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STRUCTURED CONDITION NUMBERS FOR A LINEAR FUNCTION OF THE SOLUTION OF THE GENERALIZED SADDLE POINT PROBLEM*

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Abstract. This paper addresses structured normwise, mixed, and componentwise condition numbers (*CNs*) for a linear function of the solution to the generalized saddle point problem (*GSPP*). We present a general framework that enables us to measure structured *CNs* of the individual components of the solution. Then, we derive their explicit formulae when the input matrices have symmetric, Toeplitz, or some general linear structures. In addition, compact formulae for unstructured *CNs* are obtained, which recover previous results on *CNs* for *GSPPs* for specific choices of the linear function. Furthermore, applications of the derived structured *CNs* are provided to determine the structured *CNs* for the weighted Toeplitz regularized least-squares problems and Tikhonov regularization problems, which recovers some previous studies in the literature.

Key words. generalized saddle point problems, condition number, perturbation analysis, weighted Toeplitz regularized least-squares problem, Toeplitz matrices

AMS subject classifications. 65F35, 65F20, 65F99, 15A12

1. Introduction.

Generalized saddle point problems (*GSPPs*) have received significant attention owing to their extensive applications across numerous fields in scientific computing, such as computational fluid dynamics [13, 16], constrained optimization [20, 38], and so on. Consider the following two-by-two block linear system:

$$\mathcal{M}z \equiv \begin{bmatrix} A & B^{\mathsf{T}} \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv \mathbf{d}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$, $B, C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times n}$, $x, f \in \mathbb{R}^m$, $y, g \in \mathbb{R}^n$, and B^{T} represents the transpose of the matrix B. Then (1.1) is referred to as a *GSPP* if the block matrices A, B, C, and D satisfy some special properties, such as $B = C$, symmetric, Toeplitz, or have some other linear structures [10]. Recently, many efficient iteration methods have been proposed to solve the linear system (1.1), such as inexact Uzawa schemes [8], Krylov subspace methods [33], and so on. For a comprehensive survey of applications, algebraic properties, and iterative methods for *GSPPs*, we refer to [6] and the references therein.

The *GSPP* or its special cases originate from a wide range of applications. For example: (i) The Karush-Kuhn-Tucker (KKT) system ($A = A^{\mathsf{T}}$, $B = C$, and $D = \mathbf{0}$, here $\mathbf{0}$ denotes the zero matrix of appropriate dimension) is one of the simplest versions of (1.1) and arises from the KKT first-order optimality condition in constrained optimization problems [10, 36]. (ii) The sinc-Galerkin discretization of ordinary differential equations (ODEs) leads to a problem of the form (1.1) [3, 4, 7]. (iii) The system (1.1) also comes from a finite element discretization of time-harmonic eddy current models [2]. (iv) The finite difference discretization of time-dependent Stokes equations generates systems in the form of (1.1) [9]. (v) *GSPPs* emerge in the weighted Toeplitz regularized least-squares (*WTRLS*) problems [11] arising from image restoration and reconstruction problems [17, 30] of the form

$$\min_{y\in\mathbb{R}^n} \|\mathbf{M}y - \tilde{\mathbf{d}}\|_2^2, \tag{1.2}$$

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where $\mathbf{M} = \begin{bmatrix} W^{1/2}Q \\ \sqrt{\lambda} I_n \end{bmatrix} \in \mathbb{R}^{(m+n)\times n}$, $\widetilde{\mathbf{d}} = \begin{bmatrix} W^{1/2}f \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{m\times n} (m \geq n)$ is a full rank Toeplitz matrix and $W \in \mathbb{R}^{m\times m}$ is a symmetric positive definite weighting matrix. The equivalent augmented system has the structure of a *GSPP* of the form (1.1) with A being symmetric and $B = C$ a Toeplitz matrix (see Section 5).

Perturbation theory is extensively used in numerical analysis to examine the sensitivity of numerical techniques and the error analysis of a computed solution [21]. Condition numbers (*CNs*) and backward errors are the two most important tools in perturbation theory. For a given problem, the *CN* is a measure of the worst-case sensitivity of a numerical solution with respect to a tiny perturbation in the input data, whereas the backward error reveals the stability of any numerical approach. Combined with the backward error, *CNs* can provide a first-order estimate of the forward error of an approximated solution.

Rice in [31] presented the classical theory of *CNs*. It essentially deals with the normwise condition number (*NCN*) by employing norms to measure both the input perturbation and the error in the output data. A notable drawback associated with the *NCN* lies in its inability to capture the inherent structure of badly scaled or sparse input data. Consequently, the *NCN* occasionally overestimates the true conditioning of the numerical solution. As remedies for this, the mixed condition number (*MCN*) and componentwise condition number (*CCN*) have seen a growing interest in the literature [18, 32, 35]. The former measures perturbation in the input data componentwise and the output data error by norms, while the latter measures both the input perturbation and the output data error componentwise.

Perturbation theory and *CNs* for the *GSPP* (1.1) have been widely studied in the literature. A brief review of the literature on *CNs* for the *GSPP* (1.1) is as follows: In [37] the *NCN* for the solution $z = [x^\top, y^\top]^\top$ for the KKT system, i.e., the *GSPP* (1.1) with $A = A^\top$, $B = C$, and $D = \mathbf{0}$ was analyzed. In [39], the authors discussed perturbation bounds for the *GSPP* when $B = C$ and $D = \mathbf{0}$, and they derived closed formulae for the *NCN*, *MCN*, and *CCN* of the solutions $z = [x^\top, y^\top]^\top$ and the individual solution components x and y. The *NCN* and perturbation bounds have been investigated in [40] for the solution $z = [x^\top, y^\top]^\top$ of the *GSPP* (1.1) with the conditions $B = C$ and $D \neq \mathbf{0}$. Later, in [28], the *MCN* and *CCN* for $z = [x^\top, y^\top]^\top$ was studied. Additionally, the authors explored the *NCN*, *MCN*, and *CCN* for the individual solution components x and y. Recently, new perturbation bounds have been derived for the *GSPP* (1.1) under the condition $B \neq C$, without imposing any special structure on A and D [41].

In many applications, blocks of the coefficient matrix $\mathcal{M}$ of the system (1.1) exhibit linear structures (for example, symmetric, Toeplitz or symmetric-Toeplitz) [12, 16, 34, 42]. Therefore, it is reasonable to ask: how sensitive is the solution when structure-preserving perturbations are introduced to the coefficient matrix of *GSPPs*? To address the aforementioned query, we explore the notion of structured *CNs* by restricting the perturbations that preserve the structures inherent in the block matrices of $\mathcal{M}$.

Furthermore, in many instances, x and y represent two distinct physical entities, for example in the Stokes equation, x denotes the velocity vector, and y stands for the scalar pressure field [16]. Therefore, it is important to assess their individual conditioning properties. To accomplish this, we propose a general framework for assessing the conditioning of x, y, $z = [x^\top, y^\top]^\top$ and each solution component. In the proposed general framework, we consider the structured *CNs* of a linear function $\mathbf{L}[x^\top, y^\top]^\top$ of the solution to the *GSPP* (1.1), where $\mathbf{L} \in \mathbb{R}^{k\times(m+n)}$. The matrix $\mathbf{L}$ serves as a tool for the purpose of selecting solution components. For example, (i) $\mathbf{L} = I_{m+n}$ gives the *CNs* for $[x^\top, y^\top]^\top$, (ii) $\mathbf{L} = \begin{bmatrix} I_m & \mathbf{0} \end{bmatrix}$ gives the *CNs* for x, and (iii) $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}$ gives the *CNs* for y. Here, I_m stands for the identity matrix of order m.

The key contributions of this paper are summarized as follows:

- We study the *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\top,\ y^\top]^\top$, which in turn provides a general framework enabling us to derive *CNs* for the solutions $[x^\top,\ y^\top]^\top$, x, y, and each solution component.
- We investigate unstructured *CNs* for $\mathbf{L}[x^\top,\ y^\top]^\top$ by considering $B = C$ and then structured *CNs* when the (1,1)-block A is symmetric and the (1,2)-block B is Toeplitz. We derive explicit formulae for both the unstructured and structured *CNs*. For appropriate choices of $\mathbf{L}$, our derived unstructured *CN* formulae generalize the results given in the literature [28, 40].
- By considering linear structures of the block matrices A and D with $B \neq C$, we provide compact formulae of the structured *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\top,\ y^\top]^\top$ of the *GSPP* (1.1).
- Utilizing the structured *CN* formulae, we derive the structured *CNs* for the *WTRLS* problem and generalize some of the previous structured *CN* formulae for the Tikhonov regularization problem. This shows the generic nature of our obtained results.
- Numerical experiments demonstrate that the obtained structured *CNs* offer sharper bounds to the actual relative errors than their unstructured counterparts.

