Electronic Transactions on Numerical Analysis.
Volume 61, pp. 137–156, 2024.

ISSN 1068–9613.
DOI: 10.1553/etna_vol61s137

ETNA
Kent State University and
Johann Radon Institute (RICAM)

A STABLE NUMERICAL METHOD FOR INTEGRAL EPIDEMIC MODELS WITH BEHAVIORAL CHANGES IN CONTACT PATTERNS*

BRUNO BUONOMO†, ELEONORA MESSINA†, CLAUDIA PANICO†, AND ANTONIA VECCHIO‡

Abstract. We propose a non-standard numerical method for the solution of a system of integro-differential equations describing an epidemic of an infectious disease with behavioral changes in contact patterns. The method is constructed in order to preserve the key characteristics of the model, like the positivity of solutions, the existence of equilibria, and asymptotic behavior. We prove that the numerical solution converges to the exact solution as the step size h of the discretization tends to zero. Furthermore, the method is first-order accurate, meaning that the error in the discretization is $O(h)$, it is linearly implicit, and it preserves all the properties of the continuous problem, unconditionally with respect to h. Numerical simulations show all these properties and confirm, also by means of a case-study, that the method provides correct qualitative information at a low computational cost.

Key words. epidemic models, integro-differential equations, non-standard finite difference scheme, discrete models, perturbation theory

AMS subject classifications. 45D05, 65R20, 39A12, 92D30.

1. Introduction.

When solving a continuous mathematical model numerically, it is nowadays widely believed that convergence is no longer the only objective to be considered. Indeed, a robust numerical method should preserve the properties of the model, such as the positivity of the solutions and/or conserve certain quantities, under weak or no conditions on the step size [10, 16, 35]. In this way, the classical concept of numerical stability is generalized to include more general aspects such as the preservation of equilibria and limit cycles. In this context, a matter of interest is that the numerical method could introduce spurious outcomes, like the creation of ghost equilibrium points and change in the stability nature of equilibria, and would therefore fail to represent the dynamics of the model at hand correctly [38, 39]. The theory of *conservative* numerical methods is mostly applied to study the dynamics of differential equations [2, 18, 19, 20], and it is less developed in the case of evolution problems with memory [15, 27, 28]. Integral equations with memory are particularly well-suited to model the time evolution of epidemics of infectious diseases. In fact, the celebrated Kermack and McKendrick model is formulated in terms of a system of nonlinear integro-differential equations [23, 24]. The Kermack and McKendrick model is quite general since it allows arbitrary length distributions of infection periods, that is, it allows the infectivity of an infected individual to depend on the time elapsed since the moment of infection. Moreover, one can derive compartmental models from it, where the individuals are divided according to their stage of infection. Such models are mostly given by ordinary differential equations where the residence time in any given compartment is exponentially distributed. Recently, many extensions and variants of the original Kermack and McKendrick model have been proposed [3, 5, 7, 11] and applied to actual diseases, such as smallpox [1], cholera [6], SARS [4], COVID-19 [17, 21, 22], and AIDS [36, 37].

In our recent research [9], we introduced an integro-differential epidemic model that takes into account behavioral changes of individuals during an epidemic outbreak. Following the approach proposed in [12, 13], the model includes an information index that describes how individuals modify their risky contact according to information and rumors regarding

*Received April 6, 2024. Accepted September 29, 2024. Published online on November 12, 2024. Recommended by L. Fermo.

†Department of Mathematics and Applications "Renato Caccioppoli", University of Naples Federico II, 80126 Naples, Italy ({buonomo,eleonora.messina,claudia.panico2}@unina.it).

‡Institute for Computational Application "Mauro Picone", C.N.R. National Research Council of Italy, 80131 Naples, Italy (antonia.vecchio@cnr.it).

the disease. In [9], on the one hand, we investigated the qualitative properties of the model, including the analysis of equilibria via stability theory. Numerical simulations were crucial to describe the time profiles predicted by the model.

In this paper, we describe in detail the numerical method that we designed to deal with the behavioral integral epidemic model proposed in [9]. The method has been built according to the considerations mentioned above about preserving relevant properties of the model in the numerical solution. It is based on non-standard discretizations of the nonlinear terms in the equations, an idea originally introduced by Mickens in [32] for systems of differential equations. Only recently, the Mickens' approach has been extended to integral problems; see, e.g., the integral equation SIS model considered in [27] and the epidemic model for a closed population considered in [29, 31]. The discrete system associated with the method can be viewed, for any fixed value of the step size $h > 0$, as a discrete time model. We perform a qualitative analysis based on the linearization theory for discrete Volterra equations. As a final remark, the method is linearly implicit, thus providing correct qualitative information at a low computational cost.

The paper is organized as follows. In Section 2, we recall the integro-differential behavioral epidemic model and the main results developed in [9]. In Section 3, we formulate the numerical method, and we study its consistency and its convergence. We also prove that it preserves properties like positivity and boundedness, for any value of the step size. In Section 4, we prove that the discrete system that represents the method admits two equilibria under specific assumptions, and we study their local stability. In Section 5, we analyze how the properties of the discrete model relate to those of the continuous one. The approximation properties of the discrete model are also studied. Numerical experiments are presented in Section 6 and concluding remarks are given in Section 7.

2. The integral behavioral epidemic model.

In this section, we briefly recall the model and the related qualitative analysis presented in [9], to which we refer for all the details and proofs. We focus on the following integro-differential behavioral epidemic model describing the diffusion of an infectious disease in a susceptible population of size S:

$$
\begin{aligned}
&i) && S'(t) = \lambda - \mu S(t) - \beta\big(M(t)\big)S(t)F(t),\\
&ii) && F(t) = \int_0^\infty A_\mu(\tau)\beta\big(M(t-\tau)\big)S(t-\tau)F(t-\tau)d\tau,\\
&iii) && M(t) = \int_0^\infty K(\tau)H(t-\tau)g\big(F(t-\tau)\big)d\tau.
\end{aligned} \tag{2.1}
$$

Equation $(2.1)_i$ is the balance equation for the susceptible population, where the positive parameters λ and μ are the net inflow of susceptibles in the population and the per capita natural death rate, respectively. The term $\beta(M)F$ is the *Force of Infection* (FoI), i.e., the rate at which a susceptible becomes infected: $\beta(M)$ represents the inhibitory contribution to the FoI due to information and rumors regarding the disease status (it is assumed that $\beta(M) > 0$, $\beta(0) = 1$, and $\beta'(M) < 0$), M denotes the information index, which describes the change of opinion-driven behavior of individuals, and it will be discussed a few lines below.

Equation $(2.1)_{ii}$ is a constitutive law, where $F(t)$ is the driving factor of the FoI. The function $A_\mu(\cdot)$ is the *Infectivity Function with Demography* (IFD). It is defined as $A_\mu(\tau) = e^{-\mu\tau}A(\tau)$, where the term $e^{-\mu\tau}$ represents the probability that an individual stays alive for at least τ units of time and $A(\tau) \in L^1[0,\infty)$ is the *Infectivity Function* (IF), i.e., the expected contribution to the FoI by an individual that was itself infected τ units of time ago.

Equation $(2.1)_{iii}$ expresses the information index M as the weighted sum of past and present contributions of the FoI through a function g, such that $g(0) = 0$, $g(F) \geq 0$, and

TABLE 2.1
Description of parameters and quantities related to model (2.1). Note that $N = \lambda/\mu$.

Symbol	Meaning
λ	Net inflow of susceptibles
μ	Natural death rate
R_0	Basic Reproduction Number
N	Total population size
S_0	Initial susceptible population

TABLE 2.2
Functions involved in model (2.1) and related assumptions.

Function	Meaning	Assumptions
A	Infectivity function	$A(t) \geq 0$
A_μ	Infectivity func. with demography	$\hat{A}_\mu(0) := \int_0^\infty A_\mu(t)dt < +\infty$
β	Inhibition function	$\beta(0) = 1, \beta(M) > 0, \beta'(M) < 0$
g	Message function	$g(0) = 0, g(\cdot) \geq 0, g'(\cdot) > 0$
K	Memory kernel	$K(\tau) > 0, \hat{K}(0) := \int_0^\infty K(\tau)d\tau = 1$

$g'(F) > 0$. The weight is represented by a function $K(\tau)$ that is supposed to be positive and such that $\int_0^\infty K(\tau)dt = 1$. Functions g, called *message function*, and K, called *memory kernel*, describe the individual's perceived risks associated with the disease and how the population keeps the memory of the disease, respectively. Examples of message functions are suggested by d'Onofrio et al. in [14]. They also propose the following memory kernels:

- a Dirac delta function $K(\tau) = \delta(\tau)$, which means that the information propagates instantaneously (i.e., no delay);
- an Erlang distribution with rate parameter a and shape parameter n

$$K(\tau) = \mathrm{Erl}_{n,a}(\tau) = \frac{a^n}{(n-1)!}\tau^{n-1}e^{-a\tau}, \quad a \in \mathbb{R}^+, \quad n \in \mathbb{N}^+,$$

  which includes the case of exponentially fading memory (when $n = 1$), where the maximum weight is given to the current information, and the case of unimodal kernels (when $n > 1$), where the current information is unavailable and the maximum weight is assigned to the information that arrives at the public after a characteristic time $T = n/a$.

