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AVERAGED NYSTRÖM INTERPOLANTS FOR BIVARIATE FREDHOLM INTEGRAL EQUATIONS ON THE REAL POSITIVE SEMI-AXES*

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Dedicated to Professor Giuseppe Rodriguez on the occasion of his 60th anniversary.

Abstract. Nyström interpolants based on suitable anti-Gauss cubature formulae associated with the Laguerre weights are provided for the numerical solution of second-kind Fredholm integral equations defined on the first quadrant in the coordinate plane $(0,\infty)\times(0,\infty)$. The case when the right-hand side and the kernel may increase at the origin and/or at infinity is considered. Numerical tests illustrate the good performance of such interpolants.

Key words. second-kind Fredholm integral equations, anti-Gauss cubature formulae, averaged cubature rules, Nyström methods

AMS subject classifications. 65R20, 65D32, 65D30

1. Introduction.

Second-kind Fredholm integral equations have been extensively investigated in the one-dimensional case [1, 2, 9, 11]. Different approximation tools such as orthogonal polynomials, Bernstein polynomials, splines, and piecewise functions have been employed for the development of stable and convergent methods, also in the case when the kernel and/or the right-hand side have several pathologies [13].

The multi-dimensional case has interested researchers only recently. Among the methods presented in the literature, the Nyström-type method is a valid global approximation method, exactly as it happens in the one-dimensional case; see, for instance, [4, 17]. However, it leads to the task of solving a very large system of equations whose coefficients matrix is not structured in most cases, making the use of fast methods impracticable. Moreover, if orthogonal polynomials are involved, in addition to the cost of solving the system, one must consider the cost required for the computation of their zeros: if the procedure involves N nodes, then the Golub–Welsch algorithm needs $O(N^2)$ flops; see, for instance, [10].

Very recently, suitable averaged cubature Gauss-type formulae have been developed in [4, 5]. They are combinations of Gauss and anti-Gauss cubature rules each of which is constructed as a tensor product of the univariate versions [12]. Basically, if the Gauss and anti-Gauss rule involve N and $N+1$ nodes, respectively, then the averaged formula has $2N+1$ points. Its construction requires $O(2N^2)$ flops, and it gives a better accuracy than a single Gauss rule based on $2N$-points, whose construction requires more flops, i.e., $O(4N^2)$.

In view of these advantages, the aim of this paper is to investigate the application of such formulae to the numerical solution of integral equations. In bounded domains, such exploration has already been carried out both in the one-dimensional case [3, 7] and, very recently, in the bivariate one [4]. However, according to our knowledge, in unbounded domains such study has never been conducted.

We consider the case of the first quadrant in the coordinate plane $\mathcal{D}=(0,\infty)\times(0,\infty)$, and we are interested in computing the approximate solution of bivariate Fredholm integral

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equations of the second kind

$$f(\mathbf{y}) - \mu \int_{\mathcal{D}} k(\mathbf{x},\mathbf{y}) f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} = (y_1, y_2) \in \mathcal{D}, \tag{1.1}$$

where μ is a real parameter, f is the unknown, k and g are given functions, and

$$w(\mathbf{x}) = w_1(x_1) w_2(x_2), \qquad \mathbf{x} = (x_1, x_2) \in \mathcal{D}, \tag{1.2}$$

is the product of two generalized Laguerre weight functions

$$w_i(x) = x^{\alpha_i} e^{-x}, \qquad \alpha_i > -1, \qquad i = 1, 2. \tag{1.3}$$

We assume that the right-hand side g and the kernel k w.r.t. the variable $\mathbf{y}$ may have algebraic singularities at $\mathbf{y} \to 0$ and/or may increase as $\mathbf{y} \to \infty$. This is the reason why we consider equation (1.1) in suitably weighted spaces which allow to compensate such pathologies; see Section 2.

We develop a Nyström-type method based on the truncated anti-Gauss cubature rule presented in [5]. Such a rule uses only a fraction of the computed nodes and provides an error that is opposite in sign and has the same magnitude as the truncated Gauss cubature scheme developed in [17]. Then, we introduce a discrete operator $\mathring{K}_{\mathbf{n}+1}$ which approximates the original integral operator by using the above-mentioned rule and consider the finite-dimensional equation $(\mathcal{I} - \mathring{K}_{\mathbf{n}+1})\mathring{f}_{\mathbf{n}+1} = g$, where $\mathcal{I}$ is the identity operator and $\mathring{f}_{\mathbf{n}+1}$ is the unknown. For sufficiently large $\mathbf{n}$, we prove that the inverse operators $(\mathcal{I} - \mathring{K}_{\mathbf{n}+1})^{-1}$ exist and are uniformly bounded, testifying the numerical stability of the method. Moreover, we also show that the approximated solution tends to the exact one with an error which depends on the smoothness properties of the known functions.

Following [3, 4, 7], once the Nyström interpolant is constructed, we average it with the one obtained by the Nyström method based on the truncated Gauss cubature rule given in [17]. The resulting interpolant produces a better performance than the two initial approximations, because the single interpolants provide an upper and a lower bound for the unique solution of the equation. This new averaged Nyström interpolant leads to significant computational advantages: it allows us to reach a good accuracy by solving two linear systems of smaller size with a consequent saving of time and memory.

The paper is organized as follows. Section 2 starts with a description of the weighted spaces in which we consider equation (1.1). In the remaining part of Section 2, we summarize the cubature rules we use: truncated variants of Gauss–Laguerre and anti-Gauss–Laguerre cubature rules. We also introduce an averaged Gauss–Laguerre rule with its truncated version, providing new results about its error term in the unweighted case. Section 3 contains the Nyström-type method for numerically solving Fredholm integral equations of the second kind based on the truncated anti-Gauss cubature rule and the introduction of a new Nyström interpolant. Finally, Section 4 shows some numerical examples, and Section 5 contains some conclusive remarks.

2. Preliminaries.

2.1. Function spaces. In this section, we introduce the space of functions in which we consider equation (1.1). Such spaces are extensively studied in weighted polynomial approximation theory [13, 14, 17] and are useful to deal with functions $f(x_1, x_2)$ having algebraic singularities at the origin and being unbounded when one or both variables go to ∞. For instance, the function $f(x_1, x_2) = x_1^{-1/2} e^{x_2/4}$ is singular at $x_1 = 0$ and goes to ∞ as $x_2 \to \infty$. Therefore, we multiply the function f by a weight u such that fu is bounded.

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Obviously, we expect that the function u contains a factor of the type x^α ($\alpha > 0$) to cure the singularity at the origin and a negative exponential to compensate the growth at infinity.

Let us introduce the weight function

$$u_i(x) = (1+x)^{\eta_i} x^{\gamma_i} e^{-x/2}, \quad \eta_i, \gamma_i \geqslant 0, \quad i = 1, 2,$$

and set

$$u(x_1, x_2) = u_1(x_1) u_2(x_2). \tag{2.1}$$

Define the weighted space $C_u(\mathcal{D})$ as the set of all continuous functions such that

$$\begin{cases} \lim\limits_{\substack{x_1 \to \infty \\ x_1 \to 0^+}} (fu)(x_1, x_2) = 0, & \forall x_2 \in [0, \infty), \\ \lim\limits_{\substack{x_2 \to \infty \\ x_2 \to 0^+}} (fu)(x_1, x_2) = 0, & \forall x_1 \in [0, \infty). \end{cases}$$

If $\gamma_i > 0$ for each $i = 1, 2$ (or $\gamma_1 > 0$ or $\gamma_2 > 0$), then we will omit the limit conditions for $x_1, x_2 \to 0^+$ (or $x_1 \to 0^+$ or $x_2 \to 0^+$). The space C_u endowed with the norm

$$\|f\|_{C_u} = \|fu\|_\infty = \sup_{\mathbf{x} \in \mathcal{D}} |(fu)(\mathbf{x})|,$$

is a Banach space.