The organization of this paper is as follows. Section 2 discusses notation and preliminary results about *CNs*. In Section 3 and 4, we investigate the unstructured and structured *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\top,\ y^\top]^\top$ of the solution of the *GSPP*. Furthermore, an application of our obtained structured *CNs* is provided in Section 5 for *WTRLS* problems. Additionally, these *CNs* are used to retrieve some prior found results for Tikhonov regularization problems. In Section 6, numerical experiments are carried out to demonstrate the effectiveness of the proposed structured *CNs*. Section 7 presents some concluding remarks.

2. Notation and preliminaries.

In this section, we define some notation and review some well-known results, which play a crucial role in showing the main findings of this paper.

2.1. Notation.

Let $\mathbb{R}^{m\times n}$ be the set of all $m\times n$ real matrices, and $\|\cdot\|_2$, $\|\cdot\|_\infty$, and $\|\cdot\|_F$ stand for the Euclidean norm/matrix 2-norm, infinity norm, and Frobenius norm, respectively. For $x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$, we denote by $\mathbf{D}_x \in \mathbb{R}^{n\times n}$ the diagonal matrix with $\mathbf{D}_x(i,i) = x_i$. The symbol $A^\dagger$ denotes the Moore-Penrose inverse of A. Following [15, 26], the entrywise division of any two vectors $x, y \in \mathbb{R}^n$ is defined as $\frac{x}{y} := [\frac{x_i}{y_i}]$, where $x_i/0 = 0$ whenever $x_i = 0$ and infinity otherwise.

For any matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, we set $|A| := [|a_{ij}|]$, where $|a_{ij}|$ denotes the absolute value of the entry a_{ij}. For any two matrices $A, B \in \mathbb{R}^{m\times n}$, the notation $A \leq B$ represents the inequalities $a_{ij} \leq b_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. For the matrix $A = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{m\times n}$, where $a_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$, the linear operator $\mathrm{vec} : \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{mn}$ is defined by $\mathrm{vec}(A) := [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top$. The vec operator satisfies $\|\mathrm{vec}(A)\|_2 = \|A\|_F$. The Kronecker product [19] of two matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$ is defined by $X \otimes Y := [x_{ij}Y] \in \mathbb{R}^{mp\times nq}$, and some of its important properties are listed below [19, 23]:

$$\mathrm{vec}(XCY) = (Y^\top \otimes X)\mathrm{vec}(C), \qquad |X \otimes Y| = |X| \otimes |Y|, \tag{2.1}$$

where $C \in \mathbb{R}^{n\times p}$.

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2.2. Preliminaries. Throughout this paper, we assume that A and $\mathcal{M}$ are nonsingular. We know that if A is nonsingular, then $\mathcal{M}$ is nonsingular if and only if its Schur complement $S = D - CA^{-1}B^{\mathsf{T}}$ is nonsingular [1], and its inverse is expressed as

$$\mathcal{M}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B^{\mathsf{T}}S^{-1}CA^{-1} & -A^{-1}B^{\mathsf{T}}S^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \tag{2.2}$$

Following [15, 26], we employ the following notation: The componentwise distance between two vectors a and b in $\mathbb{R}^p$ is defined as

$$d(a,b) = \left\| \frac{a-b}{b} \right\|_\infty = \max_{i=1,2,\ldots,p} \left\{ \frac{|a_i - b_i|}{|b_i|} \right\}.$$

For $u \in \mathbb{R}^p$ and $\eta > 0$, we consider the sets: $B_1(u,\eta) = \{x \in \mathbb{R}^p : \|x-u\|_2 \leq \eta \|u\|_2\}$ and $B_2(u,\eta) = \{x \in \mathbb{R}^p : |x_i - u_i| \leq \eta |u_i|,\ i = 1,\ldots,p\}$. With the above conventions we next present the definitions of the *NCN*, *MCN*, and *CCN* for a mapping $\boldsymbol{\varphi} : \mathbb{R}^p \mapsto \mathbb{R}^q$.

DEFINITION 2.1 ([15, 18]). *Let $\boldsymbol{\varphi} : \mathbb{R}^p \mapsto \mathbb{R}^q$ be a continuous mapping defined on an open set $\Omega_{\boldsymbol{\varphi}} \subseteq \mathbb{R}^p$, and let $\mathbf{0} \neq u \in \Omega_{\boldsymbol{\varphi}}$ such that $\boldsymbol{\varphi}(u) \neq \mathbf{0}$.*

(i) *The NCN of $\boldsymbol{\varphi}$ at u is defined by*

$$\mathscr{K}(\boldsymbol{\varphi}, u) = \lim_{\eta \to 0} \sup_{\substack{x \neq u \\ x \in B_1(u,\eta)}} \frac{\|\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(u)\|_2 / \|\boldsymbol{\varphi}(u)\|_2}{\|x-u\|_2/\|u\|_2}.$$

(ii) *The MCN of $\boldsymbol{\varphi}$ at u is defined by*

$$\mathscr{M}(\boldsymbol{\varphi}, u) = \lim_{\eta \to 0} \sup_{\substack{x \neq u \\ x \in B_2(u,\eta)}} \frac{\|\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(u)\|_\infty}{\|\boldsymbol{\varphi}(u)\|_\infty} \frac{1}{d(x,u)}.$$

(iii) *Let $\boldsymbol{\varphi}(u) = [\boldsymbol{\varphi}(u)_1, \ldots, \boldsymbol{\varphi}(u)_q]^{\mathsf{T}}$ be such that $\boldsymbol{\varphi}(u)_i \neq 0$, for $i = 1, 2, \ldots, q$. Then the CCN of $\boldsymbol{\varphi}$ at u is defined by*

$$\mathscr{C}(\boldsymbol{\varphi}, u) = \lim_{\eta \to 0} \sup_{\substack{x \neq u \\ x \in B_2(u,\eta)}} \frac{d(\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(u))}{d(x,u)}.$$

DEFINITION 2.2 ([14]). *Let $\boldsymbol{\varphi} : \mathbb{R}^p \mapsto \mathbb{R}^q$ be a mapping defined on an open set $\Omega_{\boldsymbol{\varphi}} \subseteq \mathbb{R}^p$. Then $\boldsymbol{\varphi}$ is said to be Fréchet differentiable at $u \in \Omega_{\boldsymbol{\varphi}}$ if there exists a bounded linear operator $\mathbf{d}\boldsymbol{\varphi} : \mathbb{R}^p \mapsto \mathbb{R}^q$ such that*

$$\lim_{h \to \mathbf{0}} \frac{\|\boldsymbol{\varphi}(u+h) - \boldsymbol{\varphi}(u) - \mathbf{d}\boldsymbol{\varphi} h\|}{\|h\|} = 0,$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^p$ and $\mathbb{R}^q$.

When $\boldsymbol{\varphi}$ is Fréchet differentiable at u, we denote the Fréchet derivative at u as $\mathbf{d}\boldsymbol{\varphi}(u)$. The next lemma gives closed-form expressions for the above three *CNs* when the continuous mapping $\boldsymbol{\varphi}$ is Fréchet differentiable.

LEMMA 2.3 ([15, 18]). *Under the same hypothesis as in Definition 2.1, when φ is Fréchet differentiable at u, we have*

$$\mathscr{K}(\varphi;u)=\frac{\|\mathbf{d}\varphi(u)\|_2\|u\|_2}{\|\varphi(u)\|_2},\quad \mathscr{M}(\varphi;u)=\frac{\||\mathbf{d}\varphi(u)|\,|u|\|_\infty}{\|\varphi(u)\|_\infty},\quad \mathscr{C}(\varphi;u)=\left\|\frac{|\mathbf{d}\varphi(u)|\,|u|}{|\varphi(u)|}\right\|_\infty,$$

where $\mathbf{d}\varphi(u)$ denotes the Fréchet derivative of φ at u.