In $(2.1)_{iii}$, $H(\cdot)$ is the Heaviside step function. For the sake of clarity, we report in Table 2.1 the definition of the parameters and quantities related to the model (2.1) and in Table 2.2 the assumptions on the functions involved. From now on, we also assume that

$$A'_\mu(\tau) \in L^1[0,\infty), \quad K'(\tau) \in L^1[0,\infty). \tag{2.2}$$

Let us denote by $u(t) = (S(t), F(t), M(t))$ the solution of the model (2.1) and by D the domain

$$D = \{(x,y,z) \ : \ 0 < x \leq S_{\max}, \ 0 \leq y \leq F_{\max}, \ 0 \leq z \leq M_{\max}\}, \tag{2.3}$$

where

$$S_{\max} = \frac{\lambda}{\mu}, \quad F_{\max} = \lambda\hat{A}_\mu(0) + \frac{\lambda}{\mu}\int_0^\infty |A'_\mu(\tau)|d\tau, \quad M_{\max} = g(F_{\max}). \tag{2.4}$$

The following theorem provides a property of the domain D.

THEOREM 2.1. *Assume that* $\lim_{\tau\to\infty} A(\tau) = 0$. *Then the region* D *defined in* (2.3) *is positively invariant for the model* (2.1).

Let us introduce the basic reproduction number

$$R_0 = \frac{\lambda}{\mu}\hat{A}_\mu(0).$$

Model (2.1) admits a unique *Disease-Free Equilibrium* DFE $= (\bar{S}_0, 0, 0)$, where $\bar{S}_0 = \lambda/\mu$, and, for $R_0 > 1$, it admits a unique *Endemic Equilibrium* EE $= (S_e, F_e, M_e)$, where

$$S_e = \frac{\lambda}{\mu R_0 \beta(g(F_e))}, \quad M_e = g(F_e),$$

and $F_e > 0$ is the solution to

$$\Phi(x) = x - \mu R_0 + \frac{\mu}{\beta(g(x))} = 0. \tag{2.5}$$

Let us define

$$\hat{A}_\mu(w) = \int_0^\infty e^{-wt} A_\mu(\tau) d\tau, \quad \hat{K}(w) = \int_0^\infty e^{-wt} K(\tau) d\tau.$$

The following theorem gives a sufficient condition for the equilibria to be *Locally Asymptotically Stable* (LAS).

THEOREM 2.2. *Any equilibrium* $\tilde{E} = (\tilde{S}, \tilde{F}, \tilde{M})$ *of the model* (2.1) *is LAS if the characteristic equation*

$$\beta(\tilde{M})\tilde{S}\hat{A}_\mu(w)\left(1 + \frac{g'(\tilde{F})\beta'(\tilde{M})\tilde{F}\hat{K}(w)}{\beta(\tilde{M})}\right) = 1 + \frac{\beta(\tilde{M})\tilde{F}}{w+\mu} \tag{2.6}$$

has no solution for $Re(w) \geq 0$.

Then, from Theorem 2.2, it is easy to prove that, if $R_0 < 1$, the DFE is locally asymptotically stable. Furthermore, the following sufficient conditions for EE to be locally stable, written in terms of the information parameters and functions, are derived.

THEOREM 2.3. *Let* K *and* A_μ *be the functions defined above.*

- *If* $K(\tau) = \delta(\tau)$ *and* $A_\mu(\tau)$ *is positive definite, then EE is LAS for any choice of the information functions* β *and* g.
- *If* $K(\tau) = \delta(\tau)$ *or* $K(\tau) = ae^{-a\tau}$ *and the functions* β *and* g *satisfy the inequality*

$$g'(F_e)\beta'(M_e)F_e \geq -2\beta(M_e),$$

then EE is LAS for any choice of $A_\mu(\tau)$.

See [8, Section 7.3.6] for the definition and properties of positive definite functions.

REMARK 2.4. In [9], it has been observed that the choice of an unimodal memory kernel of the type $K(\tau) = a^2\tau e^{-a\tau}$ may produce instability of the equilibrium EE. This fact will be recalled in Section 4.4, in connection with the properties of its numerical approximation.

In the next sections, we introduce the numerical method, and we analyze its qualitative coherence with the model (2.1) to be approximated.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

3. The numerical method. To solve the integro-differential system (2.1), we first assume that the forcing terms in $(2.1)_{ii}$ and $(2.1)_{iii}$, given by

$$
\begin{aligned}
F_0(t) &= \int_{-\infty}^{0} A_\mu(t-\tau)\beta\big(M(\tau)\big)S(\tau)F(\tau)d\tau, \\
M_0(t) &= \int_{-\infty}^{0} K(t-\tau)H(\tau)g\big(F(\tau)\big)d\tau = g(F(0))K(t),
\end{aligned}
\tag{3.1}
$$

respectively, are known non-negative functions. The numerical method we propose is the outcome of the approaches described below:

- in equation $(2.1)_i$ we approximate the derivatives by forward differences, and we discretize the nonlinear terms $\beta(M)SF$ by a nonlocal technique, which is explicit in M and F and implicit in S;
- we solve the integral equations $(2.1)_{ii}$ and $(2.1)_{iii}$ by a direct quadrature method [8] based on nonlocal quadrature formulas for the integrals involved, which are left approximations in K, F, and M and right approximations in A_μ and S;
- we use *weighted step sizes* $h\gamma_i(h)$, $i = 1, 2, 3$, for the discretizations.

The method formulation is the following:

$$
\begin{aligned}
&i) && S_{n+1} = S_n + h\gamma_1(h)\big(\lambda - \mu S_{n+1} - \beta(M_n)S_{n+1}F_n\big),\\
&ii) && F_{n+1} = F_0(t_{n+1}) + h\gamma_2(h)\sum_{j=0}^{n} A_\mu(t_{n+1-j})\beta(M_j)S_{j+1}F_j,\\
&iii) && M_{n+1} = M_0(t_{n+1}) + h\gamma_3(h)\sum_{j=0}^{n} K(t_{n-j})g(F_j),
\end{aligned}
\tag{3.2}
$$

with $n = 0, 1, 2, \ldots$. The vector $u_n = (S_n, F_n, M_n)$ indicates the approximation to $u(t)$ at $t_n = nh$, where $h > 0$ is the step size of the discretization. The weight factors are positive functions of h defined by $\gamma_i(h) = 1 + \varphi_i(h)$, where $\varphi_i(h) = O(h)$ as $h \to 0$, for all $i = 1, 2, 3$. Here, $S_0 = S(0)$, $F_0 = F(0) = F_0(0)$, and $M_0 = M(0) = M_0(0)$.

The reasoning that led to the formulation (3.2) will be made clear in the following sections. Here we point out that, in analogy to what was said in the introduction, we will approach the analysis of the discrete problem (3.2) from two different, though related, points of view:

- numerical method: convergence of the numerical solution u_n to the solution $u(t)$ of (2.1), as $h \to 0$;
- discrete-time model: positivity properties, existence of equilibrium solutions and analysis of their asymptotic properties, for any fixed $h > 0$.

When the discrete equilibria and their properties are related to those of the continuous model, the last item can be regarded as a numerical stability study. Another point of interest is the study of the existence and behavior of equilibrium solutions of (3.2), in the limit as the step size $h \to 0$. This study, which represents an investigation into the approximation properties of numerical equilibria, allows us to deepen and enlarge the concept of convergence.

3.1. Consistency and convergence. As the main issue for numerical schemes is to prove convergence, we introduce the global error of discretization in t_n, that is, the vector $e(t_n, h) = u(t_n) - u_n$, and the local truncation error vector $\delta(t_n, h)$, which is the residual upon the substitution of the exact solution $u(t)$ in (3.2).

In what follows, we use the integral formulation of equation $(2.1)_i$

$$
S(t) = S_0 + \int_0^t \big(\lambda - \mu S(t-\tau) - \beta(M(t-\tau))S(t-\tau)F(t-\tau)\big)d\tau
\tag{3.3}
$$

and its discrete counterpart in (3.2)$_i$

$$S_{n+1} = S_0 + h\sum_{j=0}^{n}\big(\lambda - \mu S_{j+1} - \beta(M_j)S_{j+1}F_j\big), \quad n = 0,1,2\dots \tag{3.4}$$

First of all, we prove the following consistency result.