For smoother functions, we also introduce the Sobolev-type space of index $1 \leq r \in \mathbb{N}$,

$$W_r(u) = \left\{ f \in C_u : \|f\|_{W_r(u)} = \|fu\|_\infty + \max\{\|f_{x_2}^{(r)} \varphi_1^r u\|_\infty, \, \|f_{x_1}^{(r)} \varphi_2^r u\|_\infty\} < \infty \right\},$$

where $\varphi_i(x_i) = \sqrt{x_i}$, for each $i = 1, 2$, and f_{x_2} means that the bivariate function f is considered as a function of only the variable x_1. Similarly for f_{x_1}.

Let us now denote by $\mathbb{P}_{n_1, n_2}$ the set of all bivariate polynomials of degree at most n_1 in the variable x_1 and n_2 in the variable x_2, respectively, and define the error of the best polynomial approximation in C_u as [17, Section 2],

$$E_{n_1, n_2}[f]_u = \inf_{P \in \mathbb{P}_{n_1, n_2}} \|(f - P)u\|_\infty.$$

It is well known that for all functions $f \in W_r(u)$ one has [17, Theorem 2.1 and estimate (3.1)]

$$E_{n_1, n_2}[f]_u \leq \mathcal{C} \left[\frac{1}{n_1^{r/2}} + \frac{1}{n_2^{r/2}} \right] \|f\|_{W_r(u)}, \tag{2.2}$$

where $\mathcal{C}$ is a positive constant independent of n_1, n_2, and f.

2.2. Cubature rules. In this paragraph, we describe two Gauss-type rules associated to the Laguerre polynomials, to approximate integrals of the type

$$\mathcal{I}[f] = \int_0^\infty \int_0^\infty f(x_1, x_2) \, w(x_1, x_2) \, dx_1 \, dx_2, \tag{2.3}$$

where w is defined by (1.2) and (1.3). Then, we consider an average of these two schemes and explore its advantages.

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2.2.1. The truncated Gauss–Laguerre cubature rule. Let us consider the integral (2.3). It is well known that it can be approximated by the Gauss–Laguerre cubature formula $\mathcal{G}_{n_1,n_2}[f]$, which reads

$$\mathcal{G}_{n_1,n_2}[f] = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)} \lambda_{k_2}^{(2)} f\left(x_{k_1}^{(1)}, x_{k_2}^{(2)}\right), \tag{2.4}$$

where $\left\{x_{k_i}^{(i)}\right\}_{k_i=1}^{n_i}$, $i = 1, 2$, are the zeros of the Laguerre polynomials $\{p_{n_i}(w_i)\}_{n_i}$, which are orthogonal with respect to the weights w_i, and $\left\{\lambda_{k_i}^{(i)}\right\}_{k_i=1}^{n_i}$ are the corresponding weights of the Gauss–Laguerre formula. Regarding the remainder term

$$R_{n_1,n_2}[f] = \mathcal{I}[f] - \mathcal{G}_{n_1,n_2}[f],$$

it is well known that

$$R_{n_1,n_2}[f] = 0, \qquad \forall f \in \mathbb{P}_{2n_1-1,2n_2-1}.$$

From the cubature rule (2.4), following [15, 16], the truncated Gauss–Laguerre rule has been introduced in [17] as

$$\mathring{\mathcal{G}}_{n_1,n_2}[f] = \sum_{k_1=1}^{\ell_1} \sum_{k_2=1}^{\ell_2} \lambda_{k_1}^{(1)} \lambda_{k_2}^{(2)} f\left(x_{k_1}^{(1)}, x_{k_2}^{(2)}\right), \tag{2.5}$$

where the upper limit of summations ℓ_1 and ℓ_2 are the indices determined by

$$x_{\ell_i}^{(i)} = \min\left\{x_{\ell_i}^{(i)} | \, x_{\ell_i}^{(i)} \geq 4n_i\theta_i, \right\}, \qquad \forall i = 1, 2,$$

with $\theta_i \in (0,\, 1)$ fixed. As shown in [17], under suitable assumptions on the weights u and w, the cubature error of (2.5), i.e., $\mathring{R}_{n_1,n_2}[f] = \mathcal{I}[f] - \mathring{\mathcal{G}}_{n_1,n_2}[f]$, has the same magnitude as the error of (2.4). In this situation, formula (2.5) is convenient especially when the function f in (2.3) is bounded or does not increase too fast so that the last terms of the sums in (2.4) can be neglected. This property implies a remarkable advantage in applications. In fact, only a fraction of nodes is actually used, and this entails a considerable computational saving; see, for instance, [14, 17].

For the convenience of the reader, we report here the estimate for the remainder term in our weighted space; see [17] for the proof.

THEOREM 2.1. *For any $f \in C_u$, where the weight u given in (2.1) is such that*

$$\gamma_i < \alpha_i + 1, \quad \textit{for } i = 1, 2, \tag{2.6}$$

one has

$$|\mathring{R}_{n_1,n_2}[f]| \leq \mathcal{C}\left(E_{m_1,m_2}[f]_u + e^{-(n_1+n_2)\mathcal{A}}\|fu\|_\infty\right),$$

where $m_i = \left[\frac{\theta_i}{1+\theta_i} n_i\right]$, for $i = 1, 2$, and $\mathcal{C}$ and $\mathcal{A}$ are positive constants independent of n_1, n_2, and f.

Note that condition (2.6) is essential for the stability of the Gauss cubature formula in the weighted space C_u, which requires that

$$\int_{\mathcal{D}} \frac{w(\mathbf{x})}{u(\mathbf{x})} d\mathbf{x} < \infty.$$

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2.2.2. The truncated anti-Gauss–Laguerre cubature rule. Inspired by the basic paper on the anti-Gauss formula [12] and the latest developments (see, for instance, [6, 18, 19]) to estimate the errors $R_{n_1,n_2}[f]$ and $\mathring{R}_{n_1,n_2}[f]$, an anti-Gauss–Laguerre cubature formula and its truncated version have been introduced in [5].

In detail, the following rule has been defined:

$$\mathcal{G}^A_{n_1+1,n_2+1}[f] = \sum_{k_1=1}^{n_1+1}\sum_{k_2=1}^{n_2+1} \tilde{\lambda}^{(1)}_{k_1}\tilde{\lambda}^{(2)}_{k_2} f\left(\tilde{x}^{(1)}_{k_1}, \tilde{x}^{(2)}_{k_2}\right),$$

where $\left\{\tilde{\lambda}^{(i)}_{k_i}\right\}_{k_i=1}^{n_i+1}$ are the weights for each $i = 1, 2$, and the cubature nodes $\left\{\tilde{x}^{(i)}_{k_i}\right\}_{k_i=1}^{n_i+1}$ are the zeros of the polynomials

$$\tilde{p}_{n_i+1}(w_i, x) = p_{n_i+1}(w_i, x) - b^{\alpha_i}_{n_i}\, p_{n_i-1}(w_i, x), \qquad i = 1, 2, \tag{2.7}$$

with $b^{\alpha_i}_{n_i} = n_i(n_i + \alpha_i)$, $n_i \geq 1$. The cubature error $R^A_{n_1+1,n_2+1}$ is related to the remainder term of the Gauss-cubature formula R_{n_1,n_2} in the following way [5, Proposition 1]:

$$R^A_{n_1+1,n_2+1}[f] = \begin{cases} -R_{n_1,n_2}[f], & \forall f \in \mathbb{P}_{2n_1+1,2n_2-1} \cup \mathbb{P}_{2n_1-1,2n_2+1}, \\ 0, & \forall f \in \mathbb{P}_{2n_1-1,2n_2-1}. \end{cases} \tag{2.8}$$

In addition, the following estimate holds true [5, Theorem 1].