First, consider the case when $B=C$, i.e., the following *GSPP*

$$\mathcal{M}\begin{bmatrix}x\\y\end{bmatrix}:=\begin{bmatrix}A & B^\mathsf{T}\\ B & D\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}f\\g\end{bmatrix}:=\mathbf{d}, \tag{2.3}$$

and let ΔA, ΔB, ΔD, Δf, and Δg be perturbations in A, B, D, f, and g, respectively. Then, we have the following perturbed problem of (2.3):

$$(\mathcal{M}+\Delta\mathcal{M})\begin{bmatrix}x+\Delta x\\y+\Delta y\end{bmatrix}=\begin{bmatrix}A+\Delta A & (B+\Delta B)^\mathsf{T}\\ B+\Delta B & D+\Delta D\end{bmatrix}\begin{bmatrix}x+\Delta x\\y+\Delta y\end{bmatrix}=\begin{bmatrix}f+\Delta f\\g+\Delta g\end{bmatrix}, \tag{2.4}$$

which has the unique solution $\begin{bmatrix}x+\Delta x\\y+\Delta y\end{bmatrix}$ when $\|\mathcal{M}^{-1}\|_2\|\Delta\mathcal{M}\|_2<1$. Now, from (2.4), omitting the higher-order term, we obtain

$$\begin{bmatrix}\Delta x\\\Delta y\end{bmatrix}\approx\mathcal{M}^{-1}\begin{bmatrix}\Delta f\\\Delta g\end{bmatrix}-\mathcal{M}^{-1}\begin{bmatrix}\Delta A & \Delta B^\mathsf{T}\\ \Delta B & \Delta D\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}. \tag{2.5}$$

Using the properties in (2.1), we have the following important lemma:

LEMMA 2.4. *Let $\begin{bmatrix}x\\y\end{bmatrix}$ and $\begin{bmatrix}x+\Delta x\\y+\Delta y\end{bmatrix}$ be the unique solutions of the GSPPs (2.3) and (2.4), respectively. Then, we have the following perturbation expression:*

$$\begin{bmatrix}\Delta x\\\Delta y\end{bmatrix}\approx-\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\begin{bmatrix}\mathrm{vec}(\Delta A)\\\mathrm{vec}(\Delta B)\\\mathrm{vec}(\Delta D)\\\Delta f\\\Delta g\end{bmatrix},$$

where

$$\mathcal{R}=\begin{bmatrix}x^\mathsf{T}\otimes I_m & I_m\otimes y^\mathsf{T} & \mathbf{0}\\ \mathbf{0} & x^\mathsf{T}\otimes I_n & y^\mathsf{T}\otimes I_n\end{bmatrix}. \tag{2.6}$$

Proof. The proof follows from (2.5) and using the properties in (2.1). ∎

Let $\mathbf{H}=\begin{bmatrix}A & \mathbf{0}\\ B & D\end{bmatrix}$ and $\Delta\mathbf{H}=\begin{bmatrix}\Delta A & \mathbf{0}\\ \Delta B & \Delta D\end{bmatrix}$. The authors in [40] investigated unstructured *NCNs* and those of [28] studied unstructured *MCNs* and *NCNs* for the solution $[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$ to the *GSPP* (1.1) when $B=C$, which are given as follows:

$$(2.7)\quad \mathscr{K}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{\|[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}\|_2}{\eta\|[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_2} : \left\|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right\|_F \le \eta\left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F\right\}$$
$$= \frac{\left\|\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\right\|_2 \left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F}{\|[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_2},$$

$$(2.8)\quad \mathscr{M}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{\|[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}\|_\infty}{\eta\|[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_\infty} : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|\right\}$$
$$= \left\| |\mathcal{M}^{-1}\mathcal{R}|\begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty \Bigg/ \left\|\begin{bmatrix}x\\ y\end{bmatrix}\right\|_\infty,$$

$$(2.9)\quad \mathscr{C}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{1}{\eta}\left\|\frac{[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}}{[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}}\right\|_\infty : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|\right\}$$
$$= \left\|\mathbf{D}^\dagger_{[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}}|\mathcal{M}^{-1}\mathcal{R}|\begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + \mathbf{D}^\dagger_{[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}}|\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty,$$

where $\mathcal{R}$ is defined as in (2.6).

In the next section, we consider unstructured and structured *CNs* for a linear function of the solution of the *GSPP* (2.3).

3. *CNs* for a linear function of the solution to the *GSPP* when $B = C$.

In this section, we derive compact *NCN*, *MCN*, and *CCN* formulae for a linear function of the solution to the *GSPP* (1.1) when $B = C$, under both unstructured and structured perturbations. Additionally, comparisons between unstructured and structured *CNs* are provided.

3.1. Unstructured *CN* formulae.

In this section, we consider unstructured *CNs* for the linear function $\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$, where $\mathbf{L} \in \mathbb{R}^{k\times(m+n)}$, and we derive their explicit formulae. In the following, we define the unstructured *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$. Throughout the paper, we assume that $[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T} \neq \mathbf{0}$ for *MCN* and $x_i \neq 0$ $(i = 1,\dots,m)$ and $y_i \neq 0$ $(i = 1,\dots,n)$ for *CCN*.

DEFINITION 3.1. *Let* $[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$ *and* $\left[(x+\Delta x)^\mathsf{T},\, (y+\Delta y)^\mathsf{T}\right]^\mathsf{T}$ *be the unique solutions of the GSPPs* (2.3) *and* (2.4)*, respectively, and let* $\mathbf{L} \in \mathbb{R}^{k\times(m+n)}$. *Then we define the unstructured NCN, MCN, and CCN for the linear function* $\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$*, respectively, as follows:*

$$\mathscr{K}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{\|\mathbf{L}[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}\|_2}{\eta\|\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_2} : \left\|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right\|_F \le \eta\left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F\right\},$$
$$\mathscr{M}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{\|\mathbf{L}[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}\|_\infty}{\eta\|\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_\infty} : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|\right\},$$
$$\mathscr{C}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) := \lim_{\eta\to 0}\sup\left\{\frac{1}{\eta}\left\|\frac{\mathbf{L}[\Delta x^\mathsf{T},\, \Delta y^\mathsf{T}]^\mathsf{T}}{\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}}\right\|_\infty : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|\right\}.$$

Note that when $\mathbf{L} = I_{m+n}$, the above definitions reduce to (2.7)–(2.9). For using Lemma 2.3, we construct the mapping $\boldsymbol{\psi} : \mathbb{R}^{m^2+mn+n^2} \times \mathbb{R}^{m+n} \mapsto \mathbb{R}^{m+n}$ by

$$(3.1)\qquad \boldsymbol{\psi}([\Omega^\mathsf{T}, f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}) := \mathbf{L}\begin{bmatrix}x\\ y\end{bmatrix} = \mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}f\\ g\end{bmatrix},$$

where $\Omega^\mathsf{T} = [\mathrm{vec}(A)^\mathsf{T}, \mathrm{vec}(B)^\mathsf{T}, \mathrm{vec}(D)^\mathsf{T}]^\mathsf{T}$. The following result is crucial for finding the *CNs* formulae.

PROPOSITION 3.2. *Let $\Omega^\top = [\mathrm{vec}(A)^\top, \mathrm{vec}(B)^\top, \mathrm{vec}(D)^\top]^\top$. Then, for the map ψ defined in (3.1), we have*

$$\begin{aligned}\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top) &= \mathscr{K}(\psi, [\Omega^\top, f^\top, g^\top]^\top),\\ \mathscr{M}(\mathbf{L}[x^\top,\, y^\top]^\top) &= \mathscr{M}(\psi, [\Omega^\top, f^\top, g^\top]^\top),\\ \mathscr{C}(\mathbf{L}[x^\top,\, y^\top]^\top) &= \mathscr{C}(\psi, [\Omega^\top, f^\top, g^\top]^\top).\end{aligned}$$

Proof. Let $\Delta\Omega^\top = [\mathrm{vec}(\Delta A)^\top, \mathrm{vec}(\Delta B)^\top, \mathrm{vec}(\Delta D)^\top]^\top$. Then, from (3.1), we obtain

$$\begin{aligned}&\psi\left([\Omega^\top + \Delta\Omega^\top, f^\top + \Delta f^\top, g^\top + \Delta g^\top]^\top\right) - \psi\left([\Omega^\top,\, f^\top,\, g^\top]^\top\right)\\ &\quad= \mathbf{L}\begin{bmatrix} x + \Delta x\\ y + \Delta y\end{bmatrix} - \mathbf{L}\begin{bmatrix} x\\ y\end{bmatrix} = \mathbf{L}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}.\end{aligned} \tag{3.2}$$

Now, in Definition 3.1, substituting (3.2) and $\psi\left([\Omega^\top,\, f^\top,\, g^\top]^\top\right) = \mathbf{L}\left[x^\top,\, y^\top\right]^\top$, the proof follows as a consequence of Definition 2.1. □

Since the Fréchet derivative of ψ has a pivotal role in estimating the *CNs* in Definition 3.1, it is essential to derive simple expressions for $\mathbf{d}\psi$. By applying Lemma 2.4, we obtain the following results for $\mathbf{d}\psi$:

LEMMA 3.3. *The map ψ defined above is continuous and Fréchet differentiable at $[\Omega^\top, f^\top, g^\top]^\top$, and its Fréchet derivative at $[\Omega^\top, f^\top, g^\top]^\top$ is given by*

$$\mathbf{d}\psi([\Omega^\top, f^\top, g^\top]^\top) = -\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}.$$

Proof. Since $\mathcal{M}^{-1}$ is continuous in its elements, the linear map ψ is also continuous. Let $\Delta\Omega^\top = [\mathrm{vec}(\Delta A)^\top, \mathrm{vec}(\Delta B)^\top, \mathrm{vec}(\Delta D)^\top]^\top$. Then

$$\psi\left([\Omega^\top + \Delta\Omega^\top, f^\top + \Delta f^\top, g^\top + \Delta g^\top]^\top\right) - \psi\left([\Omega^\top,\, f^\top,\, g^\top]^\top\right) = \mathbf{L}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}.$$

Hence, the rest of the proof follows from Lemma 2.4. □

Applying Lemma 3.3, we obtain the following closed formulae for the unstructured *CNs* for the linear function $\mathbf{L}[x^\top,\, y^\top]^\top$.