LEMMA 3.1. *Let $\bar{n}$ be a fixed positive integer. Assume that the given functions $A_\mu, K \in C^1([0,T])$, with $T = \bar{n}h$. Then*

$$\max_{0\le n\le\bar{n}} \|\delta(t_n,h)\| \le \bar{C}h, \quad \textit{with } \bar{C} > 0. \tag{3.5}$$

That is, the method (3.2) *is consistent with model* (2.1) *of order one.*

Proof. The assumptions on $A_\mu(t)$ and $K(t)$ imply that also $S(t)$, $F(t)$, and $M(t)$ are continuously differentiable on $[0,T]$. Let us define the local truncation error $\delta(t_n,h)$ as

$$\begin{aligned}\delta(t_n,h) = \sum_{j=0}^{n-1}\Bigg(&\begin{bmatrix}\int_{t_j}^{t_{j+1}}(\lambda-\mu S(\tau)-\beta(M(\tau))S(\tau)F(\tau))\,d\tau\\ \int_{t_j}^{t_{j+1}}A_\mu(t_n-\tau)\beta(M(\tau))S(\tau)F(\tau)d\tau\\ \int_{t_j}^{t_{j+1}}K(t_n-\tau)g(F(\tau))d\tau\end{bmatrix}\\ &-h\begin{bmatrix}\gamma_1(h)\,(\lambda-\mu S(t_{j+1})-\beta(M(t_j))S(t_{j+1})F(t_j))\\ \gamma_2(h)A_\mu(t_{n-j})\beta(M(t_j))S(t_{j+1})F(t_j)\\ \gamma_3(h)K(t_{n-j-1})g(F(t_j))\end{bmatrix}\Bigg).\end{aligned} \tag{3.6}$$

Since $\gamma_i(h) = 1+\varphi_i(h)$, for $i=1,2,3$, we have that, for $n=0,\dots,\bar{n}$,

$$\delta(t_n,h) = \tilde{\delta}(t_n,h) + h\varphi(h)\sum_{j=0}^{n-1}\begin{bmatrix}(\lambda-\mu S(t_{j+1})-\beta(M(t_j))S(t_{j+1})F(t_j))\\ A_\mu(t_{n-j})\beta(M(t_j))S(t_{j+1})F(t_j)\\ K(t_{n-j-1})g(F(t_j))\end{bmatrix}, \tag{3.7}$$

where $\varphi(h) = \mathrm{diag}\,(\varphi_1(h),\varphi_2(h),\varphi_3(h))$ and $\tilde{\delta}(t_n,h)$ is the local truncation error of the *unweighted* numerical method. A slight generalization of the proof of [29, Lemma 3.1] leads to the following bound for $\tilde{\delta}(t_n,h)$:

$$\max_{0\le n\le\bar{n}}\left\|\tilde{\delta}(t_n,h)\right\| \le \tilde{C}h, \quad \tilde{C} > 0.$$

It then follows from (3.7) that

$$\|\delta(t_n,h)\| \le \tilde{C}h + h\,\|\varphi(h)\|\sum_{j=0}^{n-1}\left\|\begin{bmatrix}(\lambda-\mu S(t_{j+1})-\beta(M(t_j))S(t_{j+1})F(t_j))\\ A_\mu(t_{n-j})\beta(M(t_j))S(t_{j+1})F(t_j)\\ K(t_{n-j-1})g(F(t_j))\end{bmatrix}\right\|,$$

for $n=0,\dots,\bar{n}$. As a consequence, due to the regularity of the given functions in $[0,T]$ and also to the fact that $\varphi_i(h) = O(h)$, for $i=1,2,3$, the bound (3.5) holds, where $\bar{C}$ is a positive constant depending on the parameters of the problem but not on h. □

We now prove the convergence result.

THEOREM 3.2. *Let $\bar{n}$ be a fixed positive integer. Assume that $A_\mu, K \in C^1([0,T])$ with $T = \bar{n}h$. Then*

$$\lim_{h\to 0}\max_{0\le n\le\bar{n}}\|e(t_n,h)\| = 0.$$

Furthermore, $\|e(t_n,h)\| = O(h)$ as $h \searrow 0$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Proof. Subtracting (3.2) from (2.1), with $(3.2)_i$ and $(2.1)_i$ formulated as (3.4) and (3.3), respectively, gives the following error recursion:

$$e(t_n,h) \le \delta(t_n,h) + h\sum_{j=0}^{n-1}\left(\begin{bmatrix}\gamma_1(h)\big(\lambda-\mu S(t_{j+1})-\beta(M(t_j))S(t_{j+1})F(t_j)\big)\\ \gamma_2(h)A_\mu(t_{n-j})\beta(M(t_j))S(t_{j+1})F(t_j)\\ \gamma_3(h)K(t_{n-j-1})g(F(t_j))\end{bmatrix} - \begin{bmatrix}\gamma_1(h)\big(\lambda-\mu S_{j+1}-\beta(M_j)S_{j+1}F_j\big)\\ \gamma_2(h)A_\mu(t_{n-j})\beta(M_j)S_{j+1}F_j\\ \gamma_3(h)K(t_{n-j-1})g(F_j)\end{bmatrix}\right), \tag{3.8}$$

for $n=0,\ldots,\bar{n}$. Here, $\delta(t_n,h)$ is the local truncation error defined in (3.6). Using the bound (3.5) for $\delta(t_n,h)$ and standard techniques for the second term in the right-hand side of (3.8), we have

$$\|e(t_n,h)\| \le \bar{C}h + hL\,(1+\|\varphi(h)\|)\sum_{j=0}^{n-1}\|e(t_j,h)\|\,, \qquad n=0,\ldots\bar{n},$$

where $\varphi(h)=\mathrm{diag}(\varphi_1(h),\varphi_2(h),\varphi_3(h))$ and $\bar{C}$, L are positive constants depending on the parameters of the problem but not on h. Thanks to the Gronwall discrete inequality (see, for example, [25, p. 101]), we finally have

$$\|e(t_n,h)\| \le Ch\,e^{TL(1+\varphi(h))},$$

for $C>0$ and $n=0,\ldots,\bar{n}$, which yields the result. ∎

3.2. Basic properties. With the purpose of studying the qualitative and asymptotic behavior of the solution of (3.2) for any fixed discretization step size $h>0$ and its coherence with the properties of model (2.1), we assume from now on that all assumptions on the parameters and functions of the problem, as described in Section 2, are valid.

Two quantities that will play an important role in our analysis are

$$\bar{A}_\mu(1,h) = h\,\gamma_2(h)\sum_{n=1}^{\infty}A_\mu(t_n), \tag{3.9}$$
$$\bar{K}(1,h) = h\,\gamma_3(h)\sum_{n=0}^{\infty}K(t_n),$$

for which we prove the following result:

LEMMA 3.3. *For any $h>0$, $\bar{A}_\mu(1,h)<+\infty$ and $\bar{K}(1,h)<+\infty$. Furthermore, we have*

$$\lim_{h\to 0}\bar{A}_\mu(1,h)=\hat{A}_\mu(0), \qquad \lim_{h\to 0}\bar{K}(1,h)=\hat{K}(0)=1, \tag{3.10}$$

where $\hat{A}_\mu(0)$ and $\hat{K}(0)$ are defined in Table 2.2.

Proof. We prove the result for K. The proof similarly applies to A_μ. Consider

$$\int_{t_j}^{t_{j+1}}K(\tau)d\tau = hK(t_j) + \int_{t_j}^{t_{j+1}}K(\tau)d\tau - \int_{t_j}^{t_{j+1}}K(t_j)d\tau.$$

Thus, thanks to the assumption (2.2) and the fact that $\int_0^\infty K(\tau)d\tau=1$,

$$h\sum_{j=0}^{\infty}K(t_j) \le \int_0^\infty K(\tau)d\tau + h\int_0^\infty|K'(\tau)|d\tau < +\infty. \tag{3.11}$$

Therefore, also $\bar{K}(1,h) < +\infty$. Moreover, from (3.11), we deduce that

$$\left| \int_0^\infty K(\tau)d\tau - h\gamma_3(h) \sum_{j=0}^{\infty} K(t_j) \right| \leq h \int_0^\infty |K'(\tau)|d\tau + |\varphi_3(h)| h \sum_{j=0}^{\infty} K(t_j),$$

and, for $h \to 0$, since $h\sum_{j=0}^{\infty} K(t_j)$ is bounded, the second condition of (3.10) is satisfied. □

Let $h > 0$ and

$$D(h) = \{(x,y,z) : 0 < x \leq S_{\max}(h),\ 0 \leq y \leq F_{\max}(h),\ 0 \leq z \leq M_{\max}(h)\}, \tag{3.12}$$

where

$$\begin{aligned} &i) && S_{\max}(h) = S_{\max} = \frac{\lambda}{\mu}, \\ &ii) && F_{\max}(h) = \lambda\Big(\hat{A}_\mu(0) + \bar{A}_\mu(1,h)\Big) + \frac{\lambda}{\mu}\int_0^\infty |A'_\mu(\tau)|d\tau, \\ &iii) && M_{\max}(h) = g(F_{\max}) + g(F_{\max}(h))\bar{K}(1,h). \end{aligned} \tag{3.13}$$

The following theorem ensures that the solutions of the model (3.2) remain in $D(h)$ of any $n \geq 0$:

THEOREM 3.4. *Assume that* $\lim_{\tau\to\infty} A(\tau) = 0$ *and that the weight functions* $\varphi_1(h)$ *and* $\varphi_2(h)$ *satisfy*

$$\varphi_2(h) = c\varphi_1(h) \quad \textit{with } |c| \leq 1. \tag{HP1}$$

Then the region $D(h)$ *defined in* (3.12) *is positively invariant for the discrete model* (3.2).

Proof. We proceed by induction on n with base case $n = 0$, which is trivially true. Let us consider $(3.2)_i$ written in the following form

$$S_{n+1} = \frac{S_n + h\gamma_1(h)\lambda}{1 + h\gamma_1(h)\left(\mu + \beta(M_n)F_n\right)}. \tag{3.14}$$

If we assume that $S_j > 0$ and $F_j, M_j \geq 0$, for $j = 1, \ldots, n$, then the expression (3.14) implies that $S_{n+1} > 0$. From $(3.2)_{ii}$ and $(3.2)_{iii}$ it easily follows that $F_{n+1}, M_{n+1} \geq 0$.

