** 65F10

## 1. Introduction.

Iterative methods for the numerical solution of a linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, constitute a widely investigated field of research. Various efficient methods have been developed, taking into account the dimension of the problem and the features of the matrix $A$. Moreover, also the architecture of the employed computers is carefully taken into consideration, with particular attention to parallel implementations.

Some of the most popular iterative methods are the *Jacobi method* and the (*forward* or *backward* or *symmetric*) *Gauss–Seidel methods* (see, e.g., Stoer and Bulirsch [4] or Golub and Van Loan [3]), designed based on splittings of the coefficient matrix $A$ as

$$A = C + D + E,$$

where $C$ and $E$ are the strictly lower and the strictly upper triangular parts of $A$, respectively, and $D$ is its diagonal, whose elements are required to be all $\neq 0$.

Starting from an initial approximation $x^{(0)}$ to the solution, these methods take the form

$$x^{(k+1)} = Bx^{(k)} + c, \tag{1.2}$$

with

$$B = B_J := -D^{-1}(C + E) \qquad \text{and} \qquad c = c_J := D^{-1}b$$

for the Jacobi method,

$$B = B_{fGS} := -(D + C)^{-1}E \qquad \text{and} \qquad c = c_{fGS} := (D + C)^{-1}b$$

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for the forward Gauss–Seidel method,

$$B = B_{bGS} := -(D+E)^{-1}C \qquad \text{and} \qquad c = c_{bGS} := (D+E)^{-1}b$$

for the backward Gauss–Seidel method, and

$$B = B_{sGS} := (D+E)^{-1}C(D+C)^{-1}E \quad \text{and} \quad c = c_{sGS} := (D+E)^{-1}[I-C(D+C)^{-1}]b$$

or

$$B = B'_{sGS} := (D+C)^{-1}E(D+E)^{-1}C \quad \text{and} \quad c = c'_{sGS} := (D+C)^{-1}[I-E(D+E)^{-1}]b$$

for the symmetric Gauss–Seidel method.

We remark that, even if the forward and backward Gauss–Seidel iterations look more elaborate (as they require the solution of a triangular system, which may be solved by forward or backward substitution, respectively), their computational complexity is the same as that of the Jacobi iteration. Moreover, also the symmetric Gauss–Seidel iteration, which just gathers sequentially one iteration of the forward and one of the backward Gauss–Seidel method, can be suitably rearranged in order to reach the same computational cost (see the related discussion in Section 6.3). Therefore, in a standard sequential environment, the efficiency comparison among all the above mentioned methods is based on their respective speed of convergence.

It is well known that an iterative method of type (1.2) is convergent if and only if $\rho(B) < 1$, where $\rho(\cdot)$ denotes the spectral radius, and that the smaller $\rho(B)$ is, the faster is the convergence. It is also known that a sufficient condition for the convergence of the aforementioned methods is that the matrix $A$ be *strictly diagonally dominant* either *by rows* (i.e., if the condition $\|D^{-1}(C+E)\|_\infty = \|B_J\|_\infty < 1$ holds) or *by columns* (i.e., if $\|(C+E)D^{-1}\|_1 = \|DB_JD^{-1}\|_1 < 1$). As is customary, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ stand for the well-known matrix *infinity*- and 1-norm, respectively.

Although not being a general rule, in most cases of practical interest and when all of such methods converge, the symmetric Gauss–Seidel iteration is faster than both the forward and backward Gauss–Seidel one, and, in turn, these last two methods are faster than the Jacobi iteration. One important practical case, typically produced by a finite difference method applied to a linear system of differential equations, is that of a system characterized by a matrix $A$ (also called *L-matrix* [8]), for which the diagonal elements (those of $D$) are strictly positive and $C + E \leq O$ elementwise (see the well-known Stein–Rosenberg theorem and its generalizations [2, 7]).

Nevertheless, the Jacobi method is much better suited to exploit the potential speedup of a parallel computer environment than the Gauss–Seidel methods, and this fact may often change the balance of efficiency into its favour.

Recently, Ahmadi et al. [1] and, subsequently, Tagliaferro [5] proposed some iterative methods which are a kind of halfway house between the Jacobi and the forward and backward Gauss–Seidel iterations. More precisely, in [1] the Jacobi iteration matrix $B_J$ is split into two or more sets of rows, and one passes from the previous iterate to the next one by applying one after the other the various selected sets of rows in a certain order fixed a priori, each time working on the already modified components. Differently, in [5] the Jacobi iteration matrix $B_J$ is split into the lower and the upper triangular parts, and one passes from the previous iterate to the next one by applying first the upper triangle and then, on the so modified vector, the lower triangle. These two strategies are different from each other in the way of choosing the splittings of $B_J$, but both of them may be embedded in the more general methodology that we want to present in this paper.

The methods introduced in [1], which we shall call *AMKS-methods*, will be carefully analyzed in Section 5, whereas the method introduced in [5], which we shall call *$T_U$-method*,

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will be considered soon as an inspiring model to define our general splitting technique. This method looks like a usual iterative method (1.2) in the doubled dimension space $\mathbb{R}^{2n}$. Starting from a pair of initial approximations $\{x_1^{(0)}, x_2^{(0)}\}$, a sequence of pairs of vectors $\{x_1^{(k)}, x_2^{(k)}\}_{k\geq 1}$ is defined by setting

$$\begin{cases} x_1^{(k+1)} = Ux_1^{(k)} + Lx_2^{(k)} + c, \\ x_2^{(k+1)} = Ux_1^{(k+1)} + Lx_2^{(k)} + c, \end{cases} \tag{1.3}$$

where

$$L = -D^{-1}C, \qquad U = -D^{-1}E, \qquad c = D^{-1}b. \tag{1.4}$$

Denoting by $I$ and $O$ the identity and the zero $(n\times n)$-matrix, respectively, it is immediate to see that the iterative scheme (1.3) may be reformulated in the more compact form

$$X^{(k+1)} = \mathcal{B}_{T_U} X^{(k)} + \Gamma_{T_U}, \tag{1.5}$$

where

$$\mathcal{B}_{T_U} = \begin{bmatrix} I & O \\ -U & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} U & L \\ O & L \end{bmatrix} = \begin{bmatrix} U & L \\ U^2 & UL+L \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

and

$$X^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \in \mathbb{R}^{2n},\ k \geq 0, \qquad \text{and} \qquad \Gamma_{T_U} = \begin{bmatrix} c \\ (U+I)c \end{bmatrix} \in \mathbb{R}^{2n}.$$

The fixed point of (1.5) is given by $X = [x^T, x^T]^T$, where $x$ is the solution of (1.1). Looking carefully at (1.3) reveals that it is possible to carry out the iterations just by proceeding with the pair of $n$-vectors $\{Ux_1^{(k)}, x_2^{(k)}\}$ and that the computational complexity of one iteration still is the same as for the Jacobi or the Gauss–Seidel methods. In a parallel computer environment, the situation might instead change to the detriment of the $\mathrm{T}_U$-method (1.3) only if the number of parallel processors available becomes large enough. Just to give a rough, and also pessimistic, idea of the situation: if we assume that each processor can handle the product of a matrix row times a vector in a certain time $\tau$, with $n$ processors or more available, the Jacobi method can perform an iteration by computing a matrix-vector product all together in the same amount of time $\tau$, whereas the $\mathrm{T}_U$-method needs the computation of two matrix-vector products in succession, taking the total time of about $2\tau$. Anyway, such a number of available processors is often unrealistic when dealing with very large systems of equations arising from practical applications. However, in this paper we do not consider any parallel computation issues any longer and confine ourselves to evaluate the sequential complexity only.

Again without being a general rule, in many cases of practical interest it has been experimentally observed that, when all the three methods converge, the asymptotic rate of convergence of the $\mathrm{T}_U$-method lies between the Jacobi and Gauss–Seidel ones.

Below we introduce a general class of iterative methods based on an arbitrary *splitting* of the matrix $B_J$, that is, $B_J = B_1 + \ldots + B_d$ (in (1.3) we simply have $B_1 = U$, $B_2 = L$).

The paper is organized as follows. In Section 2 we define the iterative method corresponding to a given splitting, show the important property of *cyclicity* of the splittings, and introduce the partial order relation of *refinement*. We also give the notion of *essentiality* and define

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the concept of *potentially optimal* splittings. In Section 3 we treat the case of linear systems for which $B_J$ is nonnegative. In this setting we prove that the more refined the splitting is, the smaller the spectral radius of the corresponding iteration matrix. We also briefly treat the opposite case of $B_J$ nonpositive. In Section 4 we extend the convergence theorem under the hypothesis of strict diagonal dominance to the whole class of splitting methods that we propose. In Section 5 we prove the inclusion of the methods defined by Ahmadi et al. [1] in our general class. In Section 6 we consider again the $\mathrm{T}_U$-method and some specific types of refinements which include, in particular, all the Gauss–Seidel methods. In Section 7 we propose a new family of splitting methods, which seem to have a good potential for fast convergence, in competition with the symmetric Gauss–Seidel method. Finally, in Section 8 some numerical illustrative examples are given, and in Section 9 some conclusions are drawn.

**2. A general class of iterative methods.** To begin with, we observe that, possibly at the cost of an initial preconditioning and a final diagonal-matrix/vector multiplication, it is not restrictive to assume that the diagonal of the coefficient matrix $A$ be equal to the identity matrix $I$, that is, $D = I$ in (1.4), so that

$$A = I - (L + U) = I - B_J. \tag{2.1}$$

In fact, many important properties of the coefficient matrix $A$, if present, are not compromised by this assumption. For example, diagonal dominance by rows is preserved from $A$ to $D^{-1}A$, dominance by columns is preserved from $A$ to $AD^{-1}$, symmetry together with positive (semi-)definiteness are preserved from $A$ to $D^{-1/2}AD^{-1/2}$. In all cases it is immediate to define the modified system, equivalent to (1.1).

DEFINITION 2.1. *Given a matrix $B \in \mathbb{R}^{n\times n}$, we say that the $d$-tuple of matrices $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$, $B_p \in \mathbb{R}^{n\times n}$, is a* splitting *of $B$ of order $d$ if*

- *$B_p \neq O$ for all $p = 1, \ldots, d$;*
- *$B = \sum_{p=1}^{d} B_p$;*
- *the Hadamard product $B_p \circ B_q = O$ for all $p, q = 1, \ldots, d$ with $p \neq q$.*

Given a splitting $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ of $B_J$, we define the corresponding iterative scheme

$$x_i^{(k+1)} = \sum\nolimits_{j=1}^{i-1} B_j x_j^{(k+1)} + \sum\nolimits_{j=i}^{d} B_j x_j^{(k)} + c, \qquad i = 1, \ldots, d, \tag{2.2}$$

which starts from a given $d$-tuple $\{x_1^{(0)}, \ldots, x_d^{(0)}\}$ of $n$-vectors. It is straightforward to see that such a scheme may be reformulated in the more compact form

$$X^{(k+1)} = \mathcal{B}_{(d)} X^{(k)} + \Gamma_{(d)} \tag{2.3}$$

with

$$\mathcal{B}_{(d)} = (\mathcal{I}_{(d)} - \mathcal{L}_{(d)})^{-1}\mathcal{U}_{(d)} = \left(\mathcal{I}_{(d)} + \sum_{h=1}^{d-1} \mathcal{L}_{(d)}{}^{h}\right)\mathcal{U}_{(d)} \in \mathbb{R}^{dn\times dn} \tag{2.4}$$

(which, without any risk of misunderstanding, we denote by the same symbol used for the corresponding splitting), where $\mathcal{I}_{(d)} = \mathrm{blockdiag}(I, \ldots, I) \in \mathbb{R}^{dn\times dn}$,

$$\mathcal{L}_{(d)} = \begin{bmatrix} O & \cdots & & \cdots & O \\ B_1 & O & \cdots & \cdots & O \\ \vdots & B_2 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & O & \vdots \\ B_1 & B_2 & \cdots & B_{d-1} & O \end{bmatrix}, \qquad \mathcal{U}_{(d)} = \begin{bmatrix} B_1 & B_2 & \cdots & B_{d-1} & B_d \\ O & B_2 & \cdots & B_{d-1} & B_d \\ \vdots & O & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & \cdots & O & B_d \end{bmatrix},$$

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$\mathcal{L}_{(d)}, \mathcal{U}_{(d)} \in \mathbb{R}^{dn\times dn}$, and

$$X^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ \vdots \\ x_d^{(k)} \end{bmatrix}, \qquad \Gamma_{(d)} = (\mathcal{I}_{(d)} - \mathcal{L}_{(d)})^{-1} \begin{bmatrix} c \\ c \\ \vdots \\ \vdots \\ c \end{bmatrix} = \begin{bmatrix} c \\ (B_1 + I)c \\ \vdots \\ \vdots \\ (B_{d-1} + I)\cdots(B_1 + I)c \end{bmatrix}$$

are $dn$-vectors. The fixed point of (2.3) is given by the $dn$-vector $X = [x^T, \dots, x^T]^T$, where $x$ is the solution of (1.1), and convergence obviously holds if and only if $\rho(\mathcal{B}_{(d)}) < 1$.

We remark that the computational complexity of one iteration is the same as for the Jacobi or the Gauss–Seidel methods, at least with respect to the number of multiplications. This fact might be better realized by looking at the two-step iterative scheme

$$\begin{cases} x_1^{(k+1)} = & x_d^{(k)} + B_d(x_d^{(k)} - x_d^{(k-1)}) \\ x_i^{(k+1)}, = & x_{i-1}^{(k+1)} + B_{i-1}(x_{i-1}^{(k+1)} - x_{i-1}^{(k)}), \qquad i = 2, \dots, d, \end{cases}$$

which is equivalent to (2.2) for $k \geq 1$. At least from a formal point of view, the necessary amount of memory involved in the computations seems to strongly depend on the order $d$ of the splitting $\mathcal{B}_{(d)}$, since the dimension of the problem grows up to $dn$. However, the increase of the dimension is mostly due to the adopted formalism, and a smart organization of the data together with a smart formulation of the implementation algorithm may fix this inconvenience. In any case, in this paper we do not consider this practical aspect any more.

**2.1. Cyclicity of the splittings.** The results of this section are particularly important for analyzing the convergence properties of the proposed class of methods.

DEFINITION 2.2. *Given a matrix $B \in \mathbb{R}^{n\times n}$, we define the* cyclic shifting map

$$\mathcal{S} : \mathrm{Spl}(B) \longrightarrow \mathrm{Spl}(B)$$

*on the set* $\mathrm{Spl}(B)$ *of its splittings by setting*

$$\mathcal{S}(\mathcal{B}_{(d)}) := \{B_d, B_1, \dots, B_{d-1}\}$$

*for any splitting $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$. Moreover, we say that the splitting $\mathcal{S}(\mathcal{B}_{(d)})$ is the* cyclic shift *of $\mathcal{B}_{(d)}$.*

REMARK 2.3. Note that, for the limit case $d = 1$, we have $\mathcal{S}(\{B\}) = \{B\}$. Moreover, for each $d \geq 1$ it obviously holds that $\mathcal{S}^d(\mathcal{B}_{(d)}) = \mathcal{B}_{(d)}$.