THEOREM 3.4. *Let $[x^\top,\, y^\top]^\top$ be the unique solution of the GSPP (2.3). Then the unstructured NCN, MCN, and CCN for the linear function $\mathbf{L}[x^\top,\, y^\top]^\top$, respectively, are given by*

$$\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top) = \frac{\left\|\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\right\|_2 \left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F}{\|\mathbf{L}[x^\top,\, y^\top]^\top\|_2},$$

$$\mathscr{M}(\mathbf{L}[x^\top,\, y^\top]^\top) = \frac{\left\| |\mathbf{L}\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty}{\|\mathbf{L}[x^\top,\, y^\top]^\top\|_\infty},$$

$$\mathscr{C}(\mathbf{L}[x^\top,\, y^\top]^\top) = \left\| \mathbf{D}^\dagger_{\mathbf{L}[x^\top,\, y^\top]^\top} |\mathbf{L}\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + \mathbf{D}^\dagger_{\mathbf{L}[x^\top,\, y^\top]^\top} |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty.$$

Proof. Let $\Omega^\top = [\mathrm{vec}(A)^\top, \mathrm{vec}(B)^\top, \mathrm{vec}(D)^\top]^\top$. Then, from Proposition 3.2 and applying the *NCN* formula of Lemma 2.3 for the map ψ, we obtain

$$\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top) = \mathscr{K}(\psi, [\Omega^\top, f^\top, g^\top]^\top) = \frac{\left\|\mathbf{d}\psi\left(\Omega^\top,\, f^\top,\, g^\top]^\top\right)\right\|_2 \|[\Omega^\top, f^\top, g^\top]^\top\|_2}{\|\psi\left(\Omega^\top,\, f^\top,\, g^\top]^\top\right)\|_2}. \tag{3.3}$$

Now, substituting the expression of the Fréchet derivative of ψ at $[\Omega^\mathsf{T}, f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}$ provided in Lemma 3.3 in (3.3), we get

$$\mathscr{K}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) = \frac{\left\| \mathbf{L}\mathcal{M}^{-1} \begin{bmatrix} \mathcal{R} & -I_{m+n} \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} \mathbf{H} & \mathbf{d} \end{bmatrix} \right\|_F}{\|\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_2}.$$

Similarly, applying the *MCN* formula provided in Lemma 2.3 for ψ, we get

$$\mathscr{M}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) = \mathscr{M}(\psi, [\Omega^\mathsf{T}, f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}) = \frac{\left\| |\mathbf{d}\psi\left([\Omega^\mathsf{T},\, f^\mathsf{T},\, g^\mathsf{T}]^\mathsf{T}\right)| \, |[\Omega^\mathsf{T}, f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}| \right\|_\infty}{\|\psi\left([\Omega^\mathsf{T},\, f^\mathsf{T},\, g^\mathsf{T}]^\mathsf{T}\right)\|_\infty}. \tag{3.4}$$

Substituting the Fréchet derivative expression provided in Lemma 3.3 in (3.4), we obtain

$$\begin{aligned} \mathscr{M}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}) &= \frac{\left\| \left| \mathbf{L}\mathcal{M}^{-1} \begin{bmatrix} \mathcal{R} & -I_{m+n} \end{bmatrix} \right| |[\Omega^\mathsf{T}, f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}| \right\|_\infty}{\|\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}\|_\infty} \\ &= \frac{\left\| |\mathbf{L}\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|D|) \end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty}{\left\| \mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T} \right\|_\infty}. \end{aligned}$$

The rest of the proof follows in a similar way. ∎

REMARK 3.5. If we consider $\mathbf{L} = I_{m+n}$, then the formulae for $\mathscr{K}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$, $\mathscr{M}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$, and $\mathscr{C}(\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$ reduce to the unstructured *CNs* $\mathscr{K}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$, $\mathscr{M}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$, and $\mathscr{C}^u([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T})$ given in (2.7)–(2.9), respectively. Moreover, if we choose $\mathbf{L} = \begin{bmatrix} I_m & \mathbf{0} \end{bmatrix}$ or $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}$, after some easy calculations, we can recover the unstructured *CN* formulae of [28] for x and y, respectively.

3.2. Structured *CNs* when A is symmetric and $B = C$ is Toeplitz. In this section, we consider the structured *NCN*, *MCN*, and *CCN* of the *GSPP* (2.3) with $A = A^\mathsf{T}$ and $B \in \mathbb{R}^{n\times m}$ being a Toeplitz matrix. We denote by $\mathcal{S}_m$ and $\mathcal{T}^{n\times m}$ the set of all $m \times m$ symmetric matrices and $n \times m$ Toeplitz matrices, respectively. Now, we recall the definition of Toeplitz matrices and derive a few important lemmas.

DEFINITION 3.6 ([22]). *A matrix $T = [t_{ij}] \in \mathbb{R}^{n\times m}$ is called a Toeplitz matrix if there exists*

$$\mathbf{t} = [t_{-n+1}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{m-1}]^\mathsf{T} \in \mathbb{R}^{m+n-1}$$

such that $t_{ij} = t_{j-i}$, for all $1 \le i \le n$ and $1 \le j \le m$.

The generator vector $\mathbf{t}$ for T is denoted by $\mathrm{vec}_\mathcal{T}(T)$. Moreover, for $\mathbf{t} \in \mathbb{R}^{m+n-1}$, $\mathtt{Toep}(\mathbf{t})$ denotes the corresponding generated Toeplitz matrix.

As $\dim(\mathcal{T}^{n\times m}) = m + n - 1$, consider the basis $\{\mathcal{J}_i\}_{i=-n+1}^{m-1}$ for $\mathcal{T}^{n\times m}$ defined as

$$\mathcal{J}_i = \begin{cases} \mathtt{Toep}([(e_{n-i}^{(n)})^\mathsf{T},\, \mathbf{0}]^\mathsf{T}) & \text{for } i = -n+1, \ldots, -1, 0, \\ \mathtt{Toep}([\mathbf{0},\, (e_i^{(m)})^\mathsf{T}]^\mathsf{T}) & \text{for } i = 1, \ldots, m-1, \end{cases}$$

where $e_i^{(m)}$ is the i-th column of the identity matrix I_m. Moreover, let us define the diagonal matrix $\mathfrak{D}_{\mathcal{T}_{nm}} \in \mathbb{R}^{(m+n-1)\times(m+n-1)}$ with $\mathfrak{D}_{\mathcal{T}_{nm}}(j,j) = \boldsymbol{a}_j$, where

$$\boldsymbol{a} = [1, \sqrt{2}, \dots, \sqrt{n-1}, \sqrt{\min\{m,n\}}, \sqrt{m-1}, \dots, \sqrt{2}, 1]^{\mathsf{T}} \in \mathbb{R}^{m+n-1}$$

such that $\|T\|_F = \|\mathfrak{D}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(T)\|_2$.