Let us now prove the boundedness and assume that $S_j \leq \lambda/\mu$, for $j = 0, \ldots, n$. Taking into account that $h\gamma_1(h)\beta(M_n)F_n \geq 0$, from equation (3.14) one obtains

$$S_{n+1} \leq \frac{\frac{\lambda}{\mu} + h\gamma_1(h)\lambda}{1 + h\gamma_1(h)\mu} = \frac{\lambda}{\mu}.$$

By using $(3.2)_i$ in $(3.2)_{ii}$, since S_n is positive, it is easily seen that

$$F_{n+1} \leq F_0(t_{n+1}) + \frac{\gamma_2(h)}{\gamma_1(h)} \sum_{j=0}^{n} A_\mu(t_{n+1-j})(S_j - S_{j+1}) + \lambda h\gamma_2(h) \sum_{j=1}^{n+1} A_\mu(t_j). \tag{3.15}$$

Summation by parts in the second term of the right-hand side of (3.15) leads to

$$\begin{aligned} F_{n+1} \leq F_0(t_{n+1}) &+ \frac{\gamma_2(h)}{\gamma_1(h)}\left(A_\mu(t_{n+1})S_0 + \frac{\lambda}{\mu}\int_0^{t_n} |A'_\mu(\tau)|d\tau\right) \\ &+ \lambda h\gamma_2(h) \sum_{j=1}^{n+1} A_\mu(t_{n+1-j}). \end{aligned}$$

For the forcing term $F_0(t)$ defined in (3.1), one has

$$F_0(t_{n+1}) \leq \lambda \int_{t_{n+1}}^{\infty} A_\mu(\tau)d\tau - A_\mu(t_{n+1})S_0 + \frac{\lambda}{\mu}\int_{t_{n+1}}^{\infty} |A'_\mu(\tau)|d\tau.$$

Using the last two inequalities, we have that

$$\begin{aligned} F_{n+1} \leq \lambda \left(\int_{t_{n+1}}^{\infty} A_\mu(\tau)d\tau + h\gamma_2(h)\sum_{j=1}^{n+1} A_\mu(t_j) \right) + \left(\frac{\gamma_2(h)}{\gamma_1(h)} - 1 \right) A_\mu(t_{n+1})S_0 \\ + \frac{\lambda}{\mu}\left(\int_{t_{n+1}}^{\infty} |A'_\mu(\tau)|d\tau + \frac{\gamma_2(h)}{\gamma_1(h)}\int_0^{t_n} |A'_\mu(\tau)|d\tau \right). \end{aligned} \tag{3.16}$$

Since (HP1) holds, then the right-hand side in (3.16) is, for any $h > 0$, less than or equal to $F_{\max}(h)$, defined in $(3.13)_{ii}$, and the desired conclusion follows. Finally, by considering $(3.2)_{iii}$ and the fact that, from (3.1), it is $M_0(t_{n+1}) \leq g(F_{\max}) \int_{t_{n+1}}^{+\infty} K(\tau)d\tau$, the bound for M_n is a consequence of the following inequality:

$$M_{n+1} \leq g(F_{\max}) \int_{t_{n+1}}^{+\infty} K(\tau)d\tau + g(F_{\max}(h))h\gamma_3(h)\sum_{j=1}^{n+1} K(t_j). \qquad \square$$

4. Asymptotic properties of equilibria.

4.1. Equilibria.

We define

$$R_0(h) = \frac{\lambda}{\mu}\bar{A}_\mu(1,h)$$

as the discrete basic reproduction number. Since $\bar{A}_\mu(1,h)$, defined in (3.9), represents a weighted quadrature formula for $\hat{A}_\mu(0)$, $R_0(h)$ is the numerical approximation to the basic reproduction number R_0 introduced for the model problem (2.1) in Section 2. From the system (3.2), we derive the following equations for equilibria

$$\begin{aligned} &\lambda - \mu S - \beta(M)SF = 0, \\ &F = \bar{A}_\mu(1,h)\beta(M)SF, \\ &M = \bar{K}(1,h)g(F). \end{aligned}$$

If $F = 0$, for any $h > 0$, then there is exactly one equilibrium, the Disease-Free Equilibrium

$$\text{DFE}(h) = \left(\frac{\lambda}{\mu}, 0, 0 \right). \tag{4.1}$$

If $F > 0$, an Endemic Equilibrium

$$\text{EE}(h) = (S_e(h), F_e(h), M_e(h)) \tag{4.2}$$

exists, for any $h > 0$, if and only if there exists in $[0, F_{\max}(h)]$ a zero $F_e(h)$ of the nonlinear function

$$\Phi(h,x) = x - \mu R_0(h) + \frac{\mu}{\beta(\bar{K}(1,h)g(x))}, \tag{4.3}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

and

$$\text{(4.4)} \qquad \begin{aligned} &i) \qquad S_e(h) = \frac{\lambda}{\mu R_0(h)\beta(M_e(h))}, \\ &ii) \qquad M_e(h) = \bar{K}(1,h)g(F_e(h)). \end{aligned}$$

THEOREM 4.1. *Let $h > 0$. If $R_0(h) < 1$, then the model (3.2) has no endemic equilibrium; if $R_0(h) > 1$, then the model (3.2) has a unique endemic equilibrium.*

Proof. Consider the function $\Phi(h, x)$ defined in (4.3), and let $x \in [0, F_{\max}(h)]$. We observe that

$$\begin{aligned} \Phi(h, 0) &= -\mu(R_0(h) - 1), \\ \Phi(h, F_{\max}(h)) &= \lambda \hat{A}_\mu(0) + \frac{\lambda}{\mu}\int_0^\infty |A'_\mu(\tau)| d\tau + \frac{\mu}{\beta(\bar{K}(1,h)g(F_{\max}(h)))} > 0, \\ \Phi'(h, x) &= 1 - \frac{\mu\beta'(\bar{K}(1,h)g(x))\bar{K}(1,h)g'(x)}{[\beta(\bar{K}(1,h)g(x))]^2} > 0, \end{aligned}$$

where the last inequality holds since $\beta'(\cdot) < 0$. Thus, when $R_0(h) > 1$, it is $\Phi(h, 0) < 0$, and, being $\Phi(h, x)$ in (4.3) an increasing function, it has a unique zero in $[0, F_{\max}(h)]$. For the same reason, it is clear that if $R_0(h) < 1$, then $\Phi(h, x)$ has no zeros in $[0, F_{\max}(h)]$. ☐

All the results reported in this and in the previous section apply whatever the choice of the weight factors γ_1, γ_2, and γ_3. However, in order to match the numerical endemic equilibrium $\mathrm{EE}(h)$ with the equilibrium EE of system (2.1), we choose

$$\text{(HP2)} \qquad \gamma_3(h) = \frac{1}{h\sum_{n=0}^\infty K(t_n)},$$

that acts as a normalization factor, yielding $\bar{K}(1, h) = \hat{K}(0) = 1$. From the proof of Lemma 3.3 we conclude that also $\lim_{h\to 0} h \sum_{n=0}^\infty K(t_n) = 1$, therefore the weight defined by (HP2) satisfies the property $\gamma_3(h) = 1 + O(h)$, as required in the definition of the weight factors.

Since the form of γ_1 and γ_2 does not play a significant role, we set

$$\text{(HP3)} \qquad \gamma_1(h) = \gamma_2(h) = 1.$$

4.2. Linearized stability: the characteristic equation. The aim of this section is to derive the characteristic equation in order to study the local asymptotic stability of the equilibria of the discrete model (3.2), for any $h > 0$. In this regard, we assume that (HP3) holds for the weight factors $\gamma_1(h)$ and $\gamma_2(h)$. Thus, we refer to the following formulation of the numerical method (3.2)

$$\text{(4.5)} \qquad \begin{aligned} S_{n+1} &= S_0 + h\sum_{j=0}^n \big(\lambda - \mu S_{j+1} - \beta(M_j)S_{j+1}F_j\big), \\ F_{n+1} &= F_0(t_{n+1}) + h\sum_{j=0}^n A_\mu(t_{n+1-j})\beta(M_j)S_{j+1}F_j, \\ M_{n+1} &= M_0(t_{n+1}) + h\gamma_3(h)\sum_{j=0}^n K(t_{n-j})g(F_j), \end{aligned}$$

for $n = 0, 1, \dots,$ where the functions $F_0(t)$ and $M_0(t)$ are defined in (3.1). Here and in the following we assume that the weight factor $\gamma_3(h)$ is the one given in (HP2). Consider the perturbations (W_n, V_n, Z_n) to a generic equilibrium $\tilde{E}(h) = (\tilde{S}(h), \tilde{F}(h), \tilde{M}(h))$. Since,

$$S_n = \tilde{S}(h) + W_n, \quad F_n = \tilde{F}(h) + V_n, \quad M_n = \tilde{M}(h) + Z_n,$$

we have that, for $n = 0, 1, \dots,$ (W_n, V_n, Z_n) satisfy

$$\begin{bmatrix} W_{n+1} \\ V_n \\ Z_n \end{bmatrix} = \begin{bmatrix} W(0) \\ V_0(t_n) \\ Z_0(t_n) \end{bmatrix} + h \sum_{j=0}^{n} \mathbf{Q}_{n-j} \left(\begin{bmatrix} W_{j+1} \\ V_j \\ Z_j \end{bmatrix} + \mathbf{G}_j \left(\begin{bmatrix} W_{j+1} \\ V_j \\ Z_j \end{bmatrix} \right) \right), \tag{4.6}$$

where $\mathbf{G}_j(\cdot)$, $j \geq 0$, are given mappings that include all the nonlinear terms of (4.5). In (4.6) the 3×3 matrices $\mathbf{Q}_n$ are defined by