THEOREM 2.2. *For any $f \in C_u$, if condition (2.6) holds, the anti-Gauss cubature formula is stable and*

$$|R^A_{n_1+1,n_2+1}[f]| \leq \mathcal{C} E_{2n_1-1,2n_2-1}[f]_u,$$

where $\mathcal{C}$ is a positive constant independent of n_1, n_2, and f.

Now, among all the computed nodes $\left\{\tilde{x}^{(i)}_{k_i}\right\}_{k_i=1}^{n_i+1}$, let us select the node $\tilde{x}^{(i)}_{\tilde{\ell}_i}$ such that

$$\tilde{x}^{(i)}_{\tilde{\ell}_i} = \min\left\{\tilde{x}^{(i)}_{\tilde{\ell}_i} \,|\, \tilde{x}^{(i)}_{\tilde{\ell}_i} \geq 4n_i\theta_i\right\}, \qquad i = 1, 2,$$

where $\theta_i \in (0,\, 1)$ are two fixed parameters. Hence, we define the truncated anti-Gauss–Laguerre cubature formula as

$$\mathring{\mathcal{G}}^A_{n_1+1,n_2+1}[f] = \sum_{k_1=1}^{\tilde{\ell}_1}\sum_{k_2=1}^{\tilde{\ell}_2} \tilde{\lambda}^{(1)}_{k_1}\tilde{\lambda}^{(2)}_{k_2} f\left(\tilde{x}^{(1)}_{k_1}, \tilde{x}^{(2)}_{k_2}\right). \tag{2.9}$$

Denoting by $\mathring{R}^A_{n_1+1,n_2+1}[f]$ its remainder error, in [5, Proposition 4] it has been proved that the following relation which tie this error with those of the truncated Gauss–Laguerre rule (2.5) hold true:

$$\mathring{R}^A_{n_1+1,n_2+1}[f] \approx -\mathring{R}_{n_1,n_2}[f], \qquad \forall f \in \mathbb{P}_{2n_1+1,2n_2-1} \cup \mathbb{P}_{2n_1-1,2n_2+1}. \tag{2.10}$$

Here, we state the following theorem thanks to relation (2.10).

THEOREM 2.3. *For any $f \in C_u$, if condition (2.6) holds, then the truncated anti-Gauss cubature formula is stable, and*

$$|\mathring{R}^A_{n_1+1,n_2+1}[f]| \leq \mathcal{C}\left(E_{m_1,m_2}[f]_u + e^{-(n_1+n_2+2)\mathcal{A}}\|fu\|_\infty\right),$$

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where $m_i = \left\lfloor \frac{\theta_i}{1+\theta_i}(n_i+1) \right\rfloor$, *for* $i = 1, 2$, *and* $\mathcal{C}$ *and* $\mathcal{A}$ *are positive constants independent of* n_1, n_2, *and* f.

Proof. The proof follows the same line as the proof of Proposition 3.1 given in [17]. Assume that Q_{m_1,m_2} with $m_i = \left\lfloor \frac{\theta_i}{1+\theta_i}(n_i+1) \right\rfloor$ is the polynomial of best approximation of $f \in C_u$. Then,

$$\mathring{R}^A_{n_1+1,n_2+1}[f] = \mathring{R}^A_{n_1+1,n_2+1}[f - Q_{m_1,m_2}] + \mathring{R}^A_{n_1+1,n_2+1}[Q_{m_1,m_2}].$$

We have

$$\mathring{R}^A_{n_1+1,n_2+1}[f - Q_{m_1,m_2}] \le \|f - Q_{m_1,m_2}\|_\infty \left(\int_{\mathcal{D}} \frac{w(\mathbf{x})}{u(\mathbf{x})} d\mathbf{x} + \sum_{k_1=1}^{\tilde{\ell}_1} \sum_{k_2=1}^{\tilde{\ell}_2} \frac{\tilde{\lambda}^{(1)}_{k_1} \tilde{\lambda}^{(2)}_{k_2}}{u\left(\tilde{x}^{(1)}_{k_1}, \tilde{x}^{(2)}_{k_2}\right)} \right),$$

and from the proof of Theorem 1 in [5] we have

$$\sum_{k_1=1}^{\tilde{\ell}_1} \sum_{k_2=1}^{\tilde{\ell}_2} \frac{\tilde{\lambda}^{(1)}_{k_1} \tilde{\lambda}^{(2)}_{k_2}}{u\left(\tilde{x}^{(1)}_{k_1}, \tilde{x}^{(2)}_{k_2}\right)} \le \mathcal{C} \int_{\mathcal{D}} \frac{w(\mathbf{x})}{u(\mathbf{x})} d\mathbf{x}, \tag{2.11}$$

and thus

$$|\mathring{R}^A_{n_1+1,n_2+1}[f - Q_{m_1,m_2}]| \le \mathcal{C} E_{2n_1-1,2n_2-1}[f]_u \int_{\mathcal{D}} \frac{w(\mathbf{x})}{u(\mathbf{x})} d\mathbf{x}. \tag{2.12}$$

Now, by using the exactness of the anti-Gauss–Laguerre formula applied to the polynomial Q_{m_1,m_2}, we can write

$$\begin{aligned} |\mathring{R}^A_{n_1+1,n_2+1}[Q_{m_1,m_2}]| &= \left| \sum_{k_1=\tilde{\ell}_1+1}^{n_1} \sum_{k_2=\tilde{\ell}_2+1}^{n_2} \tilde{\lambda}^{(1)}_{k_1} \tilde{\lambda}^{(2)}_{k_2} Q_{m_1,m_2}\left(\tilde{x}^{(1)}_{k_1}, \tilde{x}^{(2)}_{k_2}\right) \right| \\ &\le \max_{\mathbf{x} \in \mathcal{B}} |Q_{m_1,m_2}(\mathbf{x}) u(\mathbf{x})|_\infty \int_{\mathcal{D}} \frac{w(\mathbf{x})}{u(\mathbf{x})} d\mathbf{x} \end{aligned}$$

with $\mathcal{B} = [4n_1\theta_1, \infty) \times [4n_2\theta_2, \infty)$. Therefore, by applying Lemma 7.1 in [17], we have

$$|\mathring{R}^A_{n_1+1,n_2+1}[Q_{m_1,m_2}]| \le \mathcal{C}\, e^{-(n_1+n_2)\mathcal{A}} \|fu\|_\infty.$$