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LEMMA 2.4 (Cyclicity). *Let $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ be a splitting of the matrix $B_J$, and let $\mathcal{B}'_{(d)} = \mathcal{S}(\mathcal{B}_{(d)})$. If $\lambda \neq 0$ is an eigenvalue of (the iteration matrix corresponding to) $\mathcal{B}_{(d)}$ and $\left[\alpha_1^T, \dots, \alpha_d^T\right]^T$ is one of its related eigenvectors, then $\lambda$ and $\left[\alpha_d^T, \lambda\alpha_1^T, \dots, \lambda\alpha_{d-1}^T\right]^T$ are an eigenvalue and the related eigenvector, respectively, of (the iteration matrix corresponding to) $\mathcal{B}'_{(d)}$.*

*Proof.* By (2.4), the assumed hypothesis is equivalent to

$$(\lambda\mathcal{L}_{(d)} + \mathcal{U}_{(d)})\alpha = \lambda\alpha,$$

which, in expanded form, reads

$$\sum_{j=1}^{i-1} \lambda B_j\alpha_j + \sum_{j=i}^{d} B_j\alpha_j = \lambda\alpha_i, \qquad i = 1, \dots, d. \tag{2.5}$$

Rearranging this set of equations as

$$B_d\alpha_d + \sum_{j=1}^{i-1} \lambda B_j\alpha_j + \sum_{j=i}^{d-1} B_j\alpha_j = \lambda\alpha_i, \qquad i = 1, \dots, d,$$

and performing a cyclic shift of the equations, we obtain

$$\begin{cases} B_d\alpha_d + \sum_{j=1}^{d-1} \lambda B_j\alpha_j = \lambda\alpha_d, \\ B_d\alpha_d + \sum_{j=1}^{i-1} \lambda B_j\alpha_j + \sum_{j=i}^{d-2} B_j\alpha_j = \lambda\alpha_i, \qquad i = 1, \dots, d-1. \end{cases}$$

The result then follows by multiplying by $\lambda$ the set of equations for $i = 1, \dots, d-1$. ∎

In view of the forthcoming statement and similar ones that will be made throughout the paper, we need to give the following definition:

DEFINITION 2.5. *We say that two iterative methods are* spectrum-equivalent *if their respective iteration matrices $\mathcal{B}$ and $\mathcal{B}'$, possibly of different dimensions $n$ and $n'$, have the same nonzero eigenvalues.*

REMARK 2.6. Two spectrum-equivalent iterative methods have the same asymptotic speed of convergence since their respective iteration matrices $\mathcal{B}$ and $\mathcal{B}'$ are such that $\rho(\mathcal{B}) = \rho(\mathcal{B}')$.

In the light of Remark 2.3, it is clear that the iterated application of the Cyclicity Lemma $d$ times implies the following corollary:

COROLLARY 2.7. *Let $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ be a splitting of the Jacobi iteration matrix $B_J$ of order $d$. Then the iterative methods corresponding to the iteration matrices $\mathcal{B}_{(d)}$ and $\mathcal{S}^r(\mathcal{B}_{(d)})$ are spectrum-equivalent.*

It is interesting to observe that the (iterated) use of the cyclic shifting map $\mathcal{S}$ not only leads to splittings with the same nonzero eigenvalues of the corresponding iteration matrices but also leaves the original iterative methods substantially unchanged. In fact, by writing twice the equations (2.2) for $k$ and $k+1$ in sequence and then by neglecting the first $k-1$ and the last of the resulting $2k$ equations, we obtain

$$\begin{cases} x_d^{(k+1)} &= B_d x_d^{(k)} + \sum_{j=1}^{d-1} B_j x_j^{(k+1)} + c, \\ x_1^{(k+2)} &= B_d x_d^{(k+1)} + \sum_{j=1}^{d-1} B_j x_j^{(k+1)} + c, \\ x_i^{(k+2)} &= B_d x_d^{(k+1)} + \sum_{j=1}^{i-1} B_j x_j^{(k+2)} + \sum_{j=i}^{d-1} B_j x_j^{(k+1)} + c, \qquad i = 2, \dots, d-1. \end{cases}$$

Now, by setting

$$y_1^{(k)} := x_d^{(k)} \qquad \text{and} \qquad y_i^{(k)} := x_{i-1}^{(k+1)}, \qquad k \geq 0,$$

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we get the iteration (2.2) for the sequence $\{y_1^{(k)}, \dots, y_d^{(k)}\}$, with $\mathcal{B}_{(d)}$ replaced by $\mathcal{S}(\mathcal{B}_{(d)})$.

The Cyclicity Lemma also simplifies significantly the proof of the next result which considers the eigenvectors of the iteration matrices.

PROPOSITION 2.8. *Let $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ be a splitting of $B_J$. Let $\lambda \neq 0$ be an eigenvalue of (the iteration matrix) $\mathcal{B}_{(d)}$, and let $\left[\alpha_1^T, \dots, \alpha_d^T\right]^T$ be one of its related eigenvectors. Then it holds that*

$$\alpha_p \neq 0 \qquad \textit{for all} \quad p = 1, \dots, d.$$

*Proof.* Possibly by applying the Cyclicity Lemma 2.4 the necessary number of times, we can confine ourselves to prove that assuming $\alpha_1 = 0$ leads to a contradiction. Indeed, if it were so, then the first equation of (2.5) would imply

$$B_2\alpha_2 + \cdots + B_{d-1}\alpha_{d-1} + B_d\alpha_d = 0.$$

Then substituting into the second equation would yield $\lambda\alpha_2 = 0$, and thus, being $\lambda \neq 0$, $\alpha_2 = 0$ as well. By recursively repeating the analogous substitutions into all the remaining equations of (2.5), we could prove that $\alpha_p = 0$, for all $p = 1, \dots, d$, which is impossible. ∎

**2.2. Refinement of splittings and essentiality.** Starting from a given splitting $\mathcal{B}_{(d)}$, in this section, we suggest a procedure that allows us to define a new splitting $\mathcal{B}'_{(d')}$ that may lead to a faster method.

DEFINITION 2.9. *Given two splittings*

$$\mathcal{B}_{(d)} = \{B_1, \dots, B_d\} \qquad \textit{and} \qquad \mathcal{B}'_{(d+1)} = \{B'_1, \dots, B'_{d+1}\}$$

*of a matrix $B \in \mathbb{R}^{n\times n}$ of orders $d$ and $d+1$, respectively, we say that $\mathcal{B}'_{(d+1)}$ is a* refinement *of $\mathcal{B}_{(d)}$ and write $\mathcal{B}'_{(d+1)} \succ \mathcal{B}_{(d)}$, if there exist two integers $r, s$ with $0 \leq r \leq d-1$ and $0 \leq s \leq d$ such that, with*

$$\mathcal{C}_{(d)} = \{C_1, \dots, C_d\} := \mathcal{S}^r(\mathcal{B}_{(d)})$$

*and*

$$\mathcal{C}'_{(d+1)} = \{C'_1, \dots, C'_d, C'_{d+1}\} := \mathcal{S}^s(\mathcal{B}'_{(d+1)}),$$

*the following conditions are satisfied:*

*(I)* $C'_p = C_p$ *for all* $p = 1, \dots, d-1$;
*(II) the pair of matrices* $\{C'_d, C'_{d+1}\} \subseteq \mathcal{C}'_{(d+1)}$ *is a splitting of* $C_d \in \mathcal{C}_{(d)}$;
*(III)* $C'_{d+1}C'_d \neq O$.

DEFINITION 2.10. *Given two splittings $\mathcal{B}_{(d)}$ and $\mathcal{B}'_{(d')}$ of a matrix $B \in \mathbb{R}^{n\times n}$ of orders $d$ and $d'$, respectively, with $d' \geq d+2$, we say that $\mathcal{B}'_{(d')}$ is a* refinement *of $\mathcal{B}_{(d)}$ and write $\mathcal{B}'_{(d')} \succ \mathcal{B}_{(d)}$, if there exists a chain of $d'-d-1$ splittings $\mathcal{B}'_{(d+1)}$, $\mathcal{B}'_{(d+2)}$, $\dots$, $\mathcal{B}'_{(d'-1)}$ of increasing orders $d+1, d+2, \dots, d'-1$, respectively, such that*

$$\mathcal{B}'_{(d')} \succ \mathcal{B}'_{(d'-1)} \succ \cdots \succ \mathcal{B}'_{(d+1)} \succ \mathcal{B}_{(d)}$$

*in the sense of Definition 2.9.*

It is immediate to realize that the strict partial ordering introduced by Definitions 2.9 and 2.10 on $\mathrm{Spl}(B)$ is *compatible* with the cyclic shifting map $\mathcal{S}$ in the sense that

$$\mathcal{B}'_{(d')} \succ \mathcal{B}_{(d)} \iff \mathcal{S}^r(\mathcal{B}'_{(d')}) \succ \mathcal{S}^s(\mathcal{B}_{(d)})$$

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for all nonnegative integers $r, s$. Clearly, the unique element of order $d = 1$, i.e., $\mathcal{B}_{(1)} = \{B\}$, is always a minimal element in such a strict partial ordering.

Now we show that condition (III) in Definition 2.9 is essential in order to hope for an effective improvement of a given splitting $\mathcal{B}_{(d)}$.

PROPOSITION 2.11. *With reference to Definition 2.9, we assume that two splittings* $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ *and* $\mathcal{B}'_{(d+1)} = \{B'_1, \dots, B'_{d+1}\}$ *of* $B_J$ *satisfy conditions* (I) *and* (II) *(without shifts for the sake of simplicity) but do not satisfy* (III)*, i.e.,*

$$B'_{d+1}B'_d = O. \tag{2.6}$$

*Then the corresponding iterative methods are spectrum-equivalent.*

*Proof.* First assume that $\lambda \neq 0$ is an eigenvalue of $\mathcal{B}_{(d)}$, and let $\left[\alpha_1^T, \dots, \alpha_d^T\right]^T$ be a related eigenvector. Then we set

$$\alpha'_p := \alpha_p, \quad p = 1, \dots, d, \qquad \alpha'_{d+1} := \alpha_d - \frac{1-\lambda}{\lambda} B'_d \alpha_d, \tag{2.7}$$

so that by (2.6)

$$B'_{d+1}\alpha'_{d+1} = B'_{d+1}\alpha'_d. \tag{2.8}$$

On the other hand, our hypothesis is equivalent to the set of equalities (2.5) for the splitting $\mathcal{B}_{(d)}$, and hence, taking into account (I) and (II) along with (2.7) and (2.8), we get

$$\sum_{j=1}^{i-1} \lambda B'_j \alpha'_j + \sum_{j=i}^{d+1} B'_j \alpha'_j = \lambda \alpha'_i, \qquad i = 1, \dots, d+1, \tag{2.9}$$

which is equivalent to $\lambda$ being an eigenvalue of $\mathcal{B}'_{(d+1)}$.

Vice versa, now we assume that $\lambda \neq 0$ is an eigenvalue of $\mathcal{B}'_{(d+1)}$ and that the vector $\left[{\alpha'_1}^T, \dots, {\alpha'_{d+1}}^T\right]^T$ is a related eigenvector, i.e., that (2.9) holds true. Therefore, subtracting the last two equalities from one another yields

$$\lambda(\alpha'_{d+1} - \alpha'_d) = (\lambda - 1) B'_d \alpha'_d, \tag{2.10}$$

so that by setting

$$\alpha_p := \alpha'_p, \qquad \text{for all } \; p = 1, \dots, d,$$

we get (2.7). Moreover, using (2.6) and multiplying both sides of (2.7) by $B'_{d+1}$ lead us to (2.8) (with $\alpha'_d = \alpha_d$).

Finally, conditions (I) and (II) of Definition 2.9 allow us to state that the first $d$ equalities of (2.9) coincide with the set of equalities (2.5) for the splitting $\mathcal{B}_{(d)}$. In addition, by Proposition 2.8 we can claim that $\left[\alpha_1^T, \dots, \alpha_d^T\right]^T \neq \left[0^T, \dots, 0^T\right]^T$. ∎

Similarly to what happens when applying the cyclic shifting map $\mathcal{S}$, now we shall see that, under the above hypotheses on the splittings $\mathcal{B}_{(d)}$ and $\mathcal{B}'_{(d+1)}$, not only the corresponding iteration matrices share the same nonzero eigenvalues but also, from a practical point of view, that the corresponding iterative schemes coincide.

PROPOSITION 2.12. *Under the hypotheses of Proposition 2.11, with the initial choices*

$$y_p^{(0)} = x_p^{(0)}, \qquad \textit{for all } \; p = 1, \dots, d, \tag{2.11}$$

*and*

$$y_{d+1}^{(0)} = x_d^{(0)} + \mathcal{B}'_{(d)} u = y_d^{(0)} + \mathcal{B}'_{(d)} u, \quad \textit{with } \; u \in \mathbb{R}^n, \tag{2.12}$$

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*the iterative schemes corresponding to the splittings*

$$\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\} \qquad \text{and} \qquad \mathcal{B}'_{(d+1)} = \{B'_1, \ldots, B'_{d+1}\}$$

*give rise to the sequences of approximations*

$$X^{(k)} = \left[{x_1^{(k)}}^T, \ldots, {x_d^{(k)}}^T\right]^T \quad \text{and} \quad Y^{(k)} = \left[{y_1^{(k)}}^T, \ldots, {y_d^{(k)}}^T, {y_{d+1}^{(k)}}^T\right]^T, \quad k \geq 0,$$

*to the solution $x$ of (1.1), respectively, where the equalities*

$$y_p^{(k)} = x_p^{(k)}, \qquad \text{for all } p = 1, \ldots, d, \tag{2.13}$$

*and*

$$y_{d+1}^{(k)} = x_d^{(k)} + \mathcal{B}'_{(d)}\left(x_d^{(k)} - x_d^{(k-1)}\right) = y_d^{(k)} + \mathcal{B}'_{(d)}\left(y_d^{(k)} - y_d^{(k-1)}\right) \tag{2.14}$$

*hold for all $k \geq 1$.*

*Proof.* The iterative scheme associated with the splitting $\mathcal{B}'_{(d+1)}$ is given by

$$y_i^{(k+1)} = \sum_{j=1}^{i-1} B'_j y_j^{(k+1)} + \sum_{j=i}^{d+1} B'_j y_j^{(k)} + c, \qquad i = 1, \ldots, d+1. \tag{2.15}$$

Therefore, since (2.12) and (2.6) imply

$$B'_{d+1} y_{d+1}^{(0)} = B'_{d+1} y_d^{(0)},$$

substituting (2.11) in (2.15) for $k = 0$ and using the conditions (I) and (II) of Definition 2.9 make the first $d$ equalities coincide with the iterative scheme (2.2) for $k = 0$. Consequently, (2.13) is proved for $k = 1$. Furthermore, computing the difference between the last two equalities clearly yields (2.14) for $k = 1$ as well. Since the same computations may be done to perform the general induction step from $k$ to $k+1$, the proof is complete. ∎

In light of Propositions 2.11 and 2.12 it makes sense to concentrate on splittings that enjoy the following *essentiality* property:

DEFINITION 2.13. *We say that a splitting $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ of a matrix $B$ of order $d$ is* nonessential *if there exists an integer $r$ with $0 \leq r \leq d-1$ such that, with*

$$\mathcal{C}_{(d)} = \{C_1, \ldots, C_d\} := \mathcal{S}^r(\mathcal{B}_{(d)}),$$

*we have*

$$C_d\, C_{d-1} = O.$$

*Otherwise $\mathcal{B}_{(d)}$ is said to be* essential.