$$\mathbf{Q}_n = \begin{cases} \begin{bmatrix} -\mu - \beta(\tilde{M}(h))\tilde{F}(h) & -\beta(\tilde{M}(h))\tilde{S}(h) & -\beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h) \\ \beta(\tilde{M}(h))\tilde{F}(h)A_\mu(t_n) & \beta(\tilde{M}(h))\tilde{S}(h)A_\mu(t_n) & \beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h)A_\mu(t_n) \\ 0 & K(t_n)g'(\tilde{F}(h)) & 0 \end{bmatrix}, & n > 0, \\ \begin{bmatrix} -\mu - \beta(\tilde{M}(h))\tilde{F}(h) & -\beta(\tilde{M}(h))\tilde{S}(h) & -\beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & n = 0, \end{cases} \tag{4.7}$$

and the forcing terms are

$$V_0(t_{n+1}) = F_0(t_{n+1}) - \beta\left(\tilde{M}(h)\right)\tilde{S}(h)\tilde{F}(h)h \sum_{j=n+1}^{\infty} A_\mu(t_j)$$

and

$$Z_0(t_{n+1}) = \left(g(F_0) - hg\left(\tilde{F}(h)\right)\right)K(t_{n+1}).$$

The analysis of the asymptotic behavior of the solutions to the nonlinear perturbed system (4.6) goes through the linearization theory developed in [34] and described in Appendix A. Let $\mathbf{x}_n = [W_{n+1}, V_n, Z_n]^T$ be the solution of system (4.6) and $\mathbf{y}_n$ be the solution of the linear system

$$\mathbf{y}_n = \mathbf{f}_n + h \sum_{j=0}^{n} \mathbf{Q}_{n-j}\mathbf{y}_j, \quad n = 0, 1, \dots, \tag{4.8}$$

related to (4.6), with $\mathbf{Q}_n$ defined in (4.7) and $\mathbf{f}_n = [W(0), V_0(t_n), Z_0(t_n)]^T$. We observe that the condition $\det(\mathbf{I} - h\mathbf{Q}_n) \neq 0$ is always satisfied, provided that h is sufficiently small. Therefore, by Theorem A.1, the perturbation $\mathbf{x}_n$ can be expressed in terms of $\mathbf{y}_n$ as

$$\mathbf{x}_n = \mathbf{y}_n - \sum_{j=0}^{n} \mathbf{R}_{n-j}\mathbf{G}_j(\mathbf{x}_j), \quad n = 0, 1, \dots,$$

where $\mathbf{R}_n$ are the resolvent matrices of $h\mathbf{Q}_n$. Since Theorem A.2 states that $\mathbf{x}_n$ asymptotically vanishes whenever $\lim_{n\to\infty} \mathbf{y}_n = 0$, we study the asymptotic behavior of the solutions to the linear

ETNA
Kent State University and
Johann Radon Institute (RICAM)

system (4.8), and thus we refer to Theorem A.3 for the stability analysis. The system (4.8) fits into the form (A.5), (A.6), with $l = 1$, $k = 2$ and

$$\begin{aligned}
\mathrm{f}_n^0 &= W(0),\\
\mathbf{f}_n^1 &= \begin{bmatrix} V_0(t_n) \\ Z_0(t_n) \end{bmatrix},\\
\mathbf{B}_n &= h\mathbf{Q}_n,\\
\mathbf{B}_\infty^1 &= h\begin{bmatrix} -\mu - \beta(\tilde{M}(h))\tilde{F}(h) & -\beta(\tilde{M}(h))\tilde{S}(h) & -\beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h) \end{bmatrix}.
\end{aligned}$$

For $|z| \leq 1$, the characteristic equation takes the form

$$(1 - z)\cdot \det T(z,h) = 0, \tag{4.9}$$

where T denotes the matrix

$$T(z,h) = \begin{bmatrix} 1 + (\mu + \beta(\tilde{M}(h))\tilde{F}(h)\frac{h}{1-z} & \beta(\tilde{M}(h))\tilde{S}(h)\frac{h}{1-z} & \beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h)\frac{h}{1-z} \\ -\beta(\tilde{M}(h))\tilde{F}(h)\bar{A}_\mu(z,h) & 1 - \beta(\tilde{M}(h)\tilde{S}(h)\bar{A}_\mu(z,h) & -\beta'(\tilde{M}(h))\tilde{S}(h)\tilde{F}(h)\bar{A}_\mu(z,h) \\ 0 & -g'(\tilde{F}(h))\bar{K}(z,h) & 0 \end{bmatrix},$$

and

$$\bar{A}_\mu(z,h) = h\sum_{n=1}^{\infty} A_\mu(t_n)z^n, \quad \bar{K}(z,h) = h\gamma_3(h)\sum_{n=0}^{\infty} K(t_n)z^n,$$

with $\gamma_3(h)$ given by (HP2), are the discrete and the *weighted* discrete Laplace transforms of A_μ and K, respectively. The following theorem gives a sufficient condition for the local asymptotic stability of the equilibria.

THEOREM 4.2. *Any equilibrium $\tilde{E}(h) = (\tilde{S}(h), \tilde{F}(h), \tilde{M}(h))$ of the model (4.5) is locally asymptotically stable, for $h > 0$, if the characteristic equation*

$$\beta(\tilde{M}(h))\tilde{S}(h)\bar{A}_\mu(z,h)\left(1 + \frac{g'(\tilde{F}(h))\beta'(\tilde{M}(h))\tilde{F}(h)\bar{K}(z,h)}{\beta(\tilde{M}(h))}\right) = 1 + h\frac{\beta(\tilde{M}(h))\tilde{F}(h)}{1 - z + \mu h}, \tag{4.10}$$

has no solution for $|z| \leq 1$.

Proof. The proof is a straightforward application of Theorem A.2, after observing that equation (4.10) is equivalent to (4.9) and that f_n^0 and $\mathbf{f}_n^1$ satisfy the assumptions of Theorem A.2. ∎

4.3. DFE local stability analysis. At the DFE (4.1), the characteristic equation (4.10) becomes

$$\frac{\lambda}{\mu}\bar{A}_\mu(z,h) = 1.$$

Note that $\frac{\lambda}{\mu}\bar{A}_\mu(1,h) = R_0(h)$. Suppose that $R_0(h) < 1$ and there is a root z with $|z| < 1$. Then

$$\left|\frac{\lambda}{\mu}\bar{A}_\mu(z,h)\right| \leq R_0(h) < 1,$$

which is a contradiction. Thus, the stability follows.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

4.4. EE local stability analysis. In the following, we present and analyze characteristic equations for the endemic equilibria $\mathrm{EE}(h)$ (4.2) of some submodels obtained by specific choices for the IFD A_μ and the memory kernel K, which have been listed in Section 2 as the more common one in the literature and significant in the applications. Here, the step size $h > 0$ is fixed. In our analysis, we assume that $|z| < 1$; however, all the results can be easily extended to $|z| \le 1$, by passing to the limit for $\tilde{z} \to z$, with $|\tilde{z}| < 1$.

As a main consideration, we observe that the assumption $|z| < 1$ implies that

$$\left|\beta(M_e(h))S_e(h)\bar{A}_\mu(z,h)\right| \le 1, \quad Re\left(1 + h\frac{\beta(M_e(h))F_e(h)}{1 - z + \mu h}\right) > 1, \tag{4.11}$$

where the first bound is a consequence of the equilibrium condition $(4.4)_i$.

THEOREM 4.3. *If $A_\mu(\tau)$ is positive definite and $K(\tau) = \delta(\tau)$, then $\mathrm{EE}(h)$ is LAS for any choice of the functions β and g.*

Proof. When $K(\tau) = \delta(\tau)$ the characteristic equation (4.10) reads

$$\beta(M_e(h))S_e(h)\bar{A}_\mu(z,h)\left(1 + \frac{g'(F_e(h))\beta'(M_e(h))F_e(h)}{\beta(M_e(h))}\right) = 1 + h\frac{\beta(M_e(h))F_e(h)}{1 - z + \mu h}. \tag{4.12}$$

If $A_\mu(\tau)$ is a positive definite function, then $\{A_\mu(t_n)\}_n$ is a positive definite sequence and the discrete analogue of the Bochner characterization holds; see [26, Lemma 8.2]. Then

$$Re\left(\bar{A}_\mu(z,h)\right) \ge 0, \quad \text{in } |z| < 1.$$

Since $g' < 0$ and $\beta' > 0$, considering (4.11), the real part of the left-hand side of (4.12) is smaller than or equal to 1, while the real part of the right-hand side is strictly larger than 1. It is not possible for (4.12) to have root inside the unit circle, and then the local stability is guaranteed by Theorem 4.2. □

Examples of infectivity functions that satisfy Theorem 4.3 are convex, non-negative, and non-increasing functions on $[0, \infty)$ [26]. Among them, a notable case is $A_\mu(\tau) = e^{-(\mu+\nu)\tau}$, which leads to the *SIR-M* compartment model introduced in [12].

In a more general context for A_μ, given (4.11), we derive that a sufficient condition for equation (4.10) not to have roots with $|z| < 1$ is

$$\left|1 + \frac{g'(F_e(h))\beta'(M_e(h))F_e(h)\bar{K}(z,h)}{\beta(M_e(h))}\right| \le 1. \tag{4.13}$$

In the following, we analyze how condition (4.13) affects the model parameters when the memory kernel K is a Dirac delta or a weak Erlang distribution.