LEMMA 3.7. *Let* $T \in \mathcal{T}^{n\times m}$. *Then* $\mathrm{vec}(T) = \mathbf{\Phi}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(T)$, *where*

$$\mathbf{\Phi}_{\mathcal{T}_{nm}} = \left[\mathrm{vec}(\mathcal{J}_{-n+1}), \dots, \mathrm{vec}(\mathcal{J}_{m-1})\right] \in \mathbb{R}^{mn\times(m+n-1)}.$$

Proof. Assume that $\mathrm{vec}_{\mathcal{T}}(T) = [t_{-n+1}, \dots t_0, \dots, t_{m-1}]^{\mathsf{T}}$. Then

$$T = \sum_{i=-n+1}^{m-1} t_i \mathcal{J}_i \Longleftrightarrow \mathrm{vec}(T) = \mathbf{\Phi}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(T).$$

Hence, the proof follows. ☐

Let $A \in \mathcal{S}_m$. Then $A = A^{\mathsf{T}}$. Moreover, we have $\dim(\mathcal{S}_m) = \frac{m(m+1)}{2} =: \boldsymbol{p}$. We denote the generator vector for A by

$$\mathrm{vec}_{\mathcal{S}}(A) := [a_{11}, \dots, a_{1m}, a_{22}, \dots, a_{2m}, \dots, a_{(m-1)(m-1)}, a_{(m-1)m}, a_{mm}]^{\mathsf{T}} \in \mathbb{R}^{\boldsymbol{p}}.$$

Consider the basis $\left\{S_{ij}^{(m)}\right\}$ for $\mathcal{S}_m$ defined as

$$S_{ij}^{(m)} = \begin{cases} e_i^{(m)}(e_j^{(m)})^{\mathsf{T}} + (e_j^{(m)} e_i^{(m)})^{\mathsf{T}} & \text{for } i \neq j, \\ e_i^{(m)}(e_i^{(m)})^{\mathsf{T}} & \text{for } i = j, \end{cases}$$

where $1 \leq i \leq j \leq m$. Then we have the following immediate result for the vec-structure of A:

LEMMA 3.8. *Let* $A \in \mathcal{S}_m$. *Then* $\mathrm{vec}(A) = \mathbf{\Phi}_{\mathcal{S}_m} \mathrm{vec}_{\mathcal{S}}(A)$, *where* $\mathbf{\Phi}_{\mathcal{S}_m} \in \mathbb{R}^{m^2\times \boldsymbol{p}}$ *is given by*

$$\mathbf{\Phi}_{\mathcal{S}_m} = \begin{bmatrix}\mathrm{vec}(S_{11}^{(m)}) & \cdots & \mathrm{vec}(S_{1m}^{(m)}) & \mathrm{vec}(S_{22}^{(m)}) & \cdots & \mathrm{vec}(S_{2m}^{(m)}) & \cdots & \mathrm{vec}(S_{(m-1)m}^{(m)}) & \mathrm{vec}(S_{mm}^{(m)})\end{bmatrix}.$$

Proof. The proof follows by using a similar proof method as in Lemma 3.7. ☐

We construct the diagonal matrix $\mathfrak{D}_{\mathcal{S}_m} \in \mathbb{R}^{\boldsymbol{p}\times\boldsymbol{p}}$, where

$$\begin{cases} \mathfrak{D}_{\mathcal{S}_m}(j,j) = 1 & \text{for } j = \frac{(2m-(i-2))(i-1)}{2} + 1,\ i = 1, 2, \dots, m, \\ \mathfrak{D}_{\mathcal{S}_m}(j,j) = \sqrt{2} & \text{otherwise.} \end{cases}$$

This matrix satisfies the property $\|A\|_F = \|\mathfrak{D}_{\mathcal{S}_m} \mathrm{vec}_{\mathcal{S}}(A)\|_2$. Consider the set

$$\mathcal{E} = \left\{ \mathbf{H} = \begin{bmatrix} A & \mathbf{0} \\ B & D \end{bmatrix} :\ A \in \mathcal{S}_m, B \in \mathcal{T}^{n\times m}, D \in \mathbb{R}^{n\times n} \right\},$$

and let $\Delta\mathbf{H} = \begin{bmatrix} \Delta A & \mathbf{0} \\ \Delta B & \Delta D \end{bmatrix} \in \mathcal{E}$, i.e., $\Delta A \in \mathcal{S}_m,\ \Delta B \in \mathcal{T}^{n\times m}$, and $\Delta D \in \mathbb{R}^{n\times n}$.

Next, we define the structured *CNs* for the solution of the *GSPP* (2.3).

DEFINITION 3.9. *Let* $[x^\top,\, y^\top]^\top$ *and* $\left[(x+\Delta x)^\top,\, (y+\Delta y)^\top\right]^\top$ *be the unique solutions of the GSPPs* (2.3) *and* (2.4)*, respectively, with the structure* $\mathcal{E}$ *and* $\mathbf{L}\in\mathbb{R}^{k\times(m+n)}$*. Then the structured NCN, MCN, and CCN for the linear function* $\mathbf{L}[x^\top,\, y^\top]^\top$ *are defined as follows:*

$$\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{E})$$
$$:= \lim_{\eta\to 0}\sup\left\{\frac{\|\mathbf{L}[\Delta x^\top,\, \Delta y^\top]^\top\|_2}{\eta\|\mathbf{L}[x^\top,\, y^\top]^\top\|_2} : \left\|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right\|_F \le \eta\left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F, \Delta\mathbf{H}\in\mathcal{E}\right\},$$
$$\mathscr{M}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{E})$$
$$:= \lim_{\eta\to 0}\sup\left\{\frac{\|\mathbf{L}[\Delta x^\top,\, \Delta y^\top]^\top\|_\infty}{\eta\|\mathbf{L}[x^\top,\, y^\top]^\top\|_\infty} : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|, \Delta\mathbf{H}\in\mathcal{E}\right\},$$
$$\mathscr{C}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{E})$$
$$:= \lim_{\eta\to 0}\sup\left\{\frac{1}{\eta}\left\|\frac{\mathbf{L}[\Delta x^\top,\, \Delta y^\top]^\top}{\mathbf{L}[x^\top,\, y^\top]^\top}\right\|_\infty : \left|\begin{bmatrix}\Delta\mathbf{H} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right|, \Delta\mathbf{H}\in\mathcal{E}\right\}.$$

To find the structured *CN* formulae by employing Lemma 2.3, we define the following mapping:

$$\begin{aligned}&\boldsymbol{\zeta} : \mathbb{R}^{\boldsymbol{l}}\times\mathbb{R}^m\times\mathbb{R}^n\mapsto\mathbb{R}^{m+n}\\ &\boldsymbol{\zeta}\left([\mathfrak{D}_\mathcal{E}\boldsymbol{w}^\top,\, f^\top,\, g^\top]^\top\right) = \mathbf{L}\begin{bmatrix}x\\ y\end{bmatrix} = \mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}f\\ g\end{bmatrix},\end{aligned} \tag{3.5}$$

where

$$\boldsymbol{l} = n^2+\boldsymbol{p}+m+n-1, \quad \boldsymbol{w} = \begin{bmatrix}\mathrm{vec}_\mathcal{S}(A)\\ \mathrm{vec}_\mathcal{T}(B)\\ \mathrm{vec}(D)\end{bmatrix}, \quad \text{and} \quad \mathfrak{D}_\mathcal{E} = \begin{bmatrix}\mathfrak{D}_{\mathcal{S}_m} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathfrak{D}_{\mathcal{T}_{nm}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & I_{n^2}\end{bmatrix}.$$

In the next lemma, we provide the Fréchet derivative of the map $\boldsymbol{\zeta}$ at $\left[\mathfrak{D}_\mathcal{E}\boldsymbol{w}^\top,\, f^\top,\, g^\top\right]^\top$.

LEMMA 3.10. *The mapping* $\boldsymbol{\zeta}$ *defined in* (3.5) *is continuously Fréchet differentiable at* $\left[\mathfrak{D}_\mathcal{E}\boldsymbol{w}^\top,\, f^\top,\, g^\top\right]^\top$*, and the Fréchet derivative is given by*

$$\mathbf{d}\boldsymbol{\zeta}\left([\mathfrak{D}_\mathcal{E}\boldsymbol{w}^\top,\, f^\top,\, g^\top]^\top\right) = -\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\boldsymbol{\Phi}_\mathcal{E}\mathfrak{D}_\mathcal{E}^{-1} & -I_{m+n}\end{bmatrix},$$

where

$$\boldsymbol{\Phi}_\mathcal{E} = \begin{bmatrix}\boldsymbol{\Phi}_{\mathcal{S}_m} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Phi}_{\mathcal{T}_{nm}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & I_{n^2}\end{bmatrix}.$$

Proof. The continuity of the linear map $\boldsymbol{\zeta}$ follows from the continuity of $\mathcal{M}^{-1}$. For the second part, let

$$\Delta\boldsymbol{w} = \begin{bmatrix}\mathrm{vec}_\mathcal{S}(\Delta A)\\ \mathrm{vec}_\mathcal{T}(\Delta B)\\ \mathrm{vec}(\Delta D)\end{bmatrix},$$

and consider

$$\boldsymbol{\zeta}\left([\mathfrak{D}_\mathcal{E}(\boldsymbol{w}^\top+\Delta\boldsymbol{w}^\top), f^\top+\Delta f^\top, g^\top+\Delta g^\top]\right) - \boldsymbol{\zeta}\left([\mathfrak{D}_\mathcal{E}\boldsymbol{w}^\top,\, f^\top,\, g^\top]^\top\right) = \mathbf{L}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}. \tag{3.6}$$