Without being necessarily a general rule, later in this paper we shall see that if the spectral radius satisfies $\rho(B_J) < 1$, then two splittings $\mathcal{B}'_{(d')}$ and $\mathcal{B}_{(d)}$ of $B_J$ of orders $d'$ and $d$ (with $d' \geq d+1 \geq 2$), respectively, that satisfy the condition $\mathcal{B}'_{(d')} \succ \mathcal{B}_{(d)} \succ \{B_J\}$ very often enjoy the property that $\rho(\mathcal{B}'_{(d')}) < \rho(\mathcal{B}_{(d)}) < \rho(B_J)$. In this sense, it is worth giving the following qualitative definition:

DEFINITION 2.14. *We say that a splitting $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ of $B_J$ is* potentially optimal *if it is both essential and maximal in the strict partial ordering "$\succ$".*

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**2.3. A general necessary condition for convergence.** Now we look for a necessary and sufficient condition for the presence of the eigenvalue $\lambda = 1$ in the spectrum of the iteration matrix of a given splitting $\mathcal{B}_{(d)} \in \mathrm{Spl}(B_J)$.

LEMMA 2.15. *With reference to Definition 2.9, we assume that two splittings* $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ *and* $\mathcal{B}'_{(d+1)} = \{B'_1, \ldots, B'_{d+1}\}$ *of* $B_J$ *satisfy conditions* (I) *and* (II). *Then, no matter whether they also satisfy condition* (III) *or not, it holds that* $\lambda = 1$ *is an eigenvalue of (the iteration matrix)* $\mathcal{B}_{(d)}$ *if and only if* $\lambda = 1$ *is an eigenvalue of (the iteration matrix)* $\mathcal{B}'_{(d+1)}$.

*Proof.* If condition (III) does not hold, then the statement is just a particular case of Proposition 2.11. On the other hand, if condition (III) is satisfied, assuming $\lambda = 1$ allows us to repeat the same proof of Proposition 2.11, the only difference being that (2.7) and (2.10) directly become $\alpha'_{d+1} := \alpha_d$ and $\alpha'_{d+1} = \alpha'_d$, respectively. ∎

PROPOSITION 2.16. *Let* $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ *be a splitting of* $B_J$. *Then* $\lambda = 1$ *is an eigenvalue of (the iteration matrix)* $\mathcal{B}_{(d)}$ *if and only if* $\lambda = 1$ *is an eigenvalue of* $B_J$.

*Proof.* We consider the *chain of splittings* of increasing order

$$\mathcal{B}_{(p)} = \{B_1, \ldots, B_{p-1}, B_p + \ldots + B_d\}, \qquad p = 2, \ldots, d-1, \tag{2.16}$$

*connecting* $B_J = \mathcal{B}_{(1)}$ and $\mathcal{B}_{(d)}$. Since condition (III) may well fail to hold, it is not necessarily ordered in the sense of Definition 2.9. On the other hand, each pair of two consecutive splittings $\mathcal{B}_{(p)}$ and $\mathcal{B}_{(p+1)}$, $p = 1, \ldots, d-1$, satisfies all the other properties involved in Definition 2.9, and therefore the application of Lemma 2.15 $d-1$ times along the above chain concludes the proof. ∎

COROLLARY 2.17. *Let* $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ *be a splitting of* $B_J$. *Then a necessary condition for the convergence of the corresponding iterative scheme* (2.2) *is that* $\lambda = 1$ *be not an eigenvalue of* $B_J$.

It is worth stressing that the foregoing results consider the sole case of the eigenvalue $\lambda = 1$ and do not generalize to other eigenvalues of unitary modulus. We illustrate this fact by means of the following counterexample:

EXAMPLE 2.18. Consider the $3 \times 3$ Jacobi iteration matrix

$$B_J = \gamma \begin{bmatrix} 0 & -1 & -1 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \qquad \text{with} \quad \gamma = 1.241706082017\ldots$$

having three distinct eigenvalues, two of which being complex conjugate with unitary modulus (namely, $0.23931\ldots \pm 0.97094\ldots i$) and the third one being real (namely, $-0.47862\ldots$), so that $\rho(B_J) = 1$. A few elementary calculations show that the corresponding $\mathrm{T}_U$-method (1.5) has a $6 \times 6$ iteration matrix $\mathcal{B}_{T_U}$ with three distinct eigenvalues as well, one of which being equal to $0$ with multiplicity $4$ and the remaining two eigenvalues being complex conjugate (namely, $0.38545\ldots \pm 0.57449\ldots\ i$) with a modulus strictly less than 1, so that $\rho(\mathcal{B}_{T_U}) < 1$.

**3. The case of irreducible nonnegative Jacobi iteration matrices.** In this section we consider the particular case of matrices $A$ for which the matrix $B_J$ satisfies the *nonnegativity condition*

$$B_J \geq O \quad \text{elementwise.} \tag{3.1}$$

Assuming the matrix $A$ to be *irreducible*, it is immediate to see that the matrix $B_J$ is irreducible as well (and vice versa). Therefore, the well-known Perron-Frobenius theorem assures that the spectral radius $\rho(B_J)$ is the simple leading eigenvalue of $B_J$, whose corresponding eigenvector $\alpha$ may be chosen to be *positive*, i.e., $\alpha > 0$ elementwise.

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**3.1. Splittings of nonnegative Jacobi iteration matrices.** Let us consider a splitting $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ of $B_J$. It is immediate to see that condition (3.1) implies

$$\mathcal{B}_{(d)} \geq O \quad \text{elementwise.} \tag{3.2}$$

Unfortunately, the possible irreducibility of $B_J$ is not inherited by $\mathcal{B}_{(d)}$, so that, in principle, the general Perron-Frobenius theory only assures that $\rho(\mathcal{B}_{(d)})$ is an eigenvalue of $\mathcal{B}_{(d)}$, not necessarily simple, and that there exists a nonnegative eigenvector $\alpha_{(d)}$ associated with it, not necessarily positive. Nevertheless, we will see that, if $B_J$ is irreducible, then the positivity of $\alpha_{(d)}$ is assured.

LEMMA 3.1. *Let $B_J$ satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$. Then the iteration matrix (2.4) has a nonnegative leading eigenvector $\alpha_{(d)} = \left[\alpha_1^T, \ldots, \alpha_d^T\right]^T$ corresponding to the eigenvalue $\lambda := \rho(\mathcal{B}_{(d)})$ (assumed to be $> 0$) such that*

$$0 \leq \lambda\alpha_1 \leq \alpha_d \leq \alpha_{d-1} \leq \ldots \leq \alpha_2 \leq \alpha_1 \quad \textit{elementwise} \tag{3.3}$$

*if $\lambda \leq 1$ and*

$$0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{d-1} \leq \alpha_d \leq \lambda\alpha_1 \quad \textit{elementwise} \tag{3.4}$$

*if $\lambda \geq 1$.*

*Proof.* Taking the difference of each pair of consecutive equations in (2.5), we get

$$\lambda(\alpha_{p+1} - \alpha_p) = (\lambda - 1)B_p\alpha_p, \qquad p = 1, \ldots, d-1,$$

that is,

$$\alpha_{p+1} = (I - \sigma B_p)\alpha_p, \qquad p = 1, \ldots, d-1, \tag{3.5}$$

where $\sigma := (1 - \lambda)/\lambda$. Moreover, subtracting the last equation from the first one yields

$$\lambda\alpha_1 = (I - \sigma B_d)\alpha_d, \tag{3.6}$$

and, because of (3.2), we can assume

$$\alpha_p \geq 0, \quad p = 1, \ldots, d, \quad \text{elementwise.} \tag{3.7}$$

In conclusion, if $\lambda \leq 1$, then $\sigma \geq 0$, and consequently, since $B_p \geq O$ for all $p = 1, \ldots, d$ elementwise, the equalities (3.5) and (3.6) and the inequality (3.7) imply (3.3). Analogously, if $\lambda \geq 1$, then $\sigma \leq 0$, and consequently, (3.5), (3.6), and (3.7) imply (3.4). ∎

COROLLARY 3.2. *Let the hypotheses of Lemma 3.1 hold. Then the nonnegative leading eigenvector $\alpha_{(d)} = \left[\alpha_1^T, \ldots, \alpha_d^T\right]^T$ corresponding to the eigenvalue $\rho(\mathcal{B}_{(d)})$ satisfies the following zero components property:*

*(z.c.) if for some $i \in \{1, \ldots, n\}$ and some $q \in \{1, \ldots, d\}$ we have that the $i$-th component $(\alpha_q)_i = 0$, then $(\alpha_p)_i = 0$ for all $p = 1, \ldots, d$.*

Now we are in a position to state the mentioned positivity result for the leading eigenvector $\alpha_{(d)}$ of the iteration matrix $\mathcal{B}_{(d)}$ under the hypothesis of irreducibility of the matrix $B_J$.

PROPOSITION 3.3. *Let $B_J$ be irreducible and satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$. Then the corresponding iteration matrix has a positive leading eigenvector $\alpha_{(d)} = \left[\alpha_1^T, \ldots, \alpha_d^T\right]^T$ satisfying*

$$0 < \lambda\alpha_1 \leq \alpha_d \leq \alpha_{d-1} \leq \ldots \leq \alpha_2 \leq \alpha_1 \quad \textit{elementwise} \tag{3.8}$$

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*if* $\lambda \leq 1$ *and*

$$0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{d-1} \leq \alpha_d \leq \lambda\alpha_1 \quad \textit{elementwise} \tag{3.9}$$

*if* $\lambda \geq 1$*, which are stronger than* (3.3) *and* (3.4).

*Proof.* By contradiction, let us assume that $\alpha_{(d)}$ is not positive but just nonnegative (see Lemma 3.1). Then there exist two indices $i \in \{1, \ldots, n\}$ and $q \in \{1, \ldots, d\}$ such that $(\alpha_q)_i = 0$, and thus, Corollary 3.2 implies $(\alpha_p)_i = 0$, $p = 1, \ldots, d$. Consequently, the first equality in (2.5) yields

$$(B_1\alpha_1 + B_2\alpha_2 + \cdots + B_{d-1}\alpha_{d-1} + B_d\alpha_d)_i = 0. \tag{3.10}$$

On the other hand, the assumed irreducibility of $B_J$ assures that for any $k \in \{1, \ldots, n\}$, $k \neq i$, there exists a chain of indices $h_1, \ldots, h_r \in \{1, \ldots, n\}$, all different from each other and also from $i$ and $k$, such that

$$b_{ih_1} > 0, \quad b_{h_1h_2} > 0, \quad \ldots, \quad b_{h_{r-1}h_r} > 0, \quad b_{h_rk} > 0. \tag{3.11}$$

Hence, in view of Definition 2.1, using the fact that $B_p \geq O$ elementwise, $p = 1, \ldots, d$, using the nonnegativity of $\alpha_{(d)}$ and again Corollary 3.2, by (3.10) we get $(\alpha_p)_{h_1} = 0$, $p = 1, \ldots, d$. Therefore, using (3.11) we can repeat the same reasoning for another $r$ times and arrive at $(\alpha_p)_k = 0$, $p = 1, \ldots, d$. The arbitrariness of $k$ let us conclude that $\alpha_p = 0$, $p = 1, \ldots, d$, that is, $\alpha_{(d)} = 0$, which yields a contradiction. Finally, we just observe that the positivity of $\alpha_{(d)}$ along with (3.3) and (3.4) obviously leads to (3.8) and (3.9). □

**3.2. Monotonicity of the splittings.** Let $B_J$ and $\tilde{B}_J$ be two Jacobi iteration matrices such that

$$O \leq B_J \leq \tilde{B}_J \quad \text{elementwise.} \tag{3.12}$$

Then, let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$ and $\tilde{\mathcal{B}}_{(d)} = \{\tilde{B}_1, \ldots, \tilde{B}_d\}$ be the corresponding splitting of $\tilde{B}_J$. Clearly $B_p \leq \tilde{B}_p$, $p = 1, \ldots, d$, elementwise.

Therefore, by using (2.4) for both the related iteration matrices $\mathcal{B}_{(d)}$ and $\tilde{\mathcal{B}}_{(d)}$, since it clearly holds that $\mathcal{L}_{(d)} \leq \tilde{\mathcal{L}}_{(d)}$ and $\mathcal{U}_{(d)} \leq \tilde{\mathcal{U}}_{(d)}$ elementwise, we easily get

$$O \leq \mathcal{B}_{(d)} \leq \tilde{\mathcal{B}}_{(d)} \quad \text{elementwise.} \tag{3.13}$$

By applying the Gelfand spectral radius theorem using the matrix infinity norm $\|\cdot\|_\infty$ (which is monotone with respect to the "elementwise $\leq$" order relation) to the inequalities (3.12) and (3.13), we easily obtain

$$\rho(B_J) \leq \rho(\tilde{B}_J) \qquad \text{and} \qquad \rho(\mathcal{B}_{(d)}) \leq \rho(\tilde{\mathcal{B}}_{(d)}). \tag{3.14}$$

**3.3. Acceleration of convergence.** We have collected all the necessary tools to establish the relationships between the partial order introduced by Definition 2.9 on $\mathrm{Spl}(B_J)$ and the possible acceleration of convergence of the related iterative schemes for the case of $B_J$ irreducible and nonnegative. We first analyze the more general case in which the matrix $B_J$ may be reducible.

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LEMMA 3.4. *Let $B_J$ with $\rho(B_J) > 0$ satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$ of order $d \geq 2$. Then the following implications hold:*

*(a1)* $\rho(B_J) < 1 \iff \rho(\mathcal{B}_{(d)}) < 1$;
*(a2)* $\rho(B_J) = 1 \iff \rho(\mathcal{B}_{(d)}) = 1$;
*(a3)* $\rho(B_J) > 1 \iff \rho(\mathcal{B}_{(d)}) > 1$.