Thus, the assertion follows by combining the previous estimate with (2.12). ∎

2.2.3. A Gauss–Laguerre averaged rule and its truncated version. One of the main advantages of the cubature rules introduced in the previous section is that a combination of Gauss and anti-Gauss rules allows us to define more accurate formulae. In fact, by using (2.8) and (2.10), we have

$$\mathcal{G}^A_{n_1+1,n_2+1}[f] = 2\mathcal{I}[f] - \mathcal{G}_{n_1,n_2}[f], \qquad \forall f \in \mathbb{P}_{2n_1+1,2n_2-1} \cup \mathbb{P}_{2n_1-1,2n_2+1},$$

and

$$\mathring{\mathcal{G}}^A_{n_1+1,n_2+1}[f] = 2\mathcal{I}[f] - \mathring{\mathcal{G}}_{n_1,n_2}[f], \qquad \forall f \in \mathbb{P}_{2n_1+1,2n_2-1} \cup \mathbb{P}_{2n_1-1,2n_2+1}.$$

These identities suggest us to define the averaged Gauss–Laguerre cubature rule

$$\mathcal{G}^L_{2n_1+1,2n_2+1}[f] = \frac{1}{2}\left[\mathcal{G}_{n_1,n_2}[f] + \mathcal{G}^A_{n_1+1,n_2+1}[f]\right], \tag{2.13}$$

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as well as the more convenient truncated averaged Gauss–Laguerre cubature formula

$$\mathring{\mathcal{G}}^{L}_{2n_1+1,2n_2+1}[f] = \frac{1}{2}\left[\mathring{\mathcal{G}}_{n_1,n_2}[f] + \mathring{\mathcal{G}}^{A}_{n_1+1,n_2+1}[f]\right], \tag{2.14}$$

Formulae (2.13) and (2.14) can be used to estimate the cubature error of (2.4) and (2.5)

$$\begin{aligned} R_{n_1,n_2}[f] = \mathcal{I}[f] - \mathcal{G}_{n_1,n_2}[f] &\simeq \mathcal{G}^{L}_{2n_1+1,2n_2+1}[f] - \mathcal{G}_{n_1,n_2}[f] \\ &= \frac{1}{2}\left[\mathcal{G}^{A}_{n_1+1,n_2+1}[f] - \mathcal{G}_{n_1,n_2}[f]\right], \end{aligned}$$

and

$$\begin{aligned} \mathring{R}_{n_1,n_2}[f] = \mathcal{I}[f] - \mathring{\mathcal{G}}_{n_1,n_2}[f] &\simeq \mathring{\mathcal{G}}^{L}_{2n_1+1,2n_2+1}[f] - \mathring{\mathcal{G}}_{n_1,n_2}[f] \\ &= \frac{1}{2}\left(\mathring{\mathcal{G}}^{A}_{n_1+1,n_2+1}[f] - \mathring{\mathcal{G}}_{n_1,n_2}[f]\right). \end{aligned}$$

Similarly to the averaged rule (2.13) (see [5, Corollary 2]), by combining Theorem 2.3 and Theorem 2.1, it is possible to characterize the remainder term of the average scheme (2.14)

$$\mathring{R}^{L}_{2n_1+1,2n_2+1}[f] = \mathcal{I}[f] - \mathring{\mathcal{G}}^{L}_{2n_1+1,2n_2+1}[f]$$

in terms of the error of the best polynomial approximation as the following corollary shows.

COROLLARY 2.4. *For any $f \in C_u$, if conditions (2.6) are satisfied, then*

$$|\mathring{R}^{L}_{2n_1+1,2n_2+1}[f]| \leq \frac{\mathcal{C}}{2} E_{2n_1-1,2n_2-1}[f]_u.$$

2.2.4. Error of the averaged rules. While upper bounds for the error term in the Gauss quadrature rule are well covered in the literature, similar bounds for the averaged Gauss rule seem to have not been proved before. An upper bound for this error term is given below. In the sequel, we denote by $I[f]$ the 1D integral of the type (2.3) and by $\mathcal{G}^{L}_{2n+1}[f]$ the corresponding univariate version of the averaged rule (2.13).

THEOREM 2.5. *Let f be a $(2n+2)$-times continuously differentiable function on $[0,\infty)$. Then the error term of the univariate averaged rule can be estimated by*

$$\left|I[f] - \mathcal{G}^{L}_{2n+1}[f]\right| \leqslant \frac{1}{(2n+2)!} \max_{\xi\in[0,\infty)} |f^{(2n+2)}(\xi)| \|\tilde{p}_{n+1}\|_2^2,$$

with $\|\cdot\|_2$ being the usual L^2-norm and $\tilde{p}_{n+1}$ given in (2.7).

Proof. Let $P(x)$ be a polynomial of degree at most $2n+1$ such that $P(x_i) = f(x_i)$, for $1 \leqslant i \leqslant n$, $P(\tilde{x}_j) = f(\tilde{x}_j)$, for $1 \leqslant j \leqslant n+1$, and $P(\alpha_n) = f(\alpha_n)$. Then

$$f(x) - P(x) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi(x))(x-\alpha_n)p_n(x)\tilde{p}_{n+1}(x),$$

for some $\xi(x)$ depending on x; see, e.g., [8]. Integration yields

$$\begin{aligned} I[f] - \mathcal{G}^{L}_{2n+1}[f] = I[f] - \mathcal{G}^{L}_{2n+1}[P] &= I[f-P] \\ &\leqslant \frac{1}{(2n+2)!} \max |f^{(2n+2)}(\xi)| I[(x-\alpha_n)p_n(x)\tilde{p}_{n+1}(x)] \\ &\leqslant \frac{1}{(2n+2)!} \max |f^{(2n+2)}(\xi)| \|(x-\alpha_n)p_n(x)\|_2 \|\tilde{p}_{n+1}(x)\|_2. \end{aligned}$$

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Finally, by the Pythagoras theorem,

$$\|(x-\alpha_n)p_n(x)\|_2 = \|p_{n+1}+\beta_n p_{n-1}\|_2 = \|p_{n+1}-\beta_n p_{n-1}\|_2 = \|\tilde{p}_{n+1}\|_2,$$

so the desired inequality immediately follows. ∎

The next theorem extends the statement of the previous one to the bivariate case.

THEOREM 2.6. *If f is $(2\max\{m,n\}+2)$-times continuously differentiable, then it holds that*

$$\begin{aligned}\mathcal{I}[f]-\mathcal{G}_{2m+1,2n+1}[f] \leqslant{}& \frac{\|\tilde{p}_{n+1}\|_2^2}{(2n+2)!}\max\left|\frac{\partial^{2n+2}}{\partial x^{2n+2}}f\right|\mathcal{I}_{w_2}[1]\\ &+\frac{\|\tilde{p}_{m+1}\|_2^2}{(2m+2)!}\max\left|\frac{\partial^{2m+2}}{\partial y^{2m+2}}f\right|\mathcal{I}_{w_1}[1],\end{aligned}$$

where $\mathcal{I}_w$ is the integral with the weight function w.