THEOREM 4.4. *Assume that $K(\tau) = \delta(\tau)$ or $K(\tau) = ae^{-a\tau}$, with $a \in \mathbb{R}^+$, and that the functions β and g satisfy the inequality*

$$g'(F_e(h))\beta'(M_e(h))F_e(h) \ge -2\beta(M_e(h)). \tag{4.14}$$

Then $\mathrm{EE}(h)$ is LAS for any choice of $A_\mu(\tau)$.

Proof. When $K(\tau) = \delta(\tau)$, the sufficient condition (4.13) reads

$$\left|1 + \frac{g'(F_e(h))\beta'(M_e(h))F_e(h)}{\beta(M_e(h))}\right| \le 1,$$

which is equivalent to (4.14) since $\beta' < 0$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

If $K(\tau) = ae^{-a\tau}$, then

$$\bar{K}(z,h) = \frac{e^{ah}-1}{e^{ah}-z}.$$

Set $c = \frac{g'(F_e(h))\beta'(M_e(h))F_e(h)}{\beta(M_e(h))}$. Let $z = x + iy$, and consider

$$\left|1 + c\frac{e^{ah}-1}{e^{ah}-z}\right|^2 = 1 + \frac{c(e^{ah}-1)}{(e^{ah}-x)^2+y^2}\left(c(e^{ah}-1) + 2(e^{ah}-x)\right).$$

Condition (4.13) is satisfied if

$$\left|1 + c\frac{e^{ah}-1}{e^{ah}-z}\right|^2 \leq 1,$$

that is,

$$\frac{c(e^{ah}-1)}{(e^{ah}-x)^2+y^2}\left(c(e^{ah}-1) + 2(e^{ah}-x)\right) \leq 0. \tag{4.15}$$

The negativity of β' implies that $c < 0$, and then (4.15) is satisfied for $c \geq -2$. □

When the memory kernel has the unimodal form $K(\tau) = a^2\tau e^{-a\tau}$, condition (4.13) reduces to

$$\left|1 + \frac{z(e^{ah}-1)^2 g'(\tilde{F}(h))\beta'(\tilde{M}(h))\tilde{F}(h)}{(e^{ah}-z)^2\beta(\tilde{M}(h))}\right| \leq 1. \tag{4.16}$$

Let $z = x + iy$ and $c = \frac{g'(\tilde{F}(h))\beta'(\tilde{M}(h))\tilde{F}(h)}{\beta(\tilde{M}(h))}$. Then condition (4.16) becomes

$$\left|1 + c(e^{ah}-1)^2(X + iY)\right| \leq 1, \tag{4.17}$$

where

$$X = \frac{x[(e^{ah}-x)^2-y^2] - 2y^2(e^{ah}-x)}{[(e^{ah}-x)^2+y^2]^2 + 4y^2(e^{ah}-x)^2}, \quad Y = \frac{y[(e^{ah}-x)^2-y^2] - 2xy(e^{ah}-x)}{[(e^{ah}-x)^2+y^2]^2 + 4y^2(e^{ah}-x)^2}.$$

From (4.17) we derive

$$-1 \leq 2c(e^{ah}-1)^2X + c^2(e^{ah}-1)^4(X^2+Y^2) \leq 0,$$

which is not true for any x and y in the interval $(-1, 1)$.

REMARK 4.5. The fact that the sufficient condition (4.13) is not satisfied suggests that the choice of unimodal memory kernels may lead to instability of the equilibrium EE(h), in accordance to what is observed for the continuous model in Remark 2.4.

5. Qualitative relation and approximation properties. In the previous sections we have addressed the problem of how to construct a numerical method that, for any step size $h > 0$, replicates all the properties introduced in Section 2.

Here, we consider the effect of the approximation (3.2) on equilibria under the assumptions (2.2) on A_μ and K. First of all, we observe that, because of Lemma 3.3, $R_0(h) \to R_0$ as $h \to 0$. For the bounds (2.4) and (3.13) on the continuous and numerical solution, respectively, the following result holds:

THEOREM 5.1. *For any $h > 0$, $S_{\max} = S_{\max}(h)$, $F_{\max} < F_{\max}(h) < \bar{F}$, and $M_{\max} < M_{\max}(h) < \bar{M}$, with $\bar{F}$ and $\bar{M}$ positive constants. Furthermore,*

$$\lim_{h\to 0} F_{\max}(h) = F_{\max} + \lambda \hat{A}_\mu(0),$$
$$\lim_{h\to 0} M_{\max}(h) = M_{\max} + g(F_{\max} + \lambda \hat{A}_\mu(0)).$$

Proof. The proof is straightforward by considering in (3.13) that $\bar{A}_\mu(1,h)$, $\bar{K}(1,h) < +\infty$ and that $\bar{A}_\mu(1,h) \to \hat{A}_\mu(0)$ and $\bar{K}(1,h) \to 1$, as $h \to 0$; see Lemma 3.3. ∎

THEOREM 5.2. *Consider the endemic equilibria* $\mathrm{EE} = (S_e, F_e, M_e)$ *and* $\mathrm{EE}(h) = (S_e(h), F_e(h), M_e(h))$ *of the continuous and numerical models, respectively. Then*

$$\lim_{h\to 0} \mathrm{EE}(h) = \mathrm{EE}.$$

Proof. The function $\Phi(h,x)$ defined in (4.3), is a continuous function in $[0,+\infty) \times [0,\bar{F}]$, and, from Theorem 4.1, it has a unique zero $F_e(h)$. Furthermore, since $R_0(h) \to R_0$ and $\bar{K}(1,h) \to \hat{K}(0) = 1$, as $h \to 0$, we have $\Phi(0,x) = \Phi(x)$, where $\Phi(x)$ is the nonlinear function defined in (2.5), for which F_e is the unique zero; see Section 2. Thus, $\Phi(h,x)$ satisfies the assumptions of [30, Theorem 3.5], which show the continuous dependence of solutions to nonlinear equations with respect to parameters. Therefore, we have that $F_e(h) \to F_e$ and, from (4.4), also $M_e(h) \to M_e$ and $S_e(h) \to S_e$, as $h \to 0$. ∎

Thanks to Theorems 5.1 and 5.2, we can assert that the characteristic equation (4.10) is consistent as $h \to 0$ with the characteristic equation (2.6) of the model (2.1). This is clear upon the change of variable $z = e^{-wh}$, with $Re(w) > 0$, in (4.10) and passing to the limit as $h \to 0$.

Concerning the local stability analysis, it is clear that the sufficient conditions established in Theorems 4.3 and 4.4 for the numerical equilibria are consistent, as $h \to 0$, with the continuous analogue described by Theorem 2.3.

All the considerations and results reported in this section demonstrate that not only the numerical model behaves in agreement with the continuous model in terms of preservation of the main properties such as positivity invariance, existence of equilibrium solutions and their stability, independently of the step size, but also all the numerical parameters that describe the numerical dynamics converge, as $h \to 0$, to their continuous counterparts.

6. Numerical examples.

In this section, we present some numerical examples, based on the non-standard discretization (3.2) of the model (2.1), with the aim to illustrate experimentally the theoretical results of the previous sections. The discrete model described by equations (3.2) needs a specification of the initial condition $S_0 = S(0)$ and the forcing functions $F_0(t)$, $M_0(t)$ defined in (3.1). By the same considerations made in [9], we set

$$F_0(t) = A_\mu(t)(\lambda - \mu S_0).$$

Moreover $M_0(t)$ is defined in (3.1). In our experiments the weight factors $\gamma_1(h)$, $\gamma_2(h)$, and $\gamma_3(h)$ are chosen according to (HP2) and (HP3). For all the experiments we use

$$g(F) = F \quad \text{and} \quad \beta(M) = \frac{1}{1+\alpha M},$$

where $\alpha > 0$ is a decline factor due to voluntary social distancing.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

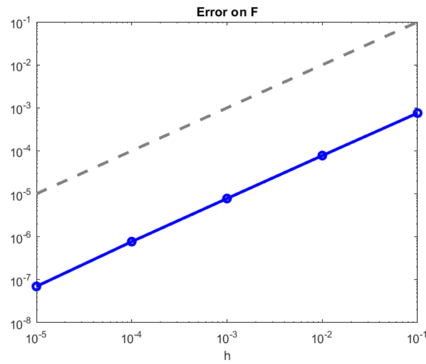

FIG. 6.1. *Problem* (2.1)-(6.1)*: norm of the errors (solid line) as functions of the step size, compared to the slope of order one (dashed line).*

As first example we integrate problem (3.2) for $t \in [0, 1]$, with

$$\begin{gathered} A_\mu(\tau) = \beta_0 e^{-(\mu+\nu)\tau}, \quad K(\tau) = a e^{-a\tau}, \quad \beta_0 = R_0 \frac{(\mu+\nu)}{N}, \\ R_0 = 3.3, \quad \nu = 1/7 \text{ days}^{-1}, \quad \mu = 1/75 \text{ years}^{-1}, \quad a = 1/30 \text{ days}^{-1}, \\ N = 5 \cdot 10^4, \quad S_0 = 0.8N, \quad \alpha = 10. \end{gathered} \tag{6.1}$$

To study the convergence of our numerical scheme, we use the discrete solution computed with step size $h = 10^{-6}$ as reference solution. The reduction of numerical errors as function of the step size, shown in Figure 6.1, confirms the linear convergence stated in Theorem 3.2.