*Proof.* For $\gamma \in [0, +\infty)$ we define the function

$$f(\gamma) := \rho(\tilde{\mathcal{B}}_{(d)}(\gamma)),$$

where $\tilde{\mathcal{B}}_{(d)}(\gamma)$ is the iteration matrix of the splitting corresponding to the Jacobi matrix $\tilde{B}_J(\gamma) := \gamma B_J$ (in particular, $f(1) = \rho(\mathcal{B}_{(d)})$). The function $f(\gamma)$ is clearly nondecreasing (see (3.13) and (3.14)), continuous, and such that $f(0) = 0$.

Now assume that $\rho(B_J) \leq 1$ and, by contradiction, that $\rho(\mathcal{B}_{(d)}) > 1$. Therefore, since $f(1) = \rho(\mathcal{B}_{(d)}) > 1$, there exists a $\bar{\gamma} < 1$ such that $f(\bar{\gamma}) = \rho(\tilde{\mathcal{B}}_{(d)}(\bar{\gamma})) = 1$. On the other hand, the nonnegativity of $\tilde{\mathcal{B}}_{(d)}(\bar{\gamma})$ implies that $f(\bar{\gamma}) = 1$ is one of its eigenvalues, and consequently, Proposition 2.16 implies the absurd chain of inequalities

$$1 \geq \rho(B_J) > \bar{\gamma}\rho(B_J) = \rho(\bar{\gamma}B_J) = \rho(\tilde{B}_J(\bar{\gamma})) \geq 1.$$

Summarizing, we have proved that

$$\rho(B_J) \leq 1 \quad \Longrightarrow \quad \rho(\mathcal{B}_{(d)}) \leq 1. \tag{3.15}$$

Now assume that $\rho(B_J) \geq 1$ and, by contradiction, that $\rho(\mathcal{B}_{(d)}) < 1$. It obviously holds that $\rho(B_J/\rho(B_J)) = 1$, and thus, like before, by the nonnegativity of $B_J$ (which implies that $\rho(B_J)$ is one of its eigenvalues) and by using Proposition 2.16 in the opposite direction, we obtain the absurd chain of inequalities

$$1 > \rho(\mathcal{B}_{(d)}) = f(1) \geq f(1/\rho(B_J)) = \rho(\tilde{\mathcal{B}}_{(d)}(1/\rho(B_J))) \geq 1.$$

Summarizing, we have also proved that

$$\rho(B_J) \geq 1 \quad \Longrightarrow \quad \rho(\mathcal{B}_{(d)}) \geq 1,$$

which, along with (3.15), the nonnegativity of $B_J$ and $\mathcal{B}_{(d)}$, and again Proposition 2.16, implies (a1), (a2), (a3). ∎

THEOREM 3.5. *Let $B_J$ with $\rho(B_J) > 0$ satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ and $\mathcal{B}'_{(d+1)} = \{B'_1, \ldots, B'_{d+1}\}$ be two splittings of $B_J$ of orders $d$ and $d+1$, respectively. Moreover, with reference to Definition 2.9, let $\mathcal{B}_{(d)}$ and $\mathcal{B}'_{(d+1)}$ satisfy conditions (I) and (II) (but not necessarily (III)).*

*Then the following implications hold:*

*(b1) if $\rho(B_J) < 1$, then $\rho(\mathcal{B}'_{(d+1)}) \leq \rho(\mathcal{B}_{(d)}) \leq \rho(B_J)$;*
*(b2) if $\rho(B_J) = 1$, then $\rho(\mathcal{B}'_{(d+1)}) = \rho(\mathcal{B}_{(d)}) = 1$;*
*(b3) if $\rho(B_J) > 1$, then $\rho(\mathcal{B}'_{(d+1)}) \geq \rho(\mathcal{B}_{(d)}) \geq \rho(B_J)$.*

*Proof.* Since $B_J$ satisfies condition (3.1), we have that $\mathcal{B}'_{(d+1)} \geq O$ elementwise. If condition (III) is not satisfied, i.e., if $B'_{d+1}B'_d = O$, by Proposition 2.11 we immediately get $\rho(\mathcal{B}'_{(d+1)}) = \rho(\mathcal{B}_{(d)})$. On the other hand, if condition (III) is satisfied, then Proposition 2.11 can not be used and we must proceed in a different way. For this aim, we observe that, by

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Lemma 3.1, $\lambda := \rho(\mathcal{B}'_{(d+1)})$ (assumed to be $> 0$ without loss of generality) is one of the eigenvalues of $\mathcal{B}'_{(d+1)}$ and that a corresponding eigenvector has the form

$$\alpha'_{(d+1)} = \left[{\alpha'_1}^T, \dots, {\alpha'_{d+1}}^T\right]^T \geq \left[0^T, \dots, 0^T\right]^T \quad \text{elementwise.}$$

So, similarly to (2.5), we get

$$\sum_{j=1}^{i-1} \lambda B'_j \alpha'_j + \sum_{j=i}^{d+1} B'_j \alpha'_j = \lambda \alpha'_i, \qquad i = 1, \dots, d+1. \tag{3.16}$$

Now we assume that $\rho(B_J) \leq 1$, so that Lemma 3.4 yields $\lambda \leq 1$ as well. Thus, by conditions (I)–(II), using (3.3) we arrive at

$$\sum_{j=1}^{i-1} \lambda B_j \alpha'_j + \sum_{j=i}^{d} B_j \alpha'_j \geq \lambda \alpha'_i, \qquad i = 1, \dots, d,$$

elementwise, which is the expanded form of

$$(\lambda \mathcal{L}_{(d)} + \mathcal{U}_{(d)})\left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \geq \lambda \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \quad \text{elementwise.}$$

By using (2.4) and the fact that condition (3.1) yields

$$(\mathcal{I}_{(d)} - \mathcal{L}_{(d)})^{-1} \geq O \qquad \text{and} \qquad \mathcal{U}_{(d)} \geq O \qquad \text{elementwise,}$$

we finally get

$$\lambda \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \leq \mathcal{B}_{(d)} \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \quad \text{elementwise.}$$

In turn, we easily obtain

$$\lambda^k \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \leq \mathcal{B}^k_{(d)} \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \quad \text{elementwise for all } k \geq 1,$$

and thus, since $\left[{\alpha'_1}^T, \dots, {\alpha'_d}^T\right]^T \neq \left[0^T, \dots, 0^T\right]^T$, by computing the infinity norm $\|\cdot\|_\infty$ of both sides and by applying the Gelfand spectral radius theorem, we can conclude that

$$\lambda = \rho(\mathcal{B}'_{(d+1)}) \leq \rho(\mathcal{B}_{(d)}).$$

Summarizing, using again Lemma 3.4 also for $\mathcal{B}_{(d)}$, we have proved that

$$\rho(B_J) \leq 1 \quad \Longrightarrow \quad \rho(\mathcal{B}'_{(d+1)}) \leq \rho(\mathcal{B}_{(d)}) \leq 1, \tag{3.17}$$

no matter whether condition (III) is satisfied or not. Now we consider again the chain of splittings (2.16) introduced in the proof of Proposition 2.16, where each pair of two consecutive splittings $\mathcal{B}_{(p)}$ and $\mathcal{B}_{(p+1)}$, $p = 1, \dots, d-1$, satisfies all the properties involved by Definition 2.9, possibly except condition (III) . Therefore, starting with the pair of splittings $\mathcal{B}_{(1)} = (\{B_J\}, \mathcal{B}_{(2)})$, if condition (III) is not satisfied, then we apply Proposition 2.11 and get $\rho(\mathcal{B}_{(2)}) = \rho(B_J)$. If condition (III) is satisfied, then we can repeat the previous procedure for such a pair and get

$$\rho(\mathcal{B}_{(2)}) \leq \rho(B_J). \tag{3.18}$$

In any case (3.18) holds true. By successively repeating the procedure for all the pairs $(\mathcal{B}_{(p)}, \mathcal{B}_{(p+1)})$, $p = 1, \dots, d-1$, we arrive at

$$\rho(\mathcal{B}_{(d)}) \leq \rho(B_J)$$

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and hence, together with (3.17), at

$$\rho(B_J) \le 1 \quad \Longrightarrow \quad \rho(\mathcal{B}'_{(d+1)}) \le \rho(\mathcal{B}_{(d)}) \le \rho(B_J). \tag{3.19}$$

Conversely, if $\rho(B_J) \ge 1$, by using (3.4), we similarly arrive at

$$\rho(\mathcal{B}'_{(d+1)}) \ge \rho(\mathcal{B}_{(d)}) \ge 1,$$

and then, by employing once again the chain of splittings (2.16), we obtain the symmetric implication of (3.19)

$$\rho(B_J) \ge 1 \quad \Longrightarrow \quad \rho(\mathcal{B}'_{(d+1)}) \ge \rho(\mathcal{B}_{(d)}) \ge \rho(B_J). \qquad \square$$

The next corollary is straightforward (its proof is included in the previous one).

COROLLARY 3.6. *Let $B_J$ with $\rho(B_J) > 0$ satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ be a splitting of $B_J$ of order $d \ge 2$. The following implications hold:*

*(c1) if $\rho(B_J) \le 1$, then $\rho(\mathcal{B}_{(d)}) \le \rho(B_J)$;*
*(c2) if $\rho(B_J) \ge 1$, then $\rho(\mathcal{B}_{(d)}) \ge \rho(B_J)$.*

Now we refine our analysis by assuming that the matrix $A$, and thus also $B_J$, is irreducible and obtain, as a corollary, the generalization of the classical Stein-Rosenberg theorem (regarding the Gauss–Seidel methods) to any generic splitting $\mathcal{B}'_{(d+1)}$ that is a refinement of $B_J$.

THEOREM 3.7. *Let $B_J$ with $\rho(B_J) > 0$ be irreducible and satisfy condition (3.1), and let $\mathcal{B}_{(d)} = \{B_1, \dots, B_d\}$ and $\mathcal{B}'_{(d+1)} = \{B'_1, \dots, B'_{d+1}\}$ be two splittings of $B_J$. Then if $\mathcal{B}'_{(d+1)} \succ \mathcal{B}_{(d)}$ the following implications hold:*
*(d1) if $\rho(B_J) < 1$, then $\rho(\mathcal{B}'_{(d+1)}) < \rho(\mathcal{B}_{(d)}) \le \rho(B_J)$;*
*(d2) if $\rho(B_J) = 1$, then $\rho(\mathcal{B}'_{(d+1)}) = \rho(\mathcal{B}_{(d)}) = 1$;*
*(d3) if $\rho(B_J) > 1$, then $\rho(\mathcal{B}'_{(d+1)}) > \rho(\mathcal{B}_{(d)}) \ge \rho(B_J)$.*

*Proof.* Without loss of generality we assume that conditions (I), (II), and (III) of Definition 2.9 are satisfied by $\mathcal{B}_{(d)}$ and $\mathcal{B}'_{(d+1)}$. It is clear that (d2) holds as it is nothing but (b2). Then, in order to prove (d1), we assume that $\rho(B_J) < 1$ and, by contradiction, that

$$\lambda := \rho(\mathcal{B}'_{(d+1)}) = \rho(\mathcal{B}_{(d)}) > 0. \tag{3.20}$$

By Proposition 3.3 both the iteration matrices $\mathcal{B}_{(d)}$ and $\mathcal{B}'_{(d+1)}$ have a positive leading eigenvector $\alpha_{(d)} = \left[{\alpha_1}^T, \dots, {\alpha_d}^T\right]^T$ and $\alpha'_{(d+1)} = \left[{\alpha'_1}^T, \dots, {\alpha'_d}^T, {\alpha'_{d+1}}^T\right]^T$, respectively, which satisfy the recursive relations

$$\alpha_{p+1} = (I - \sigma B_p)\alpha_p, \qquad p = 1, \dots, d-1, \tag{3.21}$$

and, in view of (I),

$$\alpha'_{p+1} = (I - \sigma B_p)\alpha'_p, \quad p = 1, \dots, d-1, \qquad \text{and} \qquad \alpha'_{d+1} = (I - \sigma B'_d)\alpha'_d, \tag{3.22}$$

respectively, where $\sigma = (1 - \lambda)/\lambda > 0$ (see the proof of Lemma 3.1).

Since the eigenvectors are defined up to a multiplicative constant $\neq 0$, the positivity of $\alpha_{(d)}$ allows us to assume that $\alpha'_p \le \alpha_p$, $p = 1, \dots, d$, elementwise, which implies

$$\delta_p := \alpha_p - \alpha'_p \ge 0, \quad p = 1, \dots, d, \quad \text{elementwise.} \tag{3.23}$$

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Therefore, (3.21), (3.22), and the nonnegativity of the $B_p$'s yield

$$0 \le \delta_{p+1} = (I - \sigma B_p)\delta_p \le \delta_p, \quad p = 1, \ldots, d-1, \quad \text{elementwise}$$

and, in turn,

$$0 \le \delta_d \le \ldots \le \delta_1 \quad \text{elementwise.} \tag{3.24}$$

Finally, by substituting the last equality of (3.22) into the last but first equality of (3.16), by using (II) and (3.23) and by subtracting the last equality of (2.5), we get

$$0 \le \lambda B_1\delta_1 + \cdots + \lambda B_{d-1}\delta_{d-1} + B_d\delta_d + \sigma B'_{d+1}B'_d\alpha'_d = \lambda\delta_d \quad \text{elementwise.} \tag{3.25}$$

Now, again by the arbitrariness of the multiplicative constant of $\alpha_{(d)}$, the chain of inequalities (3.24) allows us to suppose that

$$(\delta_d)_i = 0 \qquad \text{for some } \ i \in \{1, \ldots, n\},$$

and, consequently, by (3.25) we get

$$(\lambda B_1\delta_1 + \cdots + \lambda B_{d-1}\delta_{d-1} + B_d\delta_d)_i = 0$$

and

$$(B'_{d+1}B'_d\alpha'_d)_i = 0.$$

On the other hand, being $\lambda \neq 0$, the irreducibility of $B_J$ is clearly transmitted to the matrix $\lambda B_1 + \cdots + \lambda B_{d-1} + B_d$. Therefore, since (3.24) yields

$$(\delta_p)_h = 0 \quad \text{for some } \ p \in \{1, \ldots, d-1\} \quad \Longrightarrow \quad (\delta_d)_h = 0,$$

following the same steps as in the proof of Proposition 3.3, we can show that $(\delta_d)_k = 0$ for any $k \in \{1, \ldots, n\}$, $k \neq i$, and consequently that

$$(B'_{d+1}B'_d\alpha'_d)_k = 0.$$

In conclusion, since $\alpha'_d > 0$ and $B'_{d+1}B'_d \ge O$ elementwise, we get

$$B'_{d+1}B'_d = O,$$

which contradicts (III) . Hence, (3.20) cannot be true, and so, in view of (b1), the stronger implication (d1) is proved. The proof of (d3) is analogous. □

**3.4. Some remarks on the case of nonpositive Jacobi iteration matrices.** In this section we briefly consider the opposite particular case of matrices $A$ for which the Jacobi iteration matrix $B_J$ satisfies the *nonpositivity condition*

$$B_J \le O \quad \text{elementwise.} \tag{3.26}$$

As is usually done, if $B = [b_{i,j}]_{i,j=1}^n$ is an $(n \times n)$-matrix, we shall denote by

$$|B| = [|b_{i,j}|]_{i,j=1}^n$$

the matrix made up by the absolute values of the entries of $B$. Clearly $B_J \le O$ elementwise if and only if $|B_J| = -B_J$. Consequently, we have the following obvious result for the spectrum of $B_J$.