Proof. Denote $F(x)=\int_0^\infty f(x,y)dy$. Then

$$\begin{aligned}\mathcal{I}[F]-\mathcal{G}_{2n+1}^L(F) &\leqslant \frac{1}{(2n+2)!}\max|F^{(2n+2)}(\xi)|\|\tilde{p}_{n+1}\|_2^2\\ &= \frac{1}{(2n+2)!}\max\left|\int_0^\infty\frac{\partial^{2n+2}}{\partial x^{2n+2}}f(x,y)dy\right|\|\tilde{p}_{n+1}\|_2^2\\ &\leqslant \frac{1}{(2n+2)!}\max\left|\frac{\partial^{2n+2}}{\partial x^{2n+2}}f\right|\mathcal{I}_{w_2}[1]\|\tilde{p}_{n+1}\|_2^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_{2n+1}^L(F)-\mathcal{G}_{2m+1,2n+1}^L[f] &= \mathcal{G}_{2n+1}^L(\mathcal{I}[f]-\mathcal{G}_{2m+1}^L[f])\\ &\leqslant \frac{1}{(2m+2)!}\max\left|\frac{\partial^{2m+2}}{\partial y^{2m+2}}f\right|\mathcal{I}_{w_1}[1]\|\tilde{p}_{m+1}\|_2^2.\end{aligned}$$

Adding these two inequalities yields the result. ∎

In the case of the Laguerre weight functions given by (1.3), in the previous theorem we will have $\mathcal{I}_{w_i}[1]=\Gamma(\alpha_i+1)$, for $i=1,2$, where Γ represents the Gamma function.

As a consequence of the previous statement, we give a sufficient condition for the univariate Gauss rule $\mathcal{G}_n[f]$ and the univariate anti-Gauss rule $\tilde{\mathcal{G}}_{n+1}[f]$ to "bracket" the 1D integral $I[f]$.

THEOREM 2.7. *Assume that f is a $(2n+2)$-times continuously differentiable function such that $f^{(2n+2)}(x)$ does not change sign on D and*

$$\frac{\max|f^{(2n+2)}|}{\min|f^{(2n)}|} \leqslant (n+1)(2n+1)\frac{\|p_n\|_2^2}{\|\tilde{p}_{n+1}\|_2^2} = (n+1)(2n+1)\frac{\beta_n^2+\beta_{n-1}\beta_{n+1}}{\beta_{n+1}}. \tag{2.15}$$

Then, $\mathcal{G}_n[f]<\mathcal{I}[f]<\tilde{\mathcal{G}}_{n+1}[f]$.

Proof. The condition (2.15) is equivalent to

$$\frac{1}{(2n)!}\min|f^{(2n)}|\|p_n\|_2^2 \geqslant \frac{2}{(2n+2)!}\max|f^{(2n+2)}|\|\tilde{p}_{n+1}\|_2^2.$$

In the above inequality, the left-hand side is a well-known lower bound for the error term in the Gauss quadrature, $|\mathcal{G}_n[f]-\mathcal{I}[f]|$; see, e.g., [8]. On the other hand, the right-hand side is twice the upper bound for the error term $|\mathcal{G}_{2n+1}^L[f]-\mathcal{I}[f]|$ from Theorem 2.5. Therefore,

$$|\mathcal{G}_n[f]-\mathcal{I}[f]| \geqslant 2|\mathcal{G}_{2n+1}^L[f]-\mathcal{I}[f]| = \big|(\mathcal{G}_n[f]-\mathcal{I}[f])+(\tilde{\mathcal{G}}_{n+1}[f]-\mathcal{I}[f])\big|,$$

which implies that $\mathcal{G}_n[f]-\mathcal{I}[f]$ and $\tilde{\mathcal{G}}_{n+1}[f]-\mathcal{I}[f]$ are of opposite signs, as claimed. ∎

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3. A Nyström-type averaged method. Let us consider equation (1.1), which for our convenience we rewrite as

$$(\mathcal{I} - K)f = g, \tag{3.1}$$

where $\mathcal{I}$ is the identity operator and

$$(Kf)(\mathbf{y}) = \mu \int_{\mathcal{D}} k(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}. \tag{3.2}$$

In what follows, first, we present a Nyström-type method based on the truncated anti-Gauss–Laguerre formula (2.9). Then, once the Nyström interpolant is computed, we propose to combine it with the one based on the truncated Gauss rule, thus defining an averaged interpolant.

In order to simplify the notation, we denote by $\mathbf{n} = (n_1, n_2)$, $\boldsymbol{\ell} = (\ell_1, \ell_2)$, $\tilde{\boldsymbol{\ell}} = (\tilde{\ell}_1, \tilde{\ell}_2)$ and consider the set of bi-indices

$$\mathfrak{I}_{\mathbf{n}} = \{\mathbf{i} = (i_1, i_2) : 1 \le i_1 \le n_1,\ 1 \le i_2 \le n_2\}.$$

Then, setting $\mathbf{1} = (1, 1)$, for $\mathbf{i} \in \mathfrak{I}_{\mathbf{n+1}}$, consistently with the notation $\mathbf{x} = (x_1, x_2)$, we define $\tilde{\mathbf{x}}_{\mathbf{i}} = (\tilde{x}^{(1)}_{i_1}, \tilde{x}^{(2)}_{i_2})$, where $\tilde{x}^{(1)}_{i_1}$ and $\tilde{x}^{(2)}_{i_2}$ are the Laguerre anti-Gaussian nodes introduced in Section 2.2.2.

Let us consider the following functional equation in the weighted space $C_u(\mathcal{D})$:

$$(\mathcal{I} - \mathring{K}^A_{\mathbf{n+1}})\mathring{f}_{\mathbf{n+1}} u = gu, \tag{3.3}$$

where $\mathring{f}_{\mathbf{n+1}}$ is the unknown and

$$(\mathring{K}^A_{\mathbf{n+1}} f)(\mathbf{y})u(\mathbf{y}) = \mu\, u(\mathbf{y}) \sum_{\mathbf{j} \in \mathfrak{I}_{\tilde{\boldsymbol{\ell}}}} \tilde{\lambda}_{\mathbf{j}}\, k(\tilde{\mathbf{x}}_{\mathbf{j}}, \mathbf{y}) f(\tilde{\mathbf{x}}_{\mathbf{j}}), \tag{3.4}$$

with $\tilde{\lambda}_{\mathbf{j}} = \tilde{\lambda}^{(1)}_{j_1} \tilde{\lambda}^{(2)}_{j_2}$. The next lemma is essential for the proof of the stability and convergence of the method. Coherently to the notation, we denote by

$$E_{\mathbf{n}}[f] =: E_{n_1,n_2}[f]_u = \inf_{P \in \mathbb{P}_{n_1,n_2}} \|(f - P)u\|_\infty.$$

LEMMA 3.1. *Let us consider the operators (3.2) and (3.4). Then, if the weight u given in (2.1) is such that (2.6) is fulfilled and the kernel function satisfies*

$$\sup_{\mathbf{y} \in \mathcal{D}} u(\mathbf{y}) \|k_{\mathbf{y}}\|_{W_r} < \infty, \tag{3.5}$$

then

$$\lim_{\mathbf{n} \to \infty} \|(K - \mathring{K}^A_{\mathbf{n+1}}) f u\|_\infty = 0.$$

Moreover, if the weight u fulfills the condition

$$0 \le \gamma_i < \frac{1 + \alpha_i}{2}, \qquad \eta_i > \frac{1}{2}, \qquad i = 1, 2, \tag{3.6}$$

and the kernel is such that

$$\sup_{\mathbf{x} \in \mathcal{D}} u(\mathbf{x}) \|k_{\mathbf{x}}\|_{W_r(u)} < \infty, \tag{3.7}$$