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Then, from Lemma 2.3, we obtain

$$\begin{aligned}
\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} &\approx -\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\begin{bmatrix}\mathrm{vec}(\Delta A)\\ \mathrm{vec}(\Delta B)\\ \mathrm{vec}(\Delta D)\\ \Delta f\\ \Delta g\end{bmatrix}\\
&= -\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{\mathcal{S}_m} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Phi}_{\mathcal{T}_{nm}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & I_{n^2+m+n}\end{bmatrix}\begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(\Delta A)\\ \mathrm{vec}_{\mathcal{T}}(\Delta B)\\ \mathrm{vec}(\Delta D)\\ \Delta f\\ \Delta g\end{bmatrix}\\
&= -\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}} & -I_{m+n}\end{bmatrix}\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}^{-1}\mathfrak{D}_{\mathcal{E}}\Delta \boldsymbol{w}\\ \Delta f\\ \Delta g\end{bmatrix}\\
&= -\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix}\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\Delta \boldsymbol{w}\\ \Delta f\\ \Delta g\end{bmatrix}. \qquad (3.7)
\end{aligned}$$

Combining (3.7) and (3.6), the Fréchet derivative of $\boldsymbol{\zeta}$ at $\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}\\ f\\ g\end{bmatrix}$ is

$$\mathbf{d}\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^{\mathsf{T}},\, f^{\mathsf{T}},\, g^{\mathsf{T}}]^{\mathsf{T}}\right) = -\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix}.$$

Hence, the proof follows. □

Using Lemma 3.10 and Lemma 2.3, we next derive compact formulae for the structured *CNs* defined in Definition 3.9.

THEOREM 3.11. *Let $[x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}}$ be the unique solution of the GSPP (2.3) with the structure $\mathcal{E}$. Then the structured NCN, MCN, and CCN for the linear function $\mathbf{L}[x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}}$, respectively, are given by*

$$\mathscr{K}(\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}};\mathcal{E}) = \frac{\left\|\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix}\right\|_2 \left\|\begin{bmatrix}\mathbf{H} & \mathbf{d}\end{bmatrix}\right\|_F}{\left\|\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}}\right\|_2},$$

$$\mathscr{M}(\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}};\mathcal{E}) = \frac{\left\| |\mathbf{L}\mathcal{M}^{-1}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}}|\begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty}{\left\|\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}}\right\|_\infty},$$

$$\begin{aligned}
&\mathscr{C}(\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}};\mathcal{E})\\
&\quad = \left\|\mathbf{D}^{\dagger}_{\mathbf{L}[x^{\mathsf{T}},y^{\mathsf{T}}]^{\mathsf{T}}}|\mathbf{L}\mathcal{M}^{-1}\mathcal{R}\mathbf{\Phi}_{\mathcal{E}}|\begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + \mathbf{D}^{\dagger}_{\mathbf{L}[x^{\mathsf{T}},y^{\mathsf{T}}]^{\mathsf{T}}}|\mathbf{L}\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty.
\end{aligned}$$

Proof. Let $\boldsymbol{w}^{\mathsf{T}} = [\mathrm{vec}(A)_{\mathcal{S}}^{\mathsf{T}}, \mathrm{vec}(B)_{\mathcal{T}}^{\mathsf{T}}, \mathrm{vec}(D)^{\mathsf{T}}]^{\mathsf{T}}$. Following the proof method of Proposition 3.2, we have

$$\mathscr{K}(\mathbf{L}\left[x^{\mathsf{T}},\, y^{\mathsf{T}}\right]^{\mathsf{T}};\mathcal{E}) = \mathscr{K}(\boldsymbol{\zeta}, [\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^{\mathsf{T}},\, f^{\mathsf{T}},\, g^{\mathsf{T}}]^{\mathsf{T}}),$$

$$\mathscr{M}(\mathbf{L}\left[x^\top,\,y^\top\right]^\top;\,\mathcal{E}) = \mathscr{M}(\boldsymbol{\zeta},[\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top),$$
$$\mathscr{C}(\mathbf{L}\left[x^\top,\,y^\top\right]^\top;\,\mathcal{E}) = \mathscr{C}(\boldsymbol{\zeta},[\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top).$$

Applying the *NCN* formula given in Lemma 2.3 for the map $\boldsymbol{\zeta}$, we obtain

$$\mathscr{K}(\mathbf{L}[x^\top,\,y^\top]^\top;\,\mathcal{E}) = \frac{\left\|\mathbf{d}\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top\right)\right\|_2 \left\|\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}\\ f\\ g\end{bmatrix}\right\|_2}{\|\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top\right)\|_2}. \tag{3.8}$$

Now, substituting the Fréchet derivative of $\boldsymbol{\zeta}$ provided in Lemma 3.3 in (3.8), we have

$$\mathscr{K}(\mathbf{L}[x^\top,\,y^\top]^\top;\,\mathcal{E}) = \frac{\left\|\mathbf{L}\mathcal{M}^{-1}\left[\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1}\quad -I_{m+n}\right]\right\|_2 \|[\mathbf{H}\quad \mathbf{d}]\|_F}{\|\mathbf{L}[x^\top,\,y^\top]^\top\|_2}.$$

Similarly, applying the *MCN* formula provided in Lemma 2.3 for $\boldsymbol{\zeta}$, we get

$$\mathscr{M}(\mathbf{L}[x^\top,\,y^\top]^\top;\,\mathcal{E}) = \frac{\left\|\left|\mathbf{d}\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top\right)\right| \left|\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}\\ f\\ g\end{bmatrix}\right|\right\|_\infty}{\|\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top\right)\|_\infty}. \tag{3.9}$$

Now, using Lemma 3.10 in (3.9), we obtain

$$\begin{aligned}
\mathscr{M}(\mathbf{L}[x^\top,\,y^\top]^\top;\,\mathcal{E}) &= \frac{\left\|\left|\mathbf{L}\mathcal{M}^{-1}\left[\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1}\quad -I_{m+n}\right]\right| \left|\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}\\ f\\ g\end{bmatrix}\right|\right\|_\infty}{\|\mathbf{L}[x^\top,\,y^\top]^\top\|_\infty}\\
&= \frac{\left\||\mathbf{L}\mathcal{M}^{-1}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1}||\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}| + |\mathbf{L}\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty}{\|\mathbf{L}[x^\top,\,y^\top]^\top\|_\infty}\\
&= \frac{\left\||\mathbf{L}\mathcal{M}^{-1}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}|\begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty}{\left\|\mathbf{L}[x^\top,\,y^\top]^\top\right\|_\infty}.
\end{aligned}$$

In an analogous manner, we get

$$\begin{aligned}
\mathscr{C}(\mathbf{L}[x^\top,\,y^\top]^\top;\,\mathcal{E}) &= \left\|\mathbf{D}^{\dagger}_{\mathbf{L}[x^\top,y^\top]^\top}\left|\mathbf{d}\boldsymbol{\zeta}\left([\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}^\top,\,f^\top,\,g^\top]^\top\right)\right| \left|\begin{bmatrix}\mathfrak{D}_{\mathcal{E}}\boldsymbol{w}\\ f\\ g\end{bmatrix}\right|\right\|_\infty\\
&= \left\|\mathbf{D}^{\dagger}_{\mathbf{L}[x^\top,y^\top]^\top}|\mathbf{L}\mathcal{M}^{-1}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}|\begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} + \mathbf{D}^{\dagger}_{\mathbf{L}[x^\top,y^\top]^\top}|\mathbf{L}\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty.
\end{aligned}$$

Hence, the proof is completed. ∎

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REMARK 3.12. Note that the structured *MCN* and *CCN* formulae presented in Theorem 3.11 involve computing the inverse of the matrix $\mathcal{M} \in \mathbb{R}^{(m+n)\times(m+n)}$, while the structured *NCN* formula involves computing the inverse of both matrices $\mathcal{M}$ and $\mathfrak{D}_{\mathcal{E}} \in \mathbb{R}^{\boldsymbol{l}\times\boldsymbol{l}}$. However, $\mathfrak{D}_{\mathcal{E}}$ is a diagonal matrix. Therefore, its inverse can be computed using only $\mathcal{O}(\boldsymbol{l})$ operations. On the other hand, to avoid computing $\mathcal{M}^{-1}$ explicitly, motivated by [25], we adopt the following procedure. Notably, the computation of $\mathcal{M}^{-1}$ is coming in the following form:

$$\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix} \qquad \text{or} \qquad \mathbf{L}\mathcal{M}^{-1}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}} \qquad \text{or} \qquad \mathbf{L}\mathcal{M}^{-1}.$$

Thus, first, we solve the system $\mathcal{M}X = Y$, where $Y = \begin{bmatrix}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix}$ or $\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}$ and then compute $\mathbf{L}X$. The system $\mathcal{M}X = Y$ can be solved efficiently by an LU decomposition. To compute $\mathbf{L}\mathcal{M}^{-1}$, we can solve $\mathbf{L} = X\mathcal{M}$. It is worth noting that we only need to perform the LU decomposition once for all cases; this makes the procedure efficient and reliable.