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PROPOSITION 3.8. *Let $B_J$ satisfy (3.26). Then $\lambda$ is an eigenvalue of $B_J$ if and only if $-\lambda$ is an eigenvalue of $|B_J|$, and the related eigenvectors coincide. In particular, it holds that $\rho(B_J) = \rho(|B_J|)$.*

From a practical point of view, this means that passing from the nonnegative to the nonpositive case for the iteration matrix $B_J$ does not change the convergence properties of the Jacobi method. On the contrary, in general, this is not the case of any splitting method more refined than the Jacobi one. From a heuristic point of view, this fact seems to be clear enough. In fact, if $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ is a splitting of order $d$ of $B_J \leq O$ elementwise, we consider the corresponding splitting

$$\mathcal{B}'_{(d)} := \{|B_1|, \ldots, |B_d|\} \tag{3.27}$$

of $|B_J|$. By (2.4) applied to both the related iteration matrices $\mathcal{B}'_{(d)}$ and $\mathcal{B}_{(d)}$, we immediately obtain $|\mathcal{B}_{(d)}| \leq \mathcal{B}'_{(d)}$ elementwise and consequently,

$$\rho(\mathcal{B}_{(d)}) \leq \rho(\mathcal{B}'_{(d)}) \tag{3.28}$$

(see the analogous derivation of (3.14) from (3.13)).

Now, due to possible cancellation of positive and negative terms, a splitting that is more refined than the Jacobi one, starting already from those with just two elements (i.e., of the type $\mathcal{B}_{(2)} = \{B_1, B_2\}$), gives rise to an iteration matrix in which some elements of $|\mathcal{B}_{(d)}|$ are very likely strictly less than the corresponding elements in $\mathcal{B}'_{(d)}$. Therefore, the weak inequality in (3.28) is very likely to become a strict inequality. Indeed it often happens that the more refined the splitting is, the bigger is its hope of success in attaining a strict inequality in (3.28). This fact becomes particularly interesting when the Jacobi method does not converge (i.e., when $\rho(B_J) = \rho(|B_J|) \geq 1$) and when at the same time, although (obviously) not converging if applied to the system related to the opposite Jacobi iteration matrix $|B_J| = -B_J$, some more refined splitting methods do converge instead. Nevertheless, there exist also interesting examples, like that of the symmetric tridiagonal matrices $A$, in which the particular position of the few nonzero elements in the corresponding Jacobi iteration matrix $B_J$ cause the failure of the above heuristic reasoning. However, it is not a purpose of this paper to state rigorous sufficient conditions for the success of the strict inequality in (3.28). Some illustrative numerical results will be given in Section 8.

**4. The case of strictly diagonally dominant matrices.** In this section we show that the condition of *strict diagonal dominance by rows* of the coefficient matrix $A$, that is,

$$\|B_J\|_\infty = \|L + U\|_\infty < 1,$$

implies convergence of all the splitting methods of the type (2.2).

PROPOSITION 4.1. *Let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of order $d$ of $B_J$. Then the corresponding iteration matrix (2.4) is such that:*

- $\|B_J\|_\infty \leq 1 \quad \Longrightarrow \quad \|\mathcal{B}_{(d)}\|_\infty = \|B_J\|_\infty$;
- $\|B_J\|_\infty > 1 \quad \Longrightarrow \quad \|B_J\|_\infty \leq \|\mathcal{B}_{(d)}\|_\infty \leq \|B_J\|_\infty^d$.

*Proof.* The first n rows of the $(dn \times dn)$-matrix $\mathcal{B}_{(d)}$ are given by the $(n \times dn)$-matrix $[B_1 \ldots B_d]$, and hence,

$$\|\mathcal{B}_{(d)}\|_\infty \geq \|[B_1 \ldots B_d]\|_\infty.$$

On the other hand, clearly

$$\|[B_1 \ldots B_d]\|_\infty = \|B_J\|_\infty, \tag{4.1}$$

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and therefore

$$\|\mathcal{B}_{(d)}\|_\infty \geq \|B_J\|_\infty.$$

Now consider the transformation

$$y := [y_1^T, \ldots, y_d^T]^T := \mathcal{B}_{(d)} x \qquad \text{with } x := [x_1^T, \ldots, x_d^T]^T,$$

where $y_1, \ldots, y_d, x_1, \ldots, x_d \in \mathbb{R}^n$, and the iterative scheme (2.2), which suggests the equivalent system of equalities

$$y_i := \sum_{j=1}^{i-1} B_j y_j + \sum_{j=i}^{d} B_j x_j, \qquad i = 1, \ldots, d. \tag{4.2}$$

Observe that (4.1) and the first of the above equalities ensure that

$$\|y_1\|_\infty \leq \|B_J\|_\infty \, \|x\|_\infty. \tag{4.3}$$

We start with the case $\|B_J\|_\infty \leq 1$. Let $1 \leq k \leq d-1$, and assume by induction hypothesis that

$$\|y_i\|_\infty \leq \|B_J\|_\infty \, \|x\|_\infty \leq \|x\|_\infty, \qquad i = 1, \ldots, k, \tag{4.4}$$

which is true for $k = 1$ thanks to (4.3). Then (4.1) and the $(k+1)$-st equality of (4.2) clearly imply

$$\|y_{k+1}\|_\infty \leq \|B_J\|_\infty \, \|x\|_\infty,$$

too. Therefore, (4.4) holds up to $k = d$, that is,

$$\|y\|_\infty = \|\mathcal{B}_{(d)} x\|_\infty \leq \|B_J\|_\infty \, \|x\|_\infty,$$

and consequently,

$$\|\mathcal{B}_{(d)}\|_\infty = \|B_J\|_\infty.$$

Then we consider the case $\|B_J\|_\infty > 1$. As before, let $1 \leq k \leq d-1$, and assume by induction hypothesis that

$$\|y_i\|_\infty \leq \|B_J\|_\infty^k \, \|x\|_\infty, \qquad i = 1, \ldots, k, \tag{4.5}$$

which, again, is true for $k = 1$ thanks to (4.3). Then (4.1) and the $(k+1)$-st equality of (4.2) clearly imply

$$\|y_{k+1}\|_\infty \leq \|B_J\|_\infty \, \max\{\|B_J\|_\infty^k \, \|x\|_\infty, \|x\|_\infty\} = \|B_J\|_\infty^{k+1} \, \|x\|_\infty,$$

too. Therefore, (4.5) also holds for $k = d$ and this concludes the proof. □

COROLLARY 4.2. *Let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$. If $A$ is strictly diagonally dominant by rows, then $\rho(\mathcal{B}_{(d)}) < 1$.*

It is interesting to remark that a result analogous to Proposition 4.1 does not hold for the 1-norm as is shown by the following simple counterexample:

EXAMPLE 4.3. Consider the $2 \times 2$ matrix

$$B_J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

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for which $\|B_J\|_\infty = \|B_J\|_1 = 1$. The iteration matrix of the corresponding $\mathrm{T}_U$-method (1.3) is

$$\mathcal{B}_T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

for which $\|\mathcal{B}_T\|_\infty = 1$, as assured by Proposition 4.1, but $\|\mathcal{B}_T\|_1 = 3$.

However, in spite of the foregoing counterexample, an analogous result to Corollary 4.2 does hold also for the *strict diagonal dominance by columns*, i.e., when

$$\|B_J\|_1 = \|L + U\|_1 < 1. \tag{4.6}$$

PROPOSITION 4.4. *Let $\mathcal{B}_{(d)} = \{B_1, \ldots, B_d\}$ be a splitting of $B_J$. If the matrix $A$ is strictly diagonally dominant by columns, then $\rho(\mathcal{B}_{(d)}) < 1$.*

*Proof.* We consider the splitting $\mathcal{B}'_{(d)}$ defined by (3.27), for which (3.28) holds. On the other hand, by applying Corollary 3.6 to $|B_J|$ and $\mathcal{B}'_{(d)}$, from (4.6) and since $\| \, |B| \, \|_1 = \|B\|_1$, we get

$$\rho(\mathcal{B}'_{(d)}) \le \rho(|B_J|) \le \|B_J\|_1 < 1. \qquad \square$$

**5. The AMKS-methods.** In this section we consider the class of AMKS-methods introduced by Ahmadi et al. [1], and, for the sake of clarity, we briefly recall their formulation.

DEFINITION 5.1. *We say that the $d$-tuple of matrices $\mathcal{P}_{(d)} = \{P_1, \ldots, P_d\}$, $P_p \in \mathbb{R}^{n\times n}$, $1 \le d \le n$, is a* decomposition of the identity (DoI) in $\mathbb{R}^n$ *if the following three conditions hold:*

- *$P_p \ne O$ for all $p = 1, \ldots, d$;*
- *the $P_p$'s are* logical matrices, *i.e., $(P_p)_{i,j} \in \{0, 1\}$ for all $p = 1, \ldots, d$ and $i, j = 1, \ldots, n$;*
- *$\sum_{p=1}^{d} P_p = I$, where $I$ is the identity matrix.*

The corresponding iterative AMKS-method is given by

$$\begin{cases} x_0^{(k+1)} = \bar{x}^{(k)}, \\ x_p^{(k+1)} = P_p(B_J x_{p-1}^{(k+1)} + c) + (I - P_p)x_{p-1}^{(k+1)}, \qquad p = 1, \ldots, d, \\ \bar{x}^{(k+1)} = x_d^{(k+1)}, \end{cases} \tag{5.1}$$

with iteration matrix

$$\mathcal{P}_{(d)} = \prod_{p=d}^{1} (P_p B_J + I - P_p).$$

We remark that, as anticipated in Section 1, the matrices $P_p B_J$ involved in (5.1) are made up by some rows of the Jacobi iteration matrix $B_J$. Now we will show that, substantially, the above class of methods is part of the more general class of methods introduced by Definition 2.1. More precisely, we have the following result:

PROPOSITION 5.2. *The AMKS-method (5.1) for the DoI $\mathcal{P}_{(d)} = \{P_1, \ldots, P_d\}$ and the method (2.3) corresponding to the splitting $\mathcal{B}_{(d)} = \{P_1 B_J, \ldots, P_d B_J\}$ are spectrum-equivalent.*

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*Proof.* First we assume that $\lambda \neq 0$ be an eigenvalue of the iteration matrix $\mathcal{P}_{(d)}$ and $x \in \mathbb{R}^n$ a corresponding eigenvector. Let

$$\begin{aligned} &\alpha_1 := x \\ &\alpha_p := \lambda(P_1 x + \cdots + P_{p-1}x) + P_p x + \cdots + P_d x, \qquad p = 2, \ldots, d. \end{aligned}$$

By using the properties of the DoI $\mathcal{P}_{(d)}$, it is easy to verify that the above definitions all together are equivalent to

$$\lambda P_p \alpha_1 = \cdots = \lambda P_p \alpha_p = P_p \alpha_{p+1} = \cdots = P_p \alpha_d, \qquad p = 1, \ldots, d. \tag{5.2}$$

Moreover, since $\lambda \neq 0$, it is clear that

$$P_p \alpha_1 = \cdots = P_p \alpha_p, \qquad p = 2, \ldots, d. \tag{5.3}$$

Consequently, if we apply the matrices $P_p$ to the equation $\mathcal{P}_{(d)} x = \lambda x$, each of them separately for $p = 1, \ldots, d$, by using the properties of $\mathcal{P}_{(d)}$, long but easy calculations allow us to conclude that

$$P_p B_J \alpha_p = \lambda P_p \alpha_1, \qquad p = 1, \ldots, d. \tag{5.4}$$

Finally, with $B_p = P_p B_J$, $p = 1, \ldots, d$, it is immediate to see that (5.2), (5.3), and (5.4) imply the validity of (2.5), meaning that $\lambda$ is an eigenvalue of the iteration matrix $\mathcal{B}_{(d)}$ as well.

Vice versa, now assume that $\lambda \neq 0$ is an eigenvalue of the iteration matrix $\mathcal{B}_{(d)}$. Then there exists a related eigenvector $\left[\alpha_1^T, \ldots, \alpha_d^T\right]^T \in \mathbb{R}^{dn}$ with $\alpha_p \neq 0$, $p = 1, \ldots, d$, (see Proposition 2.8), and the eigenvalue equation (2.5) holds true with $B_p = P_p B_J$, $p = 1, \ldots, d$. Now, by applying all the matrices $P_p$ to (2.5), each of them separately for $p = 1, \ldots, d$, similar calculations as before allow us to prove the validity of the equalities (5.2), (5.3), and (5.4).

In turn, by using such sets of equalities in the iteration matrix $\mathcal{P}_{(d)}$ of the AMKS-method (5.1) starting from the last factor $P_1 B_J + I - P_1$ and going backward up to the first factor $P_d B_J + I - P_d$, we can easily prove that

$$\mathcal{P}_{(d)} \alpha_1 = \left[\prod_{p=d}^{q} (P_p B_J + I - P_p)\right] \alpha_q, \qquad q = 2, \ldots, d.$$

In particular, for $q = d$ we get

$$\mathcal{P}_{(d)} \alpha_1 = (P_d B_J + I - P_d)\, \alpha_d.$$

The final step shows that $\mathcal{P}_{(d)} \alpha_1 = \lambda \alpha_1$ and hence, since $\alpha_1 \neq 0$, to conclude that $\lambda$ is an eigenvalue $\mathcal{P}_{(d)}$ as well. ∎

A first interesting consequence of the foregoing result is that, substantially, even the forward Gauss–Seidel method is included in the class of methods (2.3). In fact, [1, Proposition 5] shows that it may be obtained as the AMKS-method corresponding to the complete DoI $\mathcal{P}_{(n)} = \{P_1, \ldots, P_n\}$, where $(P_p)_{p,p} = 1$ for all $p = 1, \ldots, n$ and all the other elements vanish.

Conversely, we remark that, since any splitting method (2.3) that is spectrum-equivalent to an AMKS-method (5.1) is a refinement of the Jacobi method (if different from it), Theorem 3.7 implies, for the case of irreducible nonnegative Jacobi iteration matrices $B_J$, the result of [1, Proposition 10] as a particular instance.