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then the discrete operators $\{\mathring{K}^A_{\mathbf{n+1}}\}_{\mathbf{n}}$ *are linear maps such that*

$$\lim_{\mathbf{m}\to\infty}\left(\sup_{f\in C_u}\lim_{\mathbf{n}\to\infty} E_{\mathbf{m}}(\mathring{K}^A_{\mathbf{n+1}}f)\right)=0. \tag{3.8}$$

Proof. By the definition (3.4), the conditions (3.6), and by applying Theorem 2.2, we have

$$\begin{aligned}|[(Kf)(\mathbf{y})-(\mathring{K}^A_{\mathbf{n+1}}f)(\mathbf{y})]u(\mathbf{y})| &\le \mathcal{C}u(\mathbf{y})E_{2\mathbf{n}-1}(k_{\mathbf{y}}f)_u\\ &\le \mathcal{C}u(\mathbf{y})\left[\|fu\|_\infty E_{\left[\frac{2\mathbf{n}-1}{2}\right]}(k_{\mathbf{y}})+2\|k_{\mathbf{y}}\|_\infty E_{\left[\frac{2\mathbf{n}-1}{2}\right]}[f]_u\right],\end{aligned} \tag{3.9}$$

which tends to zero by the assumption on the kernel (3.5). Let us now prove (3.8). First, we show that for each $f\in C_u(D)$, we have $\mathring{K}^A_{\mathbf{n+1}}f\in W_r(u)$

$$\begin{aligned}\left|(\mathring{K}^A_{\mathbf{n+1}}f)^{(r)}(\mathbf{y})\varphi^r(\mathbf{y})u(\mathbf{y})\right| &\le \varphi^r(\mathbf{y})u(\mathbf{y})\sum_{\mathbf{j}\in\mathfrak{I}_{\tilde{\ell}}}\frac{\tilde{\lambda}_{\mathbf{j}}}{u(\tilde{\mathbf{x}}_{\mathbf{j}})}\left|k(\tilde{\mathbf{x}}_{\mathbf{j}},\mathbf{y})^{(r)}u(\tilde{\mathbf{x}}_{\mathbf{j}})\right|\,|f(\tilde{\mathbf{x}}_{\mathbf{j}})u(\tilde{\mathbf{x}}_{\mathbf{j}})|\\ &\le \|fu\|_\infty\left(\sup_{\mathbf{x}}u(\mathbf{x})\|k^{(r)}_{\mathbf{x}}\varphi^r u\|_\infty\right)\sum_{\mathbf{j}\in\mathfrak{I}_{\tilde{\ell}}}\frac{\tilde{\lambda}_{\mathbf{j}}}{u^2(\tilde{\mathbf{x}}_{\mathbf{j}})}.\end{aligned}$$

Therefore, by applying (2.11) and using (3.7), we get that $\|\mathring{K}^A_{\mathbf{n+1}}f\|_{W_r(u)}<\infty$ for each $f\in C_u(\mathcal{D})$. At this point, (3.8) is deduced by (2.2). ∎

THEOREM 3.2. *Assume that* $\ker\{\mathcal{I}-K\}=\{0\}$ *in* $C_u(D)$*, where* u *fulfills* (3.6)*, and let us hypothesize that the kernel function* k *satisfies* (3.5) *and* (3.7)*. Then, for* $\mathbf{n}$ *sufficiently large, equations* (3.3) *admit a unique solution.*

Proof. By Lemma 3.1, it follows that

(i) $\mathring{K}^A_{\mathbf{n+1}}f$ converges to Kf for any $f\in C_u$, and, consequently,

$$\sup_m\|\mathring{K}^A_{\mathbf{n+1}}\|_{C_u\to C_u}<\infty,$$

(ii) the sequences $\{\mathring{K}^A_{\mathbf{n+1}}\}$ are collectively compact, and, hence,

$$\lim_{m\to\infty}\|(K-\mathring{K}^A_{\mathbf{n+1}})\mathring{K}^A_{\mathbf{n+1}}\|_{C_u\to C_u}=0.$$

Therefore, by using [1, Theorem 4.1.2], for $\mathbf{n}$ sufficiently large, say $\mathbf{n}>\mathbf{n}_0$, the approximate inverses $(\mathcal{I}-\mathring{K}^A_{\mathbf{n+1}})^{-1}$ exist and are uniformly bounded, i.e.,

$$\|(\mathcal{I}-\mathring{K}^A_{\mathbf{n+1}})^{-1}\|\le\frac{1+\|(\mathcal{I}-K)^{-1}\|\|\mathring{K}^A_{\mathbf{n+1}}\|}{1+\|(\mathcal{I}-K)^{-1}\|\,\|(K-\mathring{K}^A_{\mathbf{n+1}})\mathring{K}^A_{\mathbf{n+1}}\|}\le\mathcal{C},$$

where $\mathcal{C}$ does not depend on $\mathbf{n}$. ∎

To compute the unique solution of (3.3), let us now collocate equation (3.3) at the cubature nodes $\tilde{\mathbf{x}}_\ell$. In this way, we get the linear system

$$\sum_{\mathbf{j}\in\mathfrak{I}_{\tilde{\ell}}}\left[\delta_{\mathbf{i},\mathbf{j}}-\mu\tilde{\lambda}_{\mathbf{j}}\frac{u(\tilde{\mathbf{x}}_{\mathbf{i}})}{u(\tilde{\mathbf{x}}_{\mathbf{j}})}k(\tilde{\mathbf{x}}_{\mathbf{j}},\tilde{\mathbf{x}}_{\mathbf{i}})\right]\tilde{a}_{\mathbf{j}}=(gu)(\tilde{\mathbf{x}}_{\mathbf{i}}),\qquad\mathbf{i}\in\mathfrak{I}_{\tilde{\ell}}, \tag{3.10}$$

where $\delta_{\mathbf{i},\mathbf{j}}=\delta_{i_1,j_1}\delta_{i_2,j_2}$ is the product of two Kronecker delta symbols. The unique solution $\tilde{a}^*_{\mathbf{j}}$ gives us the Nyström interpolant

$$(\mathring{f}_{\mathbf{n+1}}u)(\mathbf{y})=(gu)(\mathbf{y})+u(\mathbf{y})\sum_{\mathbf{j}\in\mathfrak{I}_{\tilde{\ell}}}\frac{\tilde{\lambda}_{\mathbf{j}}}{u(\tilde{\mathbf{x}}_{\mathbf{j}})}k(\tilde{\mathbf{x}}_{\mathbf{j}},\mathbf{y})\,\tilde{a}^*_{\mathbf{j}}. \tag{3.11}$$

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THEOREM 3.3. *Let f^* be the unique solution of (3.1) in the space C_u, where u fulfills (3.6). Assume that the kernel function k satisfies (3.5) and (3.7), and the right-hand side $g \in W_r(u)$. Then, denoting by $\mathring{f}^*_{\mathbf{n+1}}$ the unique solution of (3.3), we have*

$$\|(f^* - \mathring{f}^*_{\mathbf{n+1}})u\|_\infty \le \mathcal{C} \left[\frac{1}{n_1^{r/2}} + \frac{1}{n_2^{r/2}} \right] \|f^*\|_{W_r(u)},$$

where $\mathcal{C}$ is a positive constant independent of $\mathbf{n}$ and f.