REMARK 3.13. The Toeplitz matrix B is symmetric-Toeplitz (a special case of a Toeplitz matrix) if $n = m$ and

$$b_{-n+1} = b_{n-1}, \ldots, b_{-1} = b_1, \quad \text{where} \quad \mathrm{vec}_{\mathcal{T}}(B) = [b_{-n+1}, \ldots, b_1, b_0, \ldots, b_{n-1}]^{\mathsf{T}}.$$

In this case, the basis for the set of symmetric-Toeplitz matrices is defined as $\left\{\widetilde{\mathcal{J}}_i\right\}_{i=1}^{n}$, where

$$\begin{aligned}\widetilde{\mathcal{J}}_1 &= \mathtt{Toep}([(e_n^{(n)})^{\mathsf{T}},\, \mathbf{0}]^{\mathsf{T}}),\\ \widetilde{\mathcal{J}}_{i+1} &= \mathtt{Toep}([(e_{n-i}^{(n)})^{\mathsf{T}},\, (e_i^{(n-1)})^{\mathsf{T}}]^{\mathsf{T}}), \qquad \text{for } i = 1, \ldots, n-1.\end{aligned}$$

Hence, the structured *CNs* for the *GSPP* (2.3) when A is symmetric and B is symmetric-Toeplitz is given by the formulae as in Theorem 3.11, with

$$\begin{aligned}\boldsymbol{\Phi}_{\mathcal{T}_{nm}} &= \left[\mathrm{vec}(\widetilde{\mathcal{J}}_1), \ldots, \mathrm{vec}(\widetilde{\mathcal{J}}_n)\right] \in \mathbb{R}^{n^2\times n} \qquad \text{and}\\ \mathfrak{D}_{\mathcal{T}_{nm}} &\in \mathbb{R}^{n\times n} \quad \text{with} \quad \mathfrak{D}_{\mathcal{T}_{nm}}(j,j) = \hat{\boldsymbol{a}}_j,\end{aligned}$$

where $\hat{\boldsymbol{a}} = [\sqrt{n}, \sqrt{2(n-1)}, \sqrt{2(n-2)}, \ldots, \sqrt{2}]^{\mathsf{T}} \in \mathbb{R}^n$.

Next, we compare the structured *CNs* with the unstructured ones given in (2.7)–(2.9).

THEOREM 3.14. *With the above notation, when* $\mathbf{L} = I_{m+n}$, *we have the following relations:*

$$\begin{aligned}\mathscr{K}([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}; \mathcal{E}) &\le \mathscr{K}^u([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}),\\ \mathscr{M}([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}; \mathcal{E}) &\le \mathscr{M}^u([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}),\\ \mathscr{C}([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}; \mathcal{E}) &\le \mathscr{C}^u([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}).\end{aligned}$$

Proof. Since $\mathbf{L} = I_{m+n}$, for the *NCN*, using the properties of the spectral norm, we obtain

$$\begin{aligned}\left\|\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & -I_{m+n}\end{bmatrix}\right\|_2 &\le \left\|\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\right\|_2 \left\|\begin{bmatrix}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & \mathbf{0}\\ \mathbf{0} & I_{m+n}\end{bmatrix}\right\|_2\\ &= \left\|\mathcal{M}^{-1}\begin{bmatrix}\mathcal{R} & -I_{m+n}\end{bmatrix}\right\|_2.\end{aligned}$$

The last equality follows from the fact that $\|\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1}\|_2 = 1$. Hence, the first claim is verified. Since $\boldsymbol{\Phi}_{\mathcal{E}}$ has at most one nonzero entry in each row, we find

$$|\mathcal{M}^{-1}\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}| \begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix} \le |\mathcal{M}^{-1}\mathcal{R}||\boldsymbol{\Phi}_{\mathcal{E}}| \begin{bmatrix}\mathrm{vec}_{\mathcal{S}}(|A|)\\ \mathrm{vec}_{\mathcal{T}}(|B|)\\ \mathrm{vec}(|D|)\end{bmatrix}$$

$$= |\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} |\mathbf{\Phi}_{\mathcal{S}_m}|\mathrm{vec}_{\mathcal{S}}(|A|) \\ |\mathbf{\Phi}_{\mathcal{T}_{nm}}|\mathrm{vec}_{\mathcal{T}}(|B|) \\ \mathrm{vec}(|D|) \end{bmatrix} = |\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} |\mathbf{\Phi}_{\mathcal{S}_m}\mathrm{vec}_{\mathcal{S}}(A)| \\ |\mathbf{\Phi}_{\mathcal{T}_{nm}}\mathrm{vec}_{\mathcal{T}}(B)| \\ \mathrm{vec}(|D|) \end{bmatrix}$$
$$= |\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|D|) \end{bmatrix}.$$

Therefore, from Theorem 3.4, we obtain

$$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E}) \leq \frac{\left\| |\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|D|) \end{bmatrix} + |\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty}{\|[x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}\|_\infty} = \mathscr{M}^u([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$$

and

$$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E}) \leq \left\| \mathbf{D}^\dagger_{[x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}} |\mathcal{M}^{-1}\mathcal{R}| \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|D|) \end{bmatrix} + \mathbf{D}^\dagger_{[x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}} |\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty$$
$$= \mathscr{C}^u([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}).$$

Hence, the proof is completed. □

4. Structured *CNs* when A and D have linear structures. In this section, we consider $\mathcal{L}_1 \subseteq \mathbb{R}^{m\times m}$ and $\mathcal{L}_2 \subseteq \mathbb{R}^{n\times n}$ are two distinct linear subspaces containing different classes of structured matrices. Suppose that $\dim(\mathcal{L}_1) = p$ and $\dim(\mathcal{L}_2) = s$, and the corresponding bases are $\{E_i\}_{i=1}^p$ and $\{F_i\}_{i=1}^s$, respectively. Let $A \in \mathcal{L}_1$ and $D \in \mathcal{L}_2$. Then there are unique vectors

$$\mathrm{vec}_{\mathcal{L}_1}(A) = [a_1, a_2, \ldots, a_p]^\mathsf{T} \in \mathbb{R}^p \qquad \text{and} \qquad \mathrm{vec}_{\mathcal{L}_2}(D) = [d_1, d_2, \ldots, d_s]^\mathsf{T} \in \mathbb{R}^s$$

such that

$$A = \sum_{i=1}^{p} a_i E_i \qquad \text{and} \qquad D = \sum_{i=1}^{s} d_i F_i. \tag{4.1}$$

We obtain the following lemma for the vec-structure of the matrices A and D.

LEMMA 4.1. *Let $A \in \mathcal{L}_1$ and $D \in \mathcal{L}_2$. Then it holds that* $\mathrm{vec}(A) = \mathbf{\Phi}_{\mathcal{L}_1}\mathrm{vec}_{\mathcal{L}_1}(A)$ *and* $\mathrm{vec}(D) = \mathbf{\Phi}_{\mathcal{L}_2}\mathrm{vec}_{\mathcal{L}_2}(D)$, *where*

$$\mathbf{\Phi}_{\mathcal{L}_1} = \begin{bmatrix} \mathrm{vec}(E_1) & \mathrm{vec}(E_2) & \cdots & \mathrm{vec}(E_p) \end{bmatrix} \in \mathbb{R}^{m^2\times p},$$
$$\mathbf{\Phi}_{\mathcal{L}_2} = \begin{bmatrix} \mathrm{vec}(F_1) & \mathrm{vec}(F_2) & \cdots & \mathrm{vec}(F_s) \end{bmatrix} \in \mathbb{R}^{n^2\times s}.$$

Proof. Assume that $\mathrm{vec}_{\mathcal{L}_1}(A) = [a_1, a_2, \ldots, a_p]^\mathsf{T} \in \mathbb{R}^p$. Then from (4.1), we obtain

$$\mathrm{vec}(A) = \sum_{i=1}^{p} a_i \mathrm{vec}(E_i) = \mathbf{\Phi}_{\mathcal{L}_1}\mathrm{vec}_{\mathcal{L}_1}(A).$$

Similarly, we can get $\mathrm{vec}(D) = \mathbf{\Phi}_{\mathcal{L}_2}\mathrm{vec}_{\mathcal{L}_2}(D)$. □

The matrices $\mathbf{\Phi}_{\mathcal{L}_1}$ and $\mathbf{\Phi}_{\mathcal{L}_2}$ contain the information about the structures of A and D with respect to the linear subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. For unstructured matrices,