We conclude this section by observing that an analogous class of methods could be considered, where the Jacobi matrix $B_J$ is split into sets of columns (instead of rows). This might be done by using the splitting $\mathcal{B}_{(d)} = \{B_J P_1, \ldots, B_J P_d\}$. However, we are not going to propose and study these methods in this paper.

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**6. The $T_U$-method and its refinements.** In this section we reconsider the $T_U$-method (1.3), associated with the splitting $\mathcal{B}_{T_U} = \{U, L\}$ and the method associated with its cyclic shift $\mathcal{S}(\mathcal{B}_{T_U}) = \{L, U\}$, which we call the $T_L$-method. Moreover, we introduce and analyze some of their refinements of particular interest. From now on we shall often refer to the $T_U$- and the $T_L$-method and to their refinements as *triangular methods*. Note that, in most cases, a triangular method and a splitting method equivalent to some AMKS-method are not comparable in the partial ordering "$\succ$".

LEMMA 6.1. *Let $B_J$ be such that $\rho(B_J) > 0$, and let $L \neq O$ and $U \neq O$. Then it holds $UL \neq O$ and $LU \neq O$.*

*Proof.* We assume by contradiction that $UL = O$. Then an easy calculation shows that

$$(L+U)^{2n} = \sum_{h=0}^{2n} L^h U^{2n-h},$$

and thus, since $L^n = U^n = O$ (recall that $L, U$ are triangular $n \times n$-matrices), we get $(L+U)^{2n} = O$. Consequently, we obtain the contradicting equality

$$\rho(B_J) = \rho(L+U) = 0.$$

We arrive at the same conclusion if we assume that $LU = O$. ∎

The previous lemma leads to the following result:

PROPOSITION 6.2. *Let $B_J$ be such that $\rho(B_J) > 0$, and let $L \neq O$ and $U \neq O$. Then the $T_U$-method and the $T_L$-method are refinements of the Jacobi method, i.e., $\mathcal{B}_{T_U} = \{U, L\} \succ \{B_J\}$ and $\mathcal{B}_{T_L} = \{L, U\} \succ \{B_J\}$.*

**6.1. Upper and lower triangular column methods.** We define the *upper triangular column (UTC) methods* by considering the $T_U$-method associated with $\mathcal{B}_{T_U} = \{U, L\}$ and by further splitting the upper triangle $U$ only into some subsets of columns. This idea is the basis for the possible generation of many methods. To this purpose let us consider again the complete DoI $\mathcal{P}_{(n)} = \{P_1, \ldots, P_n\}$, where $(P_j)_{j,j} = 1$, for $j = 1, \ldots, n$, and let $U_c^{(j)} = UP_j$, for $j = 2, \ldots, n$. For the sake of simplicity (and without loss of generality) we assume that $L \neq O$ and $U_c^{(j)} \neq O$, $j = 2, \ldots, n$. Since

$$U_c^{(j)} U_c^{(k)} = O, \qquad \text{for } 2 \leq k \leq j \leq n, \tag{6.1}$$

in light of Proposition 2.11, the above relations tell us that, in order to get an effective improvement by splitting the upper triangle $U$, its columns must be selected in decreasing order with respect to the column index. In particular, as a limit case, the splitting $\{U_c^{(2)}, \ldots, U_c^{(n)}, L\}$ determines a method which is equivalent to the $T_U$-method itself without any improvement. In the opposite direction, we define the *full upper triangular column (FUTC) method* when working with the splitting $\mathcal{B}_{FUTC} := \{U_c^{(n)}, \ldots, U_c^{(2)}, L\}$. We have the following equivalence result.

THEOREM 6.3. *The FUTC-method and the backward Gauss–Seidel method are spectrum-equivalent.*

*Proof.* First assume that $\lambda \neq 0$ is an eigenvalue of $\mathcal{B}_{FUTC}$ and that $\left[\alpha_1^T, \ldots, \alpha_{n-1}^T, \beta^T\right]^T$ is a related eigenvector. Then by (2.5) we get

$$\begin{cases} \sum_{j=1}^{i-1} \lambda U_c^{(n-j+1)} \alpha_j + \sum_{j=i}^{n-1} U_c^{(n-j+1)} \alpha_j + L\beta = \lambda \alpha_i, \quad i = 1, \ldots, n-1, \\ \sum_{j=1}^{n-1} \lambda U_c^{(n-j+1)} \alpha_j + L\beta = \lambda \beta. \end{cases} \tag{6.2}$$

Multiplying the first $n-1$ equalities by $U_c^{(n)}, \ldots, U_c^{(2)}$, respectively, and using (6.1) yield

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$$U_c^{(n-i+1)}\left(\sum_{j=1}^{i-1}\lambda U_c^{(n-j+1)}\alpha_j + L\beta\right) = \lambda U_c^{(n-i+1)}\alpha_i, \quad i=1,\dots,n-1. \tag{6.3}$$

Analogously, multiplying the last equality of (6.2) by $U_c^{(n)},\dots,U_c^{(2)}$ successively one after the other yields

$$U_c^{(n-i+1)}\left(\sum_{j=1}^{i-1}\lambda U_c^{(n-j+1)}\alpha_j + L\beta\right) = \lambda U_c^{(n-i+1)}\beta, \quad i=1,\dots,n-1. \tag{6.4}$$

Therefore, since $\lambda \neq 0$, comparing (6.3) to (6.4) leads to

$$U_c^{(n-i+1)}\alpha_i = U_c^{(n-i+1)}\beta, \qquad i=1,\dots,n-1. \tag{6.5}$$

Consequently, substitution into the last equality of (6.2) yields

$$\lambda\left(\sum_{j=2}^{n}U_c^{(j)}\right)\beta + L\beta = \lambda U\beta + L\beta = \lambda\beta, \tag{6.6}$$

which is equivalent to $B_{bGS}\beta = \lambda\beta$. Since $\beta \neq 0$ (see Proposition 2.8), we can conclude that $\lambda$ is an eigenvalue of the iteration matrix $B_{bGS}$.

Vice versa, let $\lambda \neq 0$ be an eigenvalue of $B_{bGS}$. Then (6.6) holds for some $\beta \neq 0$. Defining $\sigma := (1-\lambda)/\lambda$, we set

$$\alpha_i := \beta + \sigma\big(U_c^{(n-i+1)} + \cdots + U_c^{(2)}\big)\beta, \qquad i=1,\dots,n-1, \tag{6.7}$$

so that, by multiplying each equality by $U_c^{(n-i+1)}$, respectively, and by using (6.1), we get (6.5) as well. Now, it is easy to verify that (6.5), (6.6), and (6.7) together imply all the equalities of (6.2). Thus, $\lambda$ is an eigenvalue of $\mathcal{B}_{FUTC}$, and $\left[\alpha_1^T,\dots,\alpha_{n-1}^T,\beta^T\right]^T$ is the corresponding eigenvector. ∎

REMARK 6.4. Since $U$ is strictly upper triangular, $\beta \neq 0$, and $(1-\lambda)/\lambda \neq 0$, in (6.7), it must be the case that $\alpha_i \neq 0$ for all $i=1,\dots,n-1$, in perfect agreement with Proposition 2.8.

The equivalence between the FUTC-method and the backward Gauss–Seidel method is not limited to having the same spectrum of the corresponding iteration matrix. Indeed, also the sequences of the respective approximate solutions are the same.

THEOREM 6.5. *Let $\{y^{(k)}\}_{k\geq 1}$ be the sequence of approximations produced by the backward Gauss–Seidel method starting from the initial value $y^{(0)}$, i.e.,*

$$y^{(k+1)} = B_{bGS}y^{(k)} + c_{bGS} = (I-U)^{-1}\big(Ly^{(k)} + c\big), \qquad k \geq 0.$$

*Then the sequence of approximations $\left\{\left[{x_1^{(k)}}^T,\dots,{x_{n-1}^{(k)}}^T,{x_n^{(k)}}^T\right]^T\right\}_{k\geq 1}$ produced by the FUTC-method with initial value $\left[{x_1^{(0)}}^T,\dots,{x_{n-1}^{(0)}}^T,{y^{(0)}}^T\right]^T$ is such that*

$$x_n^{(k)} = y^{(k)} \qquad \textit{for all } k \geq 0 \tag{6.8}$$

*independently of the first $n-1$ initial vector-components $x_i^{(0)}$, $i=1,\dots,n-1$.*

*Proof.* The equality (6.8) is true for $k=0$ by hypothesis, and we assume, by induction, that it holds for a given $k \geq 1$. Then, as in the proof of Theorem 6.3, we multiply the first $n-1$ equalities of the FUTC-method

$$x_i^{(k+1)} = \sum_{j=1}^{i-1}U_c^{(n-j+1)}x_j^{(k+1)} + \sum_{j=i}^{n-1}U_c^{(n-j+1)}x_j^{(k)} + Lx_n^{(k)}, \quad i=1,\dots,n, \tag{6.9}$$

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by $U_c^{(n)}, \dots, U_c^{(2)}$, respectively, and the last again by $U_c^{(n)}, \dots, U_c^{(2)}$ successively one after the other to obtain

$$U_c^{(n-i+1)} x_i^{(k+1)} = U_c^{(n-i+1)} x_n^{(k+1)}, \qquad i = 1, \dots, n-1.$$

Therefore, by using again the last equality of (6.9), we can conclude that

$$x_n^{(k+1)} = \big(U_c^{(2)} + \cdots + U_c^{(n)}\big) x_n^{(k+1)} + L y^{(k)} + c = U x_n^{(k+1)} + L y^{(k)} + c,$$

which is equivalent to

$$x_n^{(k+1)} = (I - U)^{-1} \big(L y^{(k)} + c\big) = y^{(k+1)}. \qquad \square$$

Now we briefly treat the symmetric case of the *lower triangular column (LTC) methods* by considering the $\mathrm{T}_L$-method associated with $\mathcal{B}_{T_L} = \{L, U\}$ and by further splitting the lower triangle $L$ only into some subsets of columns. Everything is completely analogous to the previous case of the UTC-methods, and hence, no proof needs to be repeated. Let $L_c^{(j)} = L P_j$, $j = 1, \dots, n-1$, and assume as before that $U \neq O$ and $L_c^{(j)} \neq O$, $j = 1, \dots, n-1$. Since

$$L_c^{(j)} L_c^{(k)} = O, \qquad \text{for all } \; 1 \leq j \leq k \leq n-1, \tag{6.10}$$

using again Proposition 2.11, the above relations tell us that, in order to get an effective improvement by splitting the lower triangle $L$, its columns must be selected in increasing order with respect to the column index. It is easy to verify that the symmetric *full lower triangular column (FLTC) method*, based on the splitting $\mathcal{B}_{FLTC} := \{L_c^{(1)}, \dots, L_c^{(n-1)}, U\}$, gives rise to an iterative scheme which is spectrum-equivalent to the forward Gauss–Seidel method and that the analogous results to Theorems 6.3 and 6.5 hold.

It is worth remarking that, in the light of the results of this section, any UTC-method lies in between the $\mathrm{T}_U$- and the backward Gauss–Seidel method with respect to the partial order relation of refinement "$\succ$" introduced for the corresponding splittings in Definitions 2.9 and 2.10. Consequently, a generic UTC-method often (but not always) performs faster then the $\mathrm{T}_U$-method and slower than the backward Gauss–Seidel method. In particular, such a behaviour is assured if the methods are applied to linear systems characterized by an irreducible nonnegative Jacobi iteration matrix (see Section 3).

**6.2. Upper and lower triangular row methods.** Similarly to what we did in the previous Section 6.1, we can define the *upper triangular row (UTR) methods* by considering the $\mathrm{T}_U$-method associated with $\mathcal{B}_{T_U} = \{U, L\}$ and by further splitting the upper triangle $U$ only into some subsets of rows. We can also define the symmetric case of the *lower triangular row (LTR) method* by considering the $\mathrm{T}_L$-method associated with $\mathcal{B}_{T_L} = \{L, U\}$ and by splitting further the lower triangle $L$ only into some set of rows. Following the arguments of Section 6.1, in the former case, the splitting $\{U_r^{(1)}, \dots, U_r^{(n-1)}, L\}$, $U_r^{(i)} = P_i U$, $i = 1, \dots, n-1$, determines a method that is spectrum-equivalent to the $\mathrm{T}_U$-method itself, whereas the *full upper triangular row (FUTR)* splitting $\mathcal{B}_{FUTR} := \{U_r^{(n-1)}, \dots, U_r^{(1)}, L\}$ gives rise to a method that is spectrum-equivalent to the backward Gauss–Seidel method. As for the LTR methods, by considering the splitting $\{L_r^{(n)}, \dots, L_r^{(2)}, U\}$, $L_r^{(i)} = P_i L$, $i = 2, \dots, n$, we obtain a method that is spectrum-equivalent to the $\mathrm{T}_L$-method. On the other side, the symmetric *full lower triangular row (FLTR)* splitting $\mathcal{B}_{FLTR} := \{L_r^{(2)}, \dots, L_r^{(n)}, U\}$ leads to a method that is spectrum-equivalent to the forward Gauss–Seidel method.

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**6.3. The triangular column and row methods.** We define the *triangular column (TC) methods* by considering the $\mathrm{T}_L$-method associated with $\mathcal{B}_{T_L} = \{L, U\}$ and by further splitting both the lower and the upper triangles $L$ and $U$, the former before the latter separately, into some subsets of columns. Analogously, we define the *triangular row (TR) methods* by further splitting both the lower and the upper triangles $L$ and $U$, the former before the latter separately, into some subsets of rows.

Now we propose one of the simplest possible choices of TC- and TR-methods, but one which is already somehow significant as we shall see in Section 8 by means of some numerical experiments. In order to simplify the subsequent notation, we set

$$\nu := \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{6.11}$$

We consider the splitting

$$\mathcal{B}_{TC(2,2)} := \{L_c^{(1,\nu)}, L_c^{(\nu+1,n-1)}, U_c^{(n-\nu+1,n)}, U_c^{(2,n-\nu)}\},$$

where

$$L_c^{(1,\nu)} := L_c^{(1)} + \cdots + L_c^{(\nu)}, \qquad L_c^{(\nu+1,n-1)} := L_c^{(\nu+1)} + \cdots + L_c^{(n-1)},$$
$$U_c^{(2,n-\nu)} := U_c^{(2)} + \cdots + U_c^{(n-\nu)}, \qquad U_c^{(n-\nu+1,n)} := U_c^{(n-\nu+1)} + \cdots + U_c^{(n)}.$$

Note that

$$L_c^{(1,\nu)} L_c^{(\nu+1,n-1)} = U_c^{(n-\nu+1,n)} U_c^{(2,n-\nu)} = O,$$

and therefore the order of the elements in the splitting $\mathcal{B}_{TC(2,2)}$ can not be changed. The specification $(2,2)$ in $\mathcal{B}_{TC(2,2)}$ obviously indicates the number of divisions of $L$ and $U$.