Proof. By [1, Theorem 4.1.2], we have

$$\|(f^* - \mathring{f}^*_{\mathbf{n+1}})u\|_\infty \le \mathcal{C}\, \|(K - \mathring{K}^A_{\mathbf{n+1}})u\|_\infty.$$

By the assumptions, it follows that $f^* \in W_r(u)$, and therefore by combining (3.9) with (2.2) we get the assertion. □

Following [3, 4, 7], let us now introduce the averaged Nyström interpolant

$$(\mathfrak{f}_{\mathbf{n}}u)(\mathbf{y}) = \frac{1}{2}\left((f_{\mathbf{n}}u)(\mathbf{y}) + (\mathring{f}_{\mathbf{n+1}}u)(\mathbf{y}) \right), \quad \mathbf{y} \in \mathcal{D}, \tag{3.12}$$

where $\mathring{f}_{\mathbf{n+1}}$ is given in (3.11) and $f_{\mathbf{n}}u$ is the weighted Nyström interpolant based on the truncated Gauss rule

$$(f_{\mathbf{n}}u)(\mathbf{y}) = (gu)(\mathbf{y}) + u(\mathbf{y}) \sum_{\mathbf{j} \in \mathfrak{I}_{\boldsymbol{\ell}}} \frac{\lambda_{\mathbf{j}}}{u(\mathbf{x_j})}\, k(\mathbf{x_j}, \mathbf{y})\, a^*_{\mathbf{j}}. \tag{3.13}$$

Here, $a^*_{\mathbf{j}}$ is the unique solution of the system

$$\sum_{\mathbf{j} \in \mathfrak{I}_{\boldsymbol{\ell}}} \left[\delta_{\mathbf{i},\mathbf{j}} - \mu\lambda_{\mathbf{j}} \frac{u(\mathbf{x_i})}{u(\mathbf{x_j})}\, k(\mathbf{x_j}, \mathbf{x_i}) \right] a_{\mathbf{j}} = (gu)(\mathbf{x_i}), \qquad \mathbf{i} \in \mathfrak{I}_{\boldsymbol{\ell}}. \tag{3.14}$$

Formula (3.12) yields a more accurate approximated solution than the single Nyström interpolants (3.11) and (3.13). In fact, numerical tests show that for any $\mathbf{y} \in \mathcal{D}$, either

$$\mathring{f}_{\mathbf{n+1}}(\mathbf{y}) \le f^*(\mathbf{y}) \le f_{\mathbf{n}}(\mathbf{y}) \quad \text{or} \quad f_{\mathbf{n}}(\mathbf{y}) \le f^*(\mathbf{y}) \le \mathring{f}_{\mathbf{n+1}}(\mathbf{y}). \tag{3.15}$$

Note that, if the product $k(\mathbf{x}, \mathbf{y})f(\mathbf{x}) \in \mathbb{P}_{2n_1+1,2n_2-1} \cup \mathbb{P}_{2n_1-1,2n_2+1}$ with respect to the variable $\mathbf{x}$, then (3.15) follows by (2.8). This situation can, for instance, occur when the known functions of the integral equations are polynomials so that the solution is a polynomial too. In the cases when we handle generic functions, it is difficult to prove (3.15), or it is possible to show it under assumptions that are hard to verify; see [3, 4].

4. Numerical tests. In this section, we show the performance of the Nyström method described in the previous section and the accuracy of the averaged interpolant (3.12). To this end, in each example we solve the systems (3.10) and (3.14) to compute the Nyström interpolants (3.11), (3.13), and (3.12).

We will present four examples in which the exact solution is not known, and thus, to evaluate the relative errors, we consider as exact the Nyström interpolant (3.11) with a fixed choice of $\bar{\mathbf{n}}$ specified in each test. We point out that this choice does not affect the results since by virtue of Theorem 3.2 and Theorem 3.3 our method is stable and convergent for $\mathbf{n}$ sufficiently large. The computed relative errors are

$$\mathring{\xi}_{\mathbf{n}} = \frac{\|(f_{\bar{\mathbf{n}}} - \mathring{f}_{\mathbf{n+1}})u\|}{\|f_{\bar{\mathbf{n}}}u\|}, \qquad \xi_{\mathbf{n}} = \frac{\|(f_{\bar{\mathbf{n}}} - f_{\mathbf{n}})u\|}{\|f_{\bar{\mathbf{n}}}u\|}, \qquad \xi^{(Avg)}_{\mathbf{n}} = \frac{\|(f_{\bar{\mathbf{n}}} - \mathfrak{f}_{\mathbf{n}})u\|}{\|f_{\bar{\mathbf{n}}}u\|}, \tag{4.1}$$

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where $\|\cdot\|$ denotes the discrete infinity norm taken on a grid of 50×50 equispaced points in $(0, 15]$.

All the computations are performed on an Intel Xeon E-2244G system with 16Gb RAM, running Matlab R2023b. The software developed is only prototypical, but it is available from the authors upon request.

EXAMPLE 4.1. Let us consider the following equation:

$$f(y_1, y_2) - \int_0^\infty \int_0^\infty \frac{\sin(y_1 + x_2)}{(2 + y_2 + x_1)} f(x_1, x_2) \sqrt{x_1 x_2} e^{-(x_1+x_2)} dx_1\, dx_2 = (3y_2 + 1)e^{-y_1}.$$

It is unisolvent in the space C_u with $u_i(x_i) = (1 + x_i)\sqrt{x_i} e^{-x_i/2}$, and the exact solution is unknown. Hence, to compute the relative errors (4.1), we consider as exact the approximated one obtained with $\bar{\mathbf{n}} = (128, 128)$. Figure 4.1 displays the errors we get with the Nyström interpolants based on the Gauss and anti-Gauss rule when $\mathbf{n} = (8, 8)$. As we can observe, they are opposite in sign confirming the estimate (3.15). The kernel and the right-hand side are smooth functions, and thus we expect a fast convergence. The errors reported in Table 4.1, which are obtained with $\theta_1 = \theta_2 = 0.4$, confirm the theoretical expectation: by solving two linear systems of order 25×25 and 26×26, we get the solution of the equation with machine precision. We mention that in this case, we use a direct method to solve the two systems, due to their small size. The last two columns contain the condition numbers in the infinity norms $\kappa_\infty^{(G)}$ and $\kappa_\infty^{(A)}$ of the systems (3.14) and (3.10), respectively, showing the well-conditioning of the discrete problems.

TABLE 4.1
Numerical results for Example 4.1.

$\mathbf{n}$	$\mathbf{j}$	$\tilde{\mathbf{j}}$	$\xi_{\mathbf{n}}$	$\mathring{\xi}_{\mathbf{n}}$	$\xi_{\mathbf{n}}^{(Avg)}$	$\kappa_\infty^{(G)}$	$\kappa_\infty^{(A)}$
(8,8)	(7,7)	(8,8)	2.77e-05	2.78e-05	7.72e-07	1.55	1.71
(16,16)	(13,13)	(14,14)	3.13e-09	2.91e-09	1.11e-10	1.66	1.64
(32,32)	(25,25)	(26,26)	1.60e-13	1.51e-13	4.37e-15	1.69	1.70

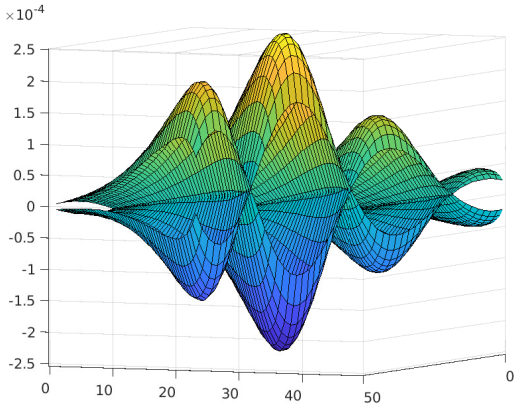

FIG. 4.1. *Error graph for Example 4.1 in the case* $\mathbf{n} = (8, 8)$.