To compare the performances of the non-standard method (3.2), proposed in this paper, with other methods, we solve the following problem:

$$\begin{gathered} A_\mu(\tau) = \beta_0 e^{-(\mu+\nu)\tau}, \quad \beta_0 = R_0 \frac{(\mu+\nu)}{N}, \quad R_0 = 20, \\ \nu = 1/4 \text{ days}^{-1}, \quad \mu = 1/75 \text{ years}^{-1}, \quad N = 5 \cdot 10^7, \quad S_0 = 0.97N. \end{gathered} \tag{6.2}$$

For the sake of simplicity, in this simulation, we do not take into account any behavioral changes within the population. We adapt standard discretization rules (see, for example, [25]) to problem (2.1). In particular, we show a first experiment where we couple the Euler method for $(2.1)_i$ and trapezoidal direct quadrature for $(2.1)_{ii,iii}$, thus resulting in an explicit method. In a second experiment we use the implicit trapezoidal discretization for the whole problem (2.1). Figure 6.2 shows the outcomes of these computations, with step size $h = 0.5$ (non-standard on the left, Euler-trapezoidal in the middle, and trapezoidal on the right). It is clear that the two standard methods do not respect the positivity property.

Finally, since our numerical scheme is robust in identifying what the theoretical results predict, we believe that our method can provide various insights where theoretical results are lacking. This allows us to analyze situations of instability and limit cycles, as illustrated in this last application motivated by SARS, which is a case-study discussed in [9]. Here, following Roberts' approach proposed in [33], we consider a trapezoidal infectivity distribution given by

$$A(\tau) = \begin{cases} p_0 \frac{\tau - \tau_a}{(\tau_b - \tau_a)}, & \tau_a < \tau < \tau_b, \\ p_0, & \tau_b < \tau < \tau_c, \\ p_0 \frac{\tau_d - \tau}{(\tau_d - \tau_c)}, & \tau_c < \tau < \tau_d, \\ 0, & \text{otherwise}. \end{cases} \tag{6.3}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

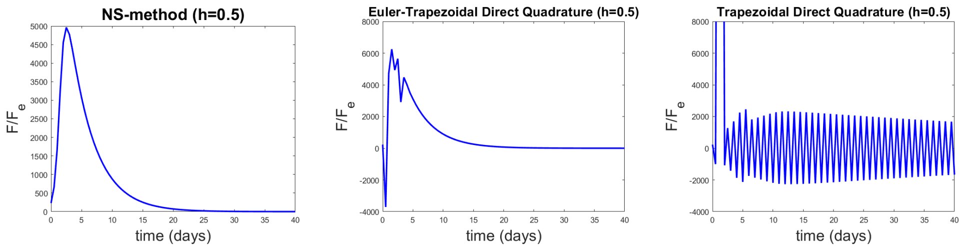

FIG. 6.2. *Problem* (2.1)-(6.2)*: comparison of numerical solutions with* $h = 0.5$*.*

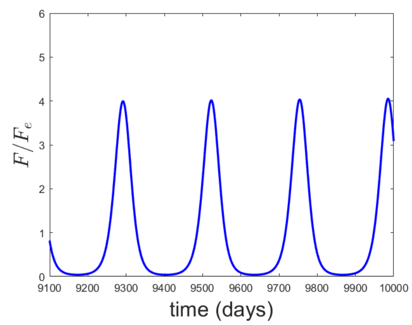

FIG. 6.3. *Time profile of the ratio* F/F_e *as predicted by model* (2.1)*: case with strong Erlang memory kernel, i.e.,* $K(\tau) = a^2\tau e^{-a\tau}$*, and IF given by* (6.3)*.*

In (6.3), it is assumed that infected individuals can transmit the infection within the time intervals τ_b and τ_c. Transmission does not occur before time τ_a or after time τ_d, and the transmission probabilities between the time intervals τ_a and τ_b, as well as between τ_c and τ_d, are estimated using linear interpolation. Here, the parameter p_0 weights the contacts between the infected and susceptible individuals. We additionally suppose that the memory of the population is represented by a strong Erlang distribution, $K(\tau) = \mathrm{Erl}_{2,a}$, implying that current information is unavailable, and the highest significance is assigned to information received by the public after a characteristic time $T = 2/a$. This aligns with the scarcity of available information about SARS when it first emerged. For this application, we take $R_0 = 3.3$, $(\tau_a, \tau_b, \tau_c, \tau_d) = (4, 7, 11, 14)$, $\alpha = 5 \cdot 10^4$, $a = 1/30$, $S_0 = 0.7N$, $\mu = 1/75$ years^{-1}, and $N = 5 \cdot 10^7$. In Figure 6.3 we observe that self-sustained oscillations are produced, and the instability, as discussed in Remarks 2.4 and 4.5, is illustrated experimentally.

7. Conclusions. In [9] we have introduced an integro-differential model describing an epidemic of an infectious disease taking into account human behavioral feedback. In this paper, we have proposed a numerical method based on a weighted non-standard technique, built ad hoc on the model proposed in [9], to preserve the positivity of the solution and the asymptotic behavior of the continuous model, without any restriction on the discretization step size. For this purpose, for any fixed value of the step size $h > 0$, we have regarded the numerical method as a discrete-time model and obtained results on the stability of numerical equilibria which replicate the ones described in Section 2 for the continuous model. Since it is natural to ask whether the equilibria of the numerical method are close to the equilibria of the integro-differential model, we have also studied their approximation properties proving the convergence as $h \to 0$. These properties make the numerical method we propose efficient and reliable in long-time simulations, which are essential to gain a deep understanding in equilibria of the system and their stability properties related to control parameters, especially when theoretical results are not available. Numerical experiments confirm this view. In fact, in Figure 6.2, where our non-standard method is compared with discretizations from the classical literature, it can be seen that, at the same discretization step size, the latter exhibits *non-physical* oscillations. Moreover, in the simulation shown in Figure 6.3, limit cycles, assumed but not demonstrated (see Remark 2.4) for an unimodal choice of the memory kernel, are evident.

Acknowledgments. We thank the anonymous reviewers whose comments helped to improve and clarify this manuscript. This work has been performed under the auspices of the Italian National Group for Scientific Computing (GNCS) and the Italian National Group for Mathematical Physics (GNFM) of the National Institute for Advanced Mathematics (INdAM). B.B. also acknowledges PRIN 2020 project (No. 2020JLWP23) “Integrated Mathematical Approaches to Socio–Epidemiological Dynamics”. The work of E.M. and C.P. is partially supported by the INdAM under the GNCS Project CUP_E53C23001670001.

Appendix A. Preliminaries of stability. Consider the nonlinear system of discrete Volterra equations

$$\mathbf{x}_n = \mathbf{f}_n + \sum_{j=0}^{n} \mathbf{B}_{n-j}\{\mathbf{x}_j + \mathbf{G}_j(\mathbf{x}_j)\}, \qquad n = 0, 1, \dots, \tag{A.1}$$

in $\mathbb{R}^d$, where the functions $\mathbf{G}_j(\cdot)$ are Lipschitz and satisfy $\mathbf{G}_j(\mathbf{0}) = \mathbf{0}$ for all j. To study the behavior of the solution of (A.1), we introduce the following linear system:

$$\mathbf{y}_n = \mathbf{f}_n + \sum_{j=0}^{n} \mathbf{B}_{n-j}\mathbf{y}_j, \qquad n = 0, 1, \dots \tag{A.2}$$

Assume that $\det(\mathbf{I} - \mathbf{B}_n) \neq 0$, $\forall n \geq 0$. Then the unique solution of the system (A.2) is given by

$$\mathbf{y}_n = \mathbf{f}_n - \sum_{j=0}^{n} \mathbf{R}_{n-j}\mathbf{f}_j, \qquad n \geq 0, \tag{A.3}$$

where $\{\mathbf{R}_n\}_n$ are the resolvent matrices of the kernel $\{\mathbf{B}_n\}_n$, such that

$$\mathbf{R}_{n-m} = \sum_{j=m}^{n} \mathbf{R}_{n-j}\mathbf{B}_{j-m} - \mathbf{B}_{n-m}, \qquad 0 \leq m \leq n. \tag{A.4}$$

Let $|\cdot|$ denote a norm of vectors in $\mathbb{R}^d$ as well as the corresponding subordinate matrix norm. The following results represent the basis for studying the stability properties of the solution to the model (3.2).

THEOREM A.1 ([34, Lemma 2.1]). *Let $\{\boldsymbol{x}_n\}_n$ be the solution of (A.1) and $\{\boldsymbol{y}_n\}_n$ the solution of (A.2) given by (A.3). Then*

$$\boldsymbol{x}_n = \boldsymbol{y}_n - \sum_{j=0}^{n} \boldsymbol{R}_{n-j}\boldsymbol{G}_j(\boldsymbol{x}_j), \qquad \forall n \geq 0.$$

THEOREM A.2 ([34, Theorem 3.24]). *Assume that*

$$\sup_{n\geq 0} |\boldsymbol{y}_n| < \infty, \qquad \lim_{n\to\infty} \boldsymbol{y}_n = 0,$$

$$\sum_{n=0}^{\infty} |\boldsymbol{R}_n| < \infty, \qquad \lim_{n\to\infty} |\boldsymbol{R}_n| = 0,$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

where $\{\boldsymbol{y}_n\}_n$ *is the solution to* (A.2) *and* $\{\boldsymbol{R}_n\}_n$ *is defined in* (A.4)*. Then the solution* $\{\boldsymbol{x}_n\}_n$ *of* (A.1) *also satisfies* $\lim_{n\to\infty}\boldsymbol{x}_n = 0$.