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$\mathbf{\Phi}_{\mathcal{L}_1} = I_{m^2}$ and $\mathbf{\Phi}_{\mathcal{L}_2} = I_{n^2}$. On the other hand, there exist diagonal matrices $\mathfrak{D}_{\mathcal{L}_1} \in \mathbb{R}^{p\times p}$ and $\mathfrak{D}_{\mathcal{L}_2} \in \mathbb{R}^{s\times s}$ with diagonal entries $\mathfrak{D}_{\mathcal{L}_j}(i,i) = \|\mathbf{\Phi}_{\mathcal{L}_j}(:,i)\|_2$, for $j = 1,2$, such that

$$\|A\|_F = \|\mathfrak{D}_{\mathcal{L}_1} a\|_2 \qquad \text{and} \qquad \|D\|_F = \|\mathfrak{D}_{\mathcal{L}_2} d\|_2.$$

To perform a structured perturbation analysis, we restrict the perturbation ΔA of A and ΔD of D to the linear subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then there are unique vectors $\mathrm{vec}_{\mathcal{L}_1}(\Delta A) \in \mathbb{R}^p$ and $\mathrm{vec}_{\mathcal{L}_2}(\Delta D) \in \mathbb{R}^s$ such that

$$\mathrm{vec}(\Delta A) = \mathbf{\Phi}_{\mathcal{L}_1}\mathrm{vec}_{\mathcal{L}_1}(\Delta A) \qquad \text{and} \qquad \mathrm{vec}(\Delta D) = \mathbf{\Phi}_{\mathcal{L}_2}\mathrm{vec}_{\mathcal{L}_1}(\Delta D).$$

Now, consider the following set:

$$\mathcal{L} = \left\{ \mathcal{M} = \begin{bmatrix} A & B^\mathsf{T} \\ C & D \end{bmatrix} : A \in \mathcal{L}_1, B, C \in \mathbb{R}^{n\times m}, D \in \mathcal{L}_2 \right\}.$$

Consider the perturbations ΔA, ΔB, ΔC, ΔD, Δf, and Δg of the matrices A, B, C, D, f, and g, respectively. Then, the following perturbed counterpart of the system (1.1)

$$(4.2) \qquad (\mathcal{M} + \Delta\mathcal{M}) \begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix} = \begin{bmatrix} A + \Delta A & (B + \Delta B)^\mathsf{T} \\ C + \Delta C & D + \Delta D \end{bmatrix} \begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix} = \begin{bmatrix} f + \Delta f \\ g + \Delta g \end{bmatrix}$$

has a unique solution $\begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix}$ when $\|\mathcal{M}\|_2 \, \|\Delta\mathcal{M}\|_2 < 1$. Consequently, neglecting higher-order terms, we can rewrite (4.2) as

$$\mathcal{M} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} A & B^\mathsf{T} \\ C & D \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \approx \begin{bmatrix} \Delta f \\ \Delta g \end{bmatrix} - \begin{bmatrix} \Delta A & \Delta B^\mathsf{T} \\ \Delta C & \Delta D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Using the properties of the Kronecker product mentioned in (2.1), we have the next lemma.

LEMMA 4.2. *Let* $\left[x^\mathsf{T}, y^\mathsf{T}\right]^\mathsf{T}$ *and* $\left[(x + \Delta x)^\mathsf{T}, (y + \Delta y)^\mathsf{T}\right]^\mathsf{T}$ *be the unique solutions of the GSPPs* (1.1) *and* (4.2)*, respectively, with the structure* $\mathcal{L}$*. Then, we have the perturbation expression:*

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \approx -\mathcal{M}^{-1} \begin{bmatrix} \mathcal{H} & -I_{m+n} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta B) \\ \mathrm{vec}(\Delta C) \\ \mathrm{vec}(\Delta D) \\ \Delta f \\ \Delta g \end{bmatrix},$$

where

$$(4.3) \qquad \mathcal{H} = \begin{bmatrix} x^\mathsf{T} \otimes I_m & I_m \otimes y^\mathsf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & x^\mathsf{T} \otimes I_n & y^\mathsf{T} \otimes I_n \end{bmatrix}.$$

Next, we define the structured *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T}$ of the solution of the *GSPP* (1.1) with the structure $\mathcal{L}$.

DEFINITION 4.3. *Let* $\left[x^{\mathsf T},\, y^{\mathsf T}\right]^{\mathsf T}$ *and* $\left[(x+\Delta x)^{\mathsf T},\, (y+\Delta y)^{\mathsf T}\right]^{\mathsf T}$ *be the unique solutions of the GSPPs* (1.1) *and* (4.2)*, respectively, with the structure* $\mathcal{L}$*. Suppose* $\mathbf{L} \in \mathbb{R}^{k\times(m+n)}$*. Then the structured NCN, MCN, and CCN for* $\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T}$*, respectively, are defined as follows:*

$$\mathscr{K}(\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T};\, \mathcal{L})$$
$$:= \lim_{\eta\to 0}\sup\left\{ \frac{\left\|\mathbf{L}[\Delta x^{\mathsf T},\, \Delta y^{\mathsf T}]^{\mathsf T}\right\|_2}{\eta\left\|\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T}\right\|_2} : \left\|\begin{bmatrix}\Delta\mathcal{M} & \Delta\mathbf{d}\end{bmatrix}\right\|_F \le \eta\left\|\begin{bmatrix}\mathcal{M} & \mathbf{d}\end{bmatrix}\right\|_F,\, \Delta\mathcal{M}\in\mathcal{L}\right\},$$
$$\mathscr{M}(\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T};\, \mathcal{L})$$
$$:= \lim_{\eta\to 0}\sup\left\{ \frac{\left\|\mathbf{L}[\Delta x^{\mathsf T},\, \Delta y^{\mathsf T}]^{\mathsf T}\right\|_\infty}{\eta\left\|\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T}\right\|_\infty} : \left|\begin{bmatrix}\Delta\mathcal{M} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathcal{M} & \mathbf{d}\end{bmatrix}\right|,\, \Delta\mathcal{M}\in\mathcal{L}\right\},$$
$$\mathscr{C}(\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T};\, \mathcal{L})$$
$$:= \lim_{\eta\to 0}\sup\left\{ \frac{1}{\eta}\left\|\frac{\mathbf{L}[\Delta x^{\mathsf T},\, \Delta y^{\mathsf T}]^{\mathsf T}}{\mathbf{L}[x^{\mathsf T},\, y^{\mathsf T}]^{\mathsf T}}\right\|_\infty : \left|\begin{bmatrix}\Delta\mathcal{M} & \Delta\mathbf{d}\end{bmatrix}\right| \le \eta\left|\begin{bmatrix}\mathcal{M} & \mathbf{d}\end{bmatrix}\right|,\, \Delta\mathcal{M}\in\mathcal{L}\right\}.$$

The main objective of this section is to develop explicit formulae for the structured *CNs* defined above. To accomplish this, let $\boldsymbol{v}$ be a vector in $\mathbb{R}^{p+2mn+s}$ defined as

$$\boldsymbol{v} = \left[\mathrm{vec}_{\mathcal{L}_1}^{\mathsf T}(A), \mathrm{vec}(B)^{\mathsf T}, \mathrm{vec}(C)^{\mathsf T}, \mathrm{vec}_{\mathcal{L}_2}^{\mathsf T}(D)\right]^{\mathsf T}.$$

To apply Lemma 2.3, we define the mapping

$$\Upsilon : \mathbb{R}^{p+2mn+s}\times\mathbb{R}^m\times\mathbb{R}^n \mapsto \mathbb{R}^k \tag{4.4}$$
$$\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^{\mathsf T},\, f^{\mathsf T},\, g^{\mathsf T}]^{\mathsf T}\right) := \mathbf{L}\begin{bmatrix}x\\ y\end{bmatrix} = \mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}f\\ g\end{bmatrix},$$

where

$$\mathfrak{D}_{\mathcal{L}} = \begin{bmatrix}\mathfrak{D}_{\mathcal{L}_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & I_{2mn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathfrak{D}_{\mathcal{L}_2}\end{bmatrix} \tag{4.5}$$

such that $\|\mathcal{M}\|_F = \|\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}\|_2$. In the next lemma, we present an explicit formula for $\mathbf{d}\Upsilon$.

LEMMA 4.4. *The mapping* Υ *defined in* (4.4) *is continuously Fréchet differentiable at* $[\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^{\mathsf T},\, f^{\mathsf T},\, g^{\mathsf T}]^{\mathsf T}$*, and the Fréchet derivative is given by*

$$\mathbf{d}\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^{\mathsf T},\, f^{\mathsf T},\, g^{\mathsf T}]^{\mathsf T}\right) = -\mathbf{L