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EXAMPLE 4.2. Let us apply our method to the following equation:

$$f(y_1,y_2)-\int_0^\infty\int_0^\infty \frac{(x_2+y_1)^{3/2}}{(5+x_1+y_2)^2} f(x_1,x_2)\sqrt{x_1}e^{-(x_1+x_2)}dx_1\,dx_2=\log(2+y_1+y_2),$$

in the weighted space C_u with $u(x_1,x_2)=(1+x_1)(1+x_2)\sqrt[5]{x_1^3x_2}e^{-\frac{1}{2}(x_1+x_2)}$. Also in this case, the exact solution is not known analytically. Then, we consider as exact the one obtained with $\bar{\mathbf{n}}=(256,256)$. In this case, the right-hand side is a smooth function whereas the non-separable kernel belongs to the Sobolev space $W_3(u)$. As a consequence, according to Theorem 3.3, we expect an order of convergence which behaves like $\mathcal{O}(\mathbf{n}^{-3/2})$. Table 4.2 contains the results we obtained with $\theta_1=\theta_2=0.4$, showing a better numerical convergence. Moreover, it is evident, by the first three columns, that the use of truncated rules allows us to reduce the size of the two systems; the last column shows that the accuracy of the averaged interpolant improves of 1–4 significant digits. We remark that in this case we solve the two systems by using an optimized version of the GMRES method which converges in few iterations reported in parentheses.

TABLE 4.2
Numerical results for Example 4.2.

$\mathbf{n}$	$\mathbf{j}$	$\tilde{\mathbf{j}}$	$\xi_{\mathbf{n}}$ (iter)	$\mathring{\xi}_{\mathbf{n}}$ (iter)	$\xi_{\mathbf{n}}^{(Avg)}$
(8,8)	(7,7)	(8,8)	3.16e-08 (6)	2.20e-08 (6)	4.80e-09
(16,16)	(13,13)	(14,14)	6.57e-09 (6)	5.74e-09 (6)	4.14e-10
(32,32)	(25,25)	(26,26)	3.96e-10 (6)	3.47e-10 (6)	2.49e-11
(64,64)	(49,49)	(50,50)	2.42e-11 (6)	2.13e-11 (6)	1.43e-12
(128,128)	(97,97)	(97,97)	1.41e-12 (6)	1.41e-12 (6)	9.70e-16

EXAMPLE 4.3. Let us now consider an equation in which the kernel is smooth whereas the right-hand side has a low smoothness with respect to one of the variables:

$$f(y_1,y_2)-\int_0^\infty\int_0^\infty \frac{x_1y_1+4}{2+x_2+y_2} f(x_1,x_2)\frac{e^{-(x_1+x_2)}}{\sqrt[4]{x_1x_2}}dx_1\,dx_2=y_1|y_2-1|^{\frac{7}{2}}.$$

We solve the above equation in the weighted space C_u with $u(\mathbf{x})=(1+\mathbf{x})^{4/5}\mathbf{x}^{3/10}e^{-\mathbf{x}/2}$. Also in this case the linear systems are solved by the GMRES method, which converges in few iterations. By inspecting Table 4.3, we can appreciate the good performance of the averaged interpolant. By solving two systems of order $14\times 53=742$ and $15\times 54=810$, we get an error of order 10^{-9}. The same error could be also reached by the standard Gauss interpolant (3.13). However, we need to solve a system of order $14\times 211=2954$ which requires a larger complexity and storage space.

EXAMPLE 4.4. Let us test the performance of the proposed approach to approximate the solution of the following equation:

$$f(y_1,y_2)-\frac{1}{32}\int_0^\infty\int_0^\infty k(x_1,x_2,y_1,y_2)f(x_1,x_2)e^{-(x_1+x_2)}dx_1\,dx_2=\frac{|\cos\left(y_1-\frac{3}{2}\right)|^{\frac{5}{2}}}{y_2+1},$$

with $k(x_1,x_2,y_1,y_2)=x_1y_1\sin(x_2+y_2+1)$ in the space C_u with $u(\mathbf{x})=(1+\mathbf{x})e^{-\mathbf{x}/2}$. In this case, the kernel is smooth, and the right-hand side is a function of $W_2(u)$. For $\mathbf{n}$ sufficiently large, the theoretical order of convergence is $\mathbf{n}^{-1}$, and then the convergence is slower than the previous examples. Table 4.4 contains the errors computed by assuming as

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TABLE 4.3
Numerical results for Example 4.3.

$\mathbf{n}$	$\mathbf{j}$	$\tilde{\mathbf{j}}$	$\xi_{\mathbf{n}}$ (iter)	$\mathring{\xi}_{\mathbf{n}}$ (iter)	$\xi_{\mathbf{n}}^{(Avg)}$
(16,8)	(14,8)	(15,8)	4.74e-06 (7)	5.70e-06 (7)	4.82e-07
(16,16)	(14,14)	(15,15)	1.28e-06 (7)	1.08e-06 (7)	1.03e-07
(16,32)	(14,27)	(15,28)	3.01e-07 (7)	2.65e-07 (7)	1.76e-08
(16,64)	(14,53)	(15,54)	5.53e-08 (7)	5.85e-08 (7)	1.57e-09
(16,128)	(14,106)	(15,107)	1.08e-08 (7)	1.14e-08 (7)	3.06e-10
(16,256)	(14,211)	(15,211)	2.37e-09 (7)	2.42e-09 (7)	2.23e-11

exact solution the Nyström interpolants with $\bar{n} = (512, 64)$ and fixing $\theta_1 = \theta_2 = 0.4$. Also in this case we can appreciate the accurate results given by the averaged interpolants with respect to the original ones.

TABLE 4.4
Numerical results for Example 4.4.

$\mathbf{n}$	$\mathbf{j}$	$\tilde{\mathbf{j}}$	$\xi_{\mathbf{n}}$ (iter)	$\mathring{\xi}_{\mathbf{n}}$ (iter)	$\xi_{\mathbf{n}}^{(Avg)}$
(16, 32)	(13,25)	(14,26)	3.12e-04 (3)	3.09e-04 (3)	1.37e-06
(32, 32)	(25,25)	(26,26)	6.64e-05 (3)	7.29e-05 (3)	3.24e-06
(64, 32)	(49,25)	(50,26)	1.93e-05 (3)	2.14e-05 (3)	1.06e-06
(128, 32)	(97,25)	(97,26)	7.25e-06 (3)	6.52e-06 (3)	3.65e-07
(256, 32)	(192,25)	(193,26)	1.85e-06 (3)	7.78e-07 (3)	5.34e-07

5. Conclusions. We developed a numerical method of Nyström type to approximate the solution of a bivariate second-kind Fredholm integral equation defined in $\mathcal{D} = (0, \infty) \times (0, \infty)$. The integral operator is approximated by a truncated anti-Gauss–Laguerre cubature formula, and the resulting Nyström interpolant is averaged with the one obtained by using a truncated Gauss–Laguerre rule. The numerical examples demonstrate that the averaged interpolant improves the accuracy given by the native component methods and avoids solving large linear systems.

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