Analogously, we can consider the splitting

$$\mathcal{B}_{TR(2,2)} := \{L_r^{(2,n-\nu)}, L_r^{(n-\nu+1,n)}, U_r^{(\nu+1,n-1)}, U_r^{(1,\nu)}\},$$

where

$$L_r^{(2,n-\nu)} := L_r^{(2)} + \cdots + L_r^{(n-\nu)}, \qquad L_r^{(n-\nu+1,n)} := L_r^{(n-\nu+1)} + \cdots + L_r^{(n)},$$
$$U_r^{(1,\nu)} := U_r^{(1)} + \cdots + U_r^{(\nu)}, \qquad U_r^{(\nu+1,n-1)} := U_r^{(\nu+1)} + \cdots + U_r^{(n-1)}.$$

Again, the order can not be changed since

$$L_r^{(2,n-\nu)} L_r^{(n-\nu+1,n)} = U_r^{(\nu+1,n-1)} U_r^{(1,\nu)} = O.$$

In light of their spectrum-equivalence with the full triangular upper/lower column/row methods, the natural idea to get a simultaneous improvement of both the backward and forward Gauss–Seidel methods is that of finding a common refinement of their corresponding splittings, hopefully maximal in our partial order relation "$\succ$". We start by observing that if we split $L_c^{(i)}$ or $L_r^{(i)}$ into a pair $\{L', L''\}$ in whatever way, then it always holds that $L'L'' = O$. Therefore, the splittings of the lower triangular matrix $L$ employed by the FLTC- and the FLTR-methods are maximal in the partial order relation. The same holds true also for the FUTC- and the FUTR-methods.

In conclusion, it is easy to see that the *full triangular column (FTC)* splitting $\mathcal{B}_{FTC} := \{L_c^{(1)}, \ldots, L_c^{(n-1)}, U_c^{(n)}, \ldots, U_c^{(2)}\}$ is a maximal refinement of the full lower

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triangular column splitting $\mathcal{B}_{FLTC}$. Furthermore, since the splitting $\{L, U_c^{(n)}, \dots, U_c^{(2)}\}$ is clearly a cyclic permutation of the full upper triangular column splitting $\mathcal{B}_{FUTC}$, the splitting $\mathcal{B}_{FTC}$ is also a maximal refinement of it modulo a cyclic permutation.

Similarly, the *full triangular row (FTR)* splitting

$$\mathcal{B}_{FTR} := \{L_r^{(2)}, \dots, L_r^{(n)}, U_r^{(n-1)}, \dots, U_r^{(1)}\}$$

is a maximal refinement of the full lower triangular row splitting $\mathcal{B}_{FLTR}$ and, modulo a cyclic permutation, of the full upper triangular row splitting $\mathcal{B}_{FUTR}$ as well.

THEOREM 6.6. *The FTC-method, the FTR-method, and the symmetric Gauss–Seidel method are all spectrum-equivalent.*

*Proof.* Writing (2.5) for the splitting $\mathcal{B}_{FTC}$ (with respect to the eigenvalue $\lambda$ and its related eigenvector $[\alpha_1^T, \dots, \alpha_{n-1}^T, \beta_1^T, \dots, \beta_{n-1}^T]^T$) leads to

$$\text{(6.12)} \quad \begin{cases} \displaystyle\sum_{j=1}^{i-1} \lambda L_c^{(j)} a_j + \sum_{j=i}^{n-1} L_c^{(j)} a_j + \sum_{j=1}^{n-1} U_c^{(n-j+1)} \beta_j = \lambda \alpha_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n-1, \\ \displaystyle\sum_{j=1}^{n-1} \lambda L_c^{(j)} a_j + \sum_{j=1}^{i-1} \lambda U_c^{(n-j+1)} \beta_j + \sum_{j=i}^{n-1} U_c^{(n-j+1)} \beta_j = \lambda \beta_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n-1. \end{cases}$$

Since $\lambda \neq 0$, we can consider the constant $\sigma = (1-\lambda)/\lambda$. Therefore, by taking the difference of each pair of consecutive equalities in (6.12), we obtain

$$\text{(6.13)} \qquad \alpha_{p+1} = \big(I - \sigma L_c^{(p)}\big)\alpha_p, \qquad p = 1, \dots, n-2,$$

$$\text{(6.14)} \qquad \beta_1 = \big(I - \sigma L_c^{(n-1)}\big)\alpha_{n-1},$$

$$\text{(6.15)} \qquad \beta_{p+1} = \big(I - \sigma U_c^{(n+1-p)}\big)\beta_p, \qquad p = 1, \dots, n-2.$$

By repeatedly and recursively using the above equalities (6.13), (6.14), and (6.15), along with (6.10), some tedious but easy calculations (which we do not report here for the sake of brevity) allow us to express all the $2n-3$ vector-components $\alpha_p$, $p = 2, \dots, n-1$, and $\beta_p$, $p = 1, \dots, n-1$, in terms of $\alpha_1$ only.

Furthermore, since $L^n = U^n = O$, it holds that

$$\text{(6.16)} \quad I + \sum_{h=1}^{n-1} (-\sigma)^h L^h = (I + \sigma L)^{-1} \quad \text{and} \quad I + \sum_{h=1}^{n-1} (-\sigma)^h U^h = (I + \sigma U)^{-1}.$$

Therefore, using again (6.1), a subsequent substitution into the first equality of (6.12) yields

$$\text{(6.17)} \qquad \big[L + \big(I + \sigma U\big)^{-1} U\big]\big(I + \sigma L\big)^{-1} \alpha_1 = \lambda \alpha_1,$$

where $\alpha_1 \neq 0$ (see Proposition 2.8).

Vice versa, we assume that (6.17) holds for some $\lambda \neq 0$ and $\alpha_1 \neq 0$. Then, by imposing the equalities (6.13), (6.14), and (6.15), we define the vectors $\alpha_p$, $p = 2, \dots, n-1$, and $\beta_p$, $p = 1, \dots, n-1$, and, starting from (6.17) and using (6.16), we can reverse the steps of the previous calculations so as to arrive at the system (6.12), proving its equivalence to (6.17).

It is immediate to see that, given the identical form of the schemes corresponding to the splittings $\mathcal{B}_{FTC}$ and $\mathcal{B}_{FTR}$, analogous calculations lead to the same equivalent eigenvalue equation (6.17) also for the iteration matrix $\mathcal{B}_{FTR}$.

To finish, we have to prove that (6.17) is also equivalent to

$$B_{sGS}\, \alpha_1 = \lambda \alpha_1,$$

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where $B_{sGS} = (I-U)^{-1}L(I-L)^{-1}U$. To this purpose, we multiply both sides of (6.17) by the nonsingular matrix $(I+\sigma L)(I+\sigma U)$, and after some more tedious but easy calculations, we arrive at the equivalent equality

$$LU\alpha_1 = \lambda(I-L)(I-U)\alpha_1.$$

Finally, multiplying both sides by $(I-U)^{-1}(I-L)^{-1}$ immediately concludes the proof. □

The equivalence among the FTC- and FTR-methods and the symmetric Gauss–Seidel method is not limited to having the same spectrum of the corresponding iteration matrix. Indeed, we shall see that the same sequence of approximations $\{y^{(k)}\}_{k\geq 1}$ produced by the symmetric Gauss–Seidel method can always be obtained also by means of the FTC- or the FTR-method.

In order to prove this fact, we first need to introduce an alternative formulation of the symmetric Gauss–Seidel iteration which, in a sequential computation environment, has the additional advantage of halving the computational cost of each step with respect to a naive standard implementation.

LEMMA 6.7. *Given the initial value $y^{(0)}$. Consider the symmetric Gauss–Seidel iterative scheme*

$$y^{(k+1)} = B_{sGS}\, y^{(k)} + c_{sGS}, \qquad k \geq 0, \tag{6.18}$$

*where $c_{sGS} = (I-U)^{-1}(I-L)^{-1}c$. Then, by setting*

$$z^{(k)} := Uy^{(k)}, \qquad k \geq 0, \tag{6.19}$$

*we get*

$$y^{(k+1)} = (I-U)^{-1}[(I-L)^{-1} - I]z^{(k)} + c_{sGS}, \qquad k \geq 0, \tag{6.20}$$

*and the* modified symmetric Gauss–Seidel *iteration*

$$z^{(k+1)} = B^*_{sGS}\, z^{(k)} + Uc_{sGS}, \qquad k \geq 0, \tag{6.21}$$

*where*

$$B^*_{sGS} := [(I-U)^{-1} - I] \cdot [(I-L)^{-1} - I].$$

*Proof.* The proof follows from the equalities

$$(I-L)^{-1} = I + L(I-L)^{-1} \qquad \text{and} \qquad (I-U)^{-1} = I + U(I-U)^{-1}. \qquad \square$$

REMARK 6.8. The naive implementation of the method (6.18) for the computation of the approximation $y^{(h)}$ (for a given $h \geq 1$) costs about twice as many $h$-steps of the Jacobi (or forward or backward Gauss–Seidel) method. Instead, it is evident that starting with (6.19) for $k = 0$, going on with the iteration scheme (6.21) up to the next to last approximation $z^{(h-1)}$ and concluding with (6.20) for $k = h-1$ almost halves the total computational cost and, thus, reduces it to about the same cost of $h$ steps of the Jacobi (or forward or backward Gauss–Seidel) method.

THEOREM 6.9. *Let $\{z^{(k)}\}_{k\geq 1}$ be the sequence of approximations produced by the modified symmetric Gauss–Seidel method* (6.21) *starting from the initial value $z^{(0)} := Uy^{(0)}$. Then the sequence of approximations*

$$\left\{ X^{(k)} = \left[ {x_1^{(k)}}^T, \ldots, {x_{n-1}^{(k)}}^T, {x_n^{(k)}}^T, \ldots, {x_{2n-2}^{(k)}}^T \right]^T \right\}_{k\geq 0}$$

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*produced by the FTC-method with initial value*

$$X^{(0)} := \left[ {x_1^{(0)}}^T, \ldots, {x_{n-1}^{(0)}}^T, {y^{(0)}}^T, \ldots, {y^{(0)}}^T \right]^T$$

*is such that*

$$U_c^{(n)} x_n^{(k)} + \cdots + U_c^{(2)} x_{2n-2}^{(k)} = z^{(k)} \qquad \textit{for all } k \geq 0, \tag{6.22}$$

*independently of the first* $n-1$ *initial vector-components* $x_i^{(0)}$, $i = 1, \ldots, n-1$. *Analogously, for the FTR-method it holds that*

$$U_r^{(n-1)} x_n^{(k)} + \cdots + U_r^{(1)} x_{2n-2}^{(k)} = z^{(k)} \qquad \textit{for all } k \geq 0. \tag{6.23}$$

*Proof.* The proof is carried out by induction and, since it is rather technical but not difficult, for the sake of brevity we give here an outline only. Since

$$x_n^{(0)} = \cdots = x_{2n-2}^{(0)} = y^{(0)},$$

the identity (6.22) holds for $k = 0$. Now we assume that it holds for $k$. By using

$$I + \sum_{h=1}^{n-1} L^h = (I - L)^{-1}$$

and by taking into account (6.10), long and tedious calculations allow us to express the second part $\left[ {x_n^{(k+1)}}^T, \ldots, {x_{2n-2}^{(k+1)}}^T \right]^T$ of $X^{(k+1)}$ in terms of the second part $\left[ {x_n^{(k)}}^T, \ldots, {x_{2n-2}^{(k)}}^T \right]^T$ of $X^{(k)}$ and of the vector $c$. In particular, as an interesting sample of the new vector components $x_i^{(k+1)}$, $i = n, \ldots, 2n-2$, we get

$$x_n^{(k+1)} = (I - L)^{-1} \big( z^{(k)} + c \big).$$

Therefore, we have just proved that the second part of all the iterates $X^{(k)}$ is independent of the first $n-1$ initial vector-components $x_i^{(0)}$, $i = 1, \ldots, n-1$. Finally, using

$$I + \sum_{h=1}^{n-1} U^h = (I - U)^{-1},$$

some further computations show that

$$U_c^{(n)} x_n^{(k+1)} + \cdots + U_c^{(2)} x_{2n-2}^{(k+1)} = B_{sGS}^* \, z^{(k)} + U c_{sGS}.$$

The proof of (6.23) is analogous. □

**7. Some further proposals of splitting methods.** In this section we propose a few particular splitting methods which have not been considered already in Section 6 and have a good potential for possibly attaining faster convergence than the symmetric Gauss–Seidel method (i.e., the FTC- or FTR-method). Such methods are not comparable in the partial order relation "$\succ$" introduced by Definitions 2.9 and 2.10, either among themselves or with the methods introduced in Section 6, including the $\mathrm{T}_U$- and the $\mathrm{T}_L$-method. Therefore, it is not possible to establish a priori which of them is the fastest, even if applied to a nonnegative Jacobi iteration matrix.

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**7.1. The alternate triangular column and row methods.** We define the *alternate triangular column (ATC) methods* by considering the $\mathrm{T}_L$- or the $\mathrm{T}_U$-method and by further splitting both the lower and the upper triangles $L$ and $U$ separately into the same number of subsets of columns, which are ordered in an alternate way, one from $L$ and one from $U$. Analogously, we define the *alternate triangular row (ATR) methods* by working by rows.