THEOREM A.3 ([26, Corollary 6.2]). *Consider the discrete system of Volterra equations*

$$\boldsymbol{y}_n = \boldsymbol{f}_n + \sum_{j=0}^{n} \boldsymbol{B}_{n-j}\boldsymbol{y}_j, \qquad n = 0, 1, \dots, \tag{A.5}$$

where $\sum_{n=0}^{\infty} |\boldsymbol{B}_n - \boldsymbol{B}_\infty| < \infty$ *for some* $\boldsymbol{B}_\infty$. $\boldsymbol{B}_\infty$ *and* $\boldsymbol{f}_n$ *can be written as*

$$\boldsymbol{B}_\infty = \begin{bmatrix} \boldsymbol{B}_\infty^1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{f}_n = \begin{bmatrix} \boldsymbol{f}_n^0 \\ \boldsymbol{f}_n^1 \end{bmatrix}, \tag{A.6}$$

where $\boldsymbol{B}_\infty$ *is an arbitrary constant matrix composed of two block, a null block with dimension* k *and a constant block with dimension* l*, denoted as* $\boldsymbol{B}_\infty^1$*, and* $\boldsymbol{f}_n^0, \boldsymbol{f}_n^1$ *have dimension* l *and* k*, respectively. Then*

$$\boldsymbol{y}_n \longrightarrow 0, \quad \textit{whenever} \quad \boldsymbol{f}_n^1, \Delta \boldsymbol{f}_n^0 \longrightarrow 0 \quad (n \to \infty),$$

if and only if

$$(1-z)^l \det\left(\boldsymbol{I} - \sum_{n=0}^{\infty} \boldsymbol{B}_n z^n\right) = 0, \tag{A.7}$$

has no solution for $|z| \le 1$.

For $|z| = 1$, (A.7) has been interpreted as the limit for $\tilde{z} \to z$, $|\tilde{z}| < 1$.

REFERENCES

[1] G. K. ALDIS AND M. G. ROBERTS, *An integral equation model for the control of a smallpox outbreak*, Math. Biosci., 195 (2005), pp. 1–22.

[2] S. BLANES, A. ISERLES, AND S. MACNAMARA, *Positivity-preserving methods for ordinary differential equations*, ESAIM Math. Model. Numer. Anal., 56 (2022), pp. 1843–1870.

[3] F. BRAUER, *Age of infection in epidemiology models*, in Proceedings of the 2004 Conference on Differential Equations and Applications in Mathematical Biology, F. Brauer and E. Braverman, eds., Electron. J. Differ. Equ. Conf., Conference 12, Nanaimo, Canada, 2005, pp. 29–37.

[4] ———, *The Kermack–McKendrick epidemic model revisited*, Math. Biosci., 198 (2005), pp. 119–131.

[5] F. BRAUER, C. CASTILLO-CHAVEZ, AND Z. FENG, *Mathematical Models in Epidemiology*, Springer, New York, 2019.

[6] F. BRAUER, Z. SHUAI, AND P. VAN DEN DRIESSCHE, *Dynamics of an age-of-infection cholera model*, Math. Biosci. Eng., 10 (2013), pp. 1335–1349.

[7] D. BREDA, O. DIEKMANN, W. F. DE GRAAF, A. PUGLIESE, AND R. VERMIGLIO, *On the formulation of epidemic models (an appraisal of Kermack and McKendrick)*, J. Biol. Dyn., 6 (2012), pp. 103–117.

[8] H. BRUNNER AND P. J. VAN DER HOUWEN, *The Numerical Solution of Volterra Equations*, CWI monographs, North-Holland, 1986.

[9] B. BUONOMO, E. MESSINA, C. PANICO, AND A. VECCHIO, *An integral renewal equation approach to behavioural epidemic models with information index*, Preprint on arXiv, 2024. https://arxiv.org/abs/2402.08618.

[10] J. CRESSON AND F. PIERRET, *Non standard finite difference scheme preserving dynamical properties*, J. Comput. Appl. Math., 303 (2016), pp. 15–30.

[11] O. DIEKMANN AND J. A. P. HEESTERBEEK, *Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation*, Wiley, Chichester, 2000.

[12] A. D'ONOFRIO AND P. MANFREDI, *Information-related changes in contact patterns may trigger oscillations in the endemic prevalence of infectious diseases*, J. Theoret. Biol., 256 (2009), pp. 473–478.

[13] ———, *Behavioral SIR models with incidence-based social-distancing*, Chaos Solitons Fractals, 159 (2022), Art. 112072 (8 pages).

[14] A. D'ONOFRIO, P. MANFREDI, AND E. SALINELLI, *Vaccinating behaviour, information, and the dynamics of SIR vaccine preventable diseases*, Theor. Popul. Biol., 71 (2007), pp. 301–317.

[15] J. T. EDWARDS, N. J. FORD, AND J. A. ROBERTS, *Bifurcations in numerical methods for Volterra integro-differential equations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13 (2003), pp. 3255–3271.

[16] I. FARAGÓ AND R. HORVÁTH, *Continuous and discrete parabolic operators and their qualitative properties*, IMA J. Numer. Anal., 29 (2009), pp. 606–631.

[17] Z. FODOR, S. D. KATZ, AND T. G. KOVACS, *Why integral equations should be used instead of differential equations to describe the dynamics of epidemics*, Preprint on arXiv, 2020. https://arxiv.org/abs/2004.07208v2.

[18] E. HAIRER, C. LUBICH, AND G. WANNER, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer, Berlin, 2013.

[19] A. R. HUMPHRIES, *Spurious solutions of numerical methods for initial value problems*, IMA J. Numer. Anal., 13 (1993), pp. 263–290.

[20] T. IZGIN, S. KOPECZ, A. MEISTER, AND A. SCHILLING, *On the non-global linear stability and spurious fixed points of MPRK schemes with negative RK parameters*, Numer. Algorithms, 96 (2024), pp. 1221–1242.

[21] A. KEIMER AND L. PFLUG, *Modeling infectious diseases using integro-differential equations: Optimal control strategies for policy decisions and applications in COVID-19*, (2020). Preprint, https://doi.org/10.13140/RG.2.2.10845.44000.

[22] A. KERGASSNER, C. BURKHARDT, D. LIPPOLD, M. KERGASSNER, L. PFLUG, D. BUDDAY, P. STEINMANN, AND S. BUDDAY, *Memory-based meso-scale modeling of Covid-19*, Comput. Mech., 66 (2020), pp. 1069–1079.

[23] W. O. KERMACK AND A. G. MCKENDRICK, *A contribution to the mathematical theory of epidemics*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 115 (1927), pp. 700–721.

[24] ———, *Contributions to the mathematical theory of epidemics–I*, Bull. Math. Biol., 53 (1991), pp. 33–55.

[25] P. LINZ, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[26] C. LUBICH, *On the stability of linear multistep methods for Volterra convolution equations*, IMA J. Numer. Anal., 3 (1983), pp. 439–465.

[27] J. M. S. LUBUMA AND Y. A. TEREFE, *A nonstandard Volterra difference equation for the SIS epidemiological model*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 109 (2015), pp. 597–602.

[28] E. MESSINA, *Numerical simulation of a SIS epidemic model based on a nonlinear Volterra integral equation*, in Conference Publications, Volume 2015, AIMS, Springfield, 2015, pp. 826–834. https://www.aimsciences.org/article/doi/10.3934/proc.2015.0826

[29] E. MESSINA, M. PEZZELLA, AND A. VECCHIO, *A non-standard numerical scheme for an age-of-infection epidemic model*, J. Comput. Dyn., 9 (2022), pp. 239–252.

[30] ———, *Asymptotic solutions of non-linear implicit Volterra discrete equations*, J. Comput. Appl. Math., 425 (2023), Art. 115068 (13 pages).

[31] ———, *Nonlocal finite difference discretization of a class of renewal equation models for epidemics*, Math. Biosci. Eng., 20 (2023), pp. 11656–11675.

[32] R. E. MICKENS, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.

[33] M. G. ROBERTS, *Modeling strategies for containing an invading infection*, Math. Popul. Stud., 13 (2006), pp. 205–214.

[34] Y. SONG AND C. T. H. BAKER, *Perturbation theory for discrete Volterra equations*, J. Difference Equ. Appl., 9 (2003), pp. 969–987.

[35] A. M. STUART AND A. R. HUMPHRIES, *Dynamical systems and numerical analysis*, Cambridge University Press, , Cambridge, 1996.

[36] H. R. THIEME AND C. CASTILLO-CHAVEZ, *On the role of variable infectivity in the dynamics of the human immunodeficiency virus epidemic*, in Mathematical and Statistical Approaches to AIDS Epidemiology, C. Castillo-Chavez, ed., Lecture Notes in Biomathematics, 83, Springer, Berlin, 1989, pp. 157–176.

[37] ———, *How may infection-age-dependent infectivity affect the dynamics of HIV/AIDS?*, SIAM J. Appl. Math., 53 (1993), pp. 1447–1479.

[38] H.-J. TIAN, L.-Q. FAN, Y.-Y. ZHANG, AND J.-X. XIANG, *Spurious numerical solutions of delay differential equations*, J. Comput. Math., 24 (2006), pp. 181–192.

[39] F. VADILLO, *On spurious fixed points of Runge-Kutta methods*, J. Comput. Phys., 132 (1997), pp. 78–90.