In particular, as the most promising choice, we consider the splittings

$$\begin{aligned}
\mathcal{B}_{AFTC_L} &:= \{L_c^{(1)}, U_c^{(n)}, \ldots, L_c^{(n-1)}, U_c^{(2)}\},\\
\mathcal{B}_{AFTC_U} &:= \{U_c^{(n)}, L_c^{(1)}, \ldots, U_c^{(2)}, L_c^{(n-1)}\},\\
\mathcal{B}_{AFTR_L} &:= \{L_r^{(2)}, U_r^{(n-1)}, \ldots, L_r^{(n)}, U_r^{(1)}\},\\
\mathcal{B}_{AFTR_U} &:= \{U_r^{(n-1)}, L_r^{(2)}, \ldots, U_r^{(1)}, L_r^{(n)}\},
\end{aligned}$$

which are obtained by alternating one column (row) of $L$ to one column (row) of $U$ in the same order used for the FTC- and FTR-methods. We call them the *alternate full triangular column splitting starting from $L$ (AFTC$_L$)* or *starting from $U$ (AFTC$_U$)* and the *alternate full triangular row splitting starting from $L$ (AFTR$_L$)* or *starting from $U$ (AFTR$_U$)*, respectively. All the above splittings are not essential since

$$\begin{cases}
U_c^{(j)} L_c^{(k)} = O, & 2 \le j \le k \le n-1,\\
L_c^{(j)} U_c^{(k)} = O, & 2 \le k \le j \le n-1,\\
U_r^{(i)} L_r^{(l)} = O, & 2 \le l \le i \le n-1,\\
L_r^{(i)} U_r^{(l)} = O, & 2 \le i \le l \le n-1.
\end{cases}$$

Therefore, using Propositions 2.11 and 2.12, we can conveniently replace them by the splittings

$$\begin{aligned}
\mathcal{B}'_{AFTC_L} &:= \{L_c^{(1)}, U_c^{(n)}, \ldots, L_c^{(n-\nu-1)}, U_c^{(\nu+2)}, L_c^{(n-\nu)} + U_c^{(\nu+1)}, \ldots, L_c^{(n-1)} + U_c^{(2)}\},\\
\mathcal{B}'_{AFTC_U} &:= \{U_c^{(n)}, L_c^{(1)}, \ldots, U_c^{(\nu+2)}, L_c^{(n-\nu-1)}, L_c^{(n-\nu)} + U_c^{(\nu+1)}, \ldots, L_c^{(n-1)} + U_c^{(2)}\},\\
\mathcal{B}'_{AFTR_L} &:= \{L_r^{(2)} + U_r^{(n-1)}, \ldots, L_r^{(\nu+1)} + U_r^{(n-\nu)}, L_r^{(\nu+2)}, U_r^{(n-\nu-1)}, \ldots, L_r^{(n)}, U_r^{(1)}\},\\
\mathcal{B}'_{AFTR_U} &:= \{L_r^{(2)} + U_r^{(n-1)}, \ldots, L_r^{(\nu+1)} + U_r^{(n-\nu)}, U_r^{(n-\nu-1)}, L_r^{(\nu+2)}, \ldots, U_r^{(1)}, L_r^{(n)}\},
\end{aligned}$$

which have $2n - 2 - \nu = O(3n/2)$ elements, where $\nu$ is given by (6.11). These splittings turn out to be maximal in the partial order relation "$\succ$". Moreover, differently from the case of the FTC- and FTR-methods, all four AFTC$_L$-, AFTC$_U$-, AFTR$_L$-, and AFTR$_U$-methods generally have different speeds of convergence.

**8. Numerical examples.** In this section we give some numerical examples based on three classes of matrices $A$, say, *Class* 1, *Class* 2, and *Class* 3, which do not necessarily play a particular role in applications but are suitable enough to illustrate the developed theory.

The matrices of *Class* 1 are constructed by choosing *uniformly distributed random elements* such that

$$a_{i,j} \in [-1, 1] \qquad \forall\, (i,j) \text{ with } i \neq j.$$

Moreover, the diagonal elements are computed by using the rule

$$a_{i,i} := \frac{1}{\phi} \sum_{j=1,\, j\neq i}^{n} |a_{i,j}| \qquad \forall\, i, \tag{8.1}$$

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TABLE 8.1
*Experiments with matrices A of* Class *1 for* $\phi = 0.9$.

| method | $\overline{\rho}(\mathcal{B})$ | s.d.$\rho(\mathcal{B})$ | $\overline{sp}(\mathcal{B})$ | s.d.$sp(\mathcal{B})$ |
|---|---|---|---|---|
| Jacobi | 0.10962 | 3.85E-03 | | |
| $T_U$ | 0.057121 | 1.90E-03 | 1.2950 | 2.12E-02 |
| fGS | 0.042714 | 1.56E-03 | 1.4296 | 2.84E-02 |
| bGS | 0.042434 | 1.54E-03 | 1.4296 | 2.84E-02 |
| TC(2,2) | 0.043724 | 1.61E-03 | 1.4160 | 2.80E-02 |
| TR(2,2) | 0.043949 | 1.58E-03 | 1.4137 | 2.76E-02 |
| sGS | 0.0075707 | 5.93E-04 | 2.2103 | 5.04E-02 |
| $AFTC_L$ | 0.032672 | 1.26E-03 | 1.5478 | 2.77E-02 |
| $AFTC_U$ | 0.032815 | 1.24E-03 | 1.5458 | 2.59E-02 |
| $AFTR_L$ | 0.032552 | 1.34E-03 | 1.5496 | 3.03E-02 |
| $AFTR_U$ | 0.032762 | 1.26E-03 | 1.5466 | 2.98E-02 |

where $\phi$ is a given parameter. In this way $\|B_J\|_\infty = \phi$.

For each matrix $A$ of *Class* 1 we define the corresponding matrix $A'$ of *Class* 2 by setting

$$a'_{i,j} := -|a_{i,j}|, \quad i \neq j, \qquad \text{and} \qquad a'_{i,i} := a_{i,i},$$

and the corresponding matrix $A''$ of *Class* 3 by setting

$$a''_{i,j} := |a_{i,j}|, \quad i \neq j, \qquad \text{and} \qquad a'_{i,i} := a_{i,i}.$$

The elements of the matrices being randomly chosen, we repeat each experiment for $N = 100$ times. For each matrix $A$, $A'$, and $A''$, we compute $\rho(B_J)$ along with the spectral radius $\rho(\mathcal{B})$ of the iteration matrix $\mathcal{B}$ of all the other considered methods. Furthermore, we compute the *speedup factors*

$$sp(\mathcal{B}) := \log(\rho(\mathcal{B}))/\log(\rho(B_J))$$

of the various methods with respect to the Jacobi iteration.

However, since we make many experiments using different matrices $A^{(r)}$, $r = 1, \ldots, N$, for each method we only show the *average values* $\overline{\rho}(\mathcal{B})$ and $\overline{sp}(\mathcal{B})$ of the spectral radii and of the speedup factors, respectively, together with the corresponding *standard deviations* s.d.$\rho(\mathcal{B})$ and s.d.$sp(\mathcal{B})$.

The results are reported in Tables 8.1, 8.2, and 8.3. The numerical values of the average spectral radii and standard deviations are given in fixed point notation, truncated to the 5th significant digit, whereas the standard deviations are given in floating point notation, truncated to the 3rd significant digit. Besides the Jacobi method we consider, in the following order, the methods $T_U$, fGS, bGS, TC(2,2), TR(2,2), sGS, $AFTC_L$, $AFTC_U$, $AFTR_L$, $AFTR_U$ in relation to the value $\phi = 0.9$ in (8.1). The dimension of the matrices is $n = 100$.

The first general observation is that the standard deviations are always small enough in comparison to the average values of the various spectral radii and speedup factors. The second is that the heuristic idea that the more the splitting is refined, the faster the convergence rate is, seems to be substantially confirmed. Nevertheless, a fixed score among the various methods cannot be established. It is clear enough that the Jacobi iteration is the slowest method, followed by the $T_U$-method with a good improvement, especially when applied to matrices of *Class* 3. However, these two methods present an evident advantage with respect to the more refined ones if they are used in a parallel computation environment. Then it looks

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TABLE 8.2
*Experiments with matrices $A'$ of* Class *2 for* $\phi = 0.9$.

| method | $\overline{\rho}(\mathcal{B})$ | s.d.$\rho(\mathcal{B})$ | $\overline{sp}(\mathcal{B})$ | s.d.$sp(\mathcal{B})$ |
|---|---|---|---|---|
| Jacobi | 0.90000 | 0 | | |
| $T_U$ | 0.85418 | 1.25E-04 | 1.4960 | 1.39E-03 |
| fGS | 0.81286 | 5.82E-04 | 1.9670 | 6.77E-03 |
| bGS | 0.81282 | 5.79E-04 | 1.9670 | 6.77E-03 |
| TC(2,2) | 0.82388 | 3.22E-04 | 1.8387 | 3.71E-03 |
| TR(2,2) | 0.82385 | 3.53E-04 | 1.8391 | 4.07E-03 |
| sGS | 0.73472 | 3.50E-04 | 2.9259 | 5.08E-03 |
| $AFTC_L$ | 0.78174 | 5.80E-04 | 2.3370 | 7.04E-03 |
| $AFTC_U$ | 0.78179 | 5.88E-04 | 2.3365 | 7.14E-03 |
| $AFTR_L$ | 0.78167 | 5.43E-04 | 2.3379 | 6.59E-03 |
| $AFTR_U$ | 0.78162 | 5.45E-04 | 2.3385 | 6.62E-03 |

TABLE 8.3
*Experiments with matrices $A''$ of* Class *3 for* $\phi = 0.9$.

| method | $\overline{\rho}(\mathcal{B})$ | s.d.$\rho(\mathcal{B})$ | $\overline{sp}(\mathcal{B})$ | s.d.$sp(\mathcal{B})$ |
|---|---|---|---|---|
| Jacobi | 0.90000 | 0 | | |
| $T_U$ | 0.40932 | 8.36E-04 | 8.4782 | 1.94E-02 |
| fGS | 0.19544 | 2.23E-03 | 15.498 | 1.07E-01 |
| bGS | 0.19537 | 2.20E-03 | 15.498 | 1.07E-01 |
| TC(2,2) | 0.22573 | 1.87E-03 | 14.127 | 7.88E-02 |
| TR(2,2) | 0.22555 | 1.94E-03 | 14.135 | 8.19E-02 |
| sGS | 0.17146 | 9.75E-04 | 16.737 | 5.39E-02 |
| $AFTC_L$ | 0.098689 | 1.78E-03 | 21.981 | 1.72E-01 |
| $AFTC_U$ | 0.098751 | 1.76E-03 | 21.975 | 1.70E-01 |
| $AFTR_L$ | 0.098327 | 1.72E-03 | 22.016 | 1.66E-01 |
| $AFTR_U$ | 0.098319 | 1.70E-03 | 22.017 | 1.65E-01 |

like that the TC(2,2) and TR(2,2) methods could be a good alternative to the forward and backward Gauss–Seidel method since, although being a little bit slower, they also present a clear advantage if used in a parallel computation environment.

It also seems to be quite frequent that the symmetric Gauss–Seidel iteration is the fastest method, closely followed by the four variants of AFTC- and AFTR- methods (see the experiments with matrices of *Class* 1 and *Class* 2). The second ones perform better on matrices of *Class* 3. In any case, the general better performance of these two types of methods is consistent with the fact that, among those we have considered, these are the only ones that are potentially optimal (in the sense of Definition 2.14).

Finally, we remark that the experiments we made within *Class* 2 clearly confirm the theoretical speed score implied by Theorem 3.7 and that, given the mutual relation among the matrices of *Class* 2 and *Class* 3, the expected inequality (3.28) is largely verified in the strict form for all the considered methods except, of course, for the Jacobi iteration (see Proposition 3.8).

In connection to the observations we made at the end of Section 3.4, now we present some results that illustrate the evident superiority of all the considered splitting methods with respect to the Jacobi iteration when they are applied to matrices of *Class* 3, for different increasing values of the parameter $\phi$. Again, we consider matrices of dimension $n = 100$. The results are

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TABLE 8.4
*$\rho(\mathcal{B})$ as function of $\phi = \rho(B_J)$ for a matrix $A''$ of* Class *3.*

| method | $\phi = 2.0$ | $\phi = 3.5$ | $\phi = 14.0$ | $\phi = 15.0$ | $\phi = 15.5$ |
|---|---|---|---|---|---|
| $T_U$ | **1.0175** | **1.8977** | **67.415** | **78.376** | **84.160** |
| fGS | 0.40715 | 0.60249 | 0.95948 | **1.0509** | **1.1194** |
| bGS | 0.39091 | 0.57263 | **1.0043** | **1.0621** | **1.1080** |
| TC(2,2) | 0.52197 | **1.2888** | **387.61** | **530.03** | **614.41** |
| TR(2,2) | 0.51804 | **1.2774** | **378.11** | **517.14** | **599.52** |
| sGS | 0.43217 | 0.65676 | 0.95397 | 0.94779 | **1.0011** |
| $AFTC_L$ | 0.16267 | 0.45879 | 0.92226 | **1.0288** | **1.0993** |
| $AFTC_U$ | 0.17824 | 0.45429 | 0.92915 | **1.0644** | **1.1364** |
| $AFTR_L$ | 0.18127 | 0.44260 | 0.92957 | **1.0577** | **1.1346** |
| $AFTR_U$ | 0.17969 | 0.44696 | 0.92392 | **1.0539** | **1.1319** |

TABLE 8.5
*$\rho(\mathcal{B})$ for example* (8.2).

| method | $T_U$ | fGS | TC(2,2) | sGS | $AFTC_L$ |
|---|---|---|---|---|---|
| $\rho(\mathcal{B})$ | 0.68383 | 0.56821 | 0.68087 | 0.35876 | 0.38260 |

reported in Table 8.4. We highlight in boldface all the values $\rho(\mathcal{B}) > 1$. This time we do not need to make any statistics because our goal is to analyze the spectral radii $\rho(\mathcal{B})$ as functions of the sole parameter $\phi$.

Our experiments suggest that, as the parameter $\phi$ increases, the resistance of the methods to exceed the critical value $\rho(\mathcal{B}) = 1$ is generally stronger for those that are more refined.

Finally, we give some numerical results referred to an application to B-spline approximation (Wang et al. [6]). The underlying matrix $A$, still of dimension $n = 100$, is a particular instance of matrices of *Class* 3. In fact, it is symmetric and 9-diagonal with elements

$$[\,0 \cdots 0 \;\; 1 \;\; 4 \;\; 1 \;\; 4 \;\; 16 \;\; 4 \;\; 1 \;\; 4 \;\; 1 \;\; 0 \cdots 0\,] \tag{8.2}$$

in each row (apart from the first and the last four rows, which are modified in an obvious way). The particular form of $A$ makes many of the spectral radii of the iteration matrices of the considered splitting methods to coincide among them. More precisely,

$$\begin{gathered}\rho(\mathcal{B}_{fGS}) = \rho(\mathcal{B}_{bGS}),\\ \rho(\mathcal{B}_{TC(2,2)}) = \rho(\mathcal{B}_{TR(2,2)}),\\ \rho(\mathcal{B}_{AFTC_L}) = \rho(\mathcal{B}_{AFTC_U}) = \rho(\mathcal{B}_{AFTR_L}) = \rho(\mathcal{B}_{AFTR_U}).\end{gathered}$$

The matrix $A$ is not diagonally dominant and $\rho(B_J) = 1.2464\ldots$ The spectral radii of the respective iteration matrices are given in Table 8.5.

**9. Conclusions.** In this paper we have defined a general class of *splitting* methods which, starting from the Jacobi method, include, among others, the $T_U$-method and all the well-known methods of Gauss–Seidel (forward, backward, and symmetric) and have the property to share all the same cost in a sequential computation environment.

A theoretical ranking of the convergence properties of all these methods is given on the basis of a certain *refinement* partial order relation which definitely works when they are applied to matrices $A$ such that the corresponding Jacobi iteration matrix $B_J$ is nonnegative (often called L-matrices).

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Two particular proposals of new methods have been presented, namely the TC(2,2)- and TR(2,2)-method and the four variants of AFTC- and AFTR-methods that seem to be promising enough. It is clear that further work could still be done to improve the potential of the proposed splitting methods such as, for example, the use of a relaxation parameter $\omega$.

Finally, also an accurate comparison of the performances of the various splitting methods would be worth considering when assuming to work in parallel computation environments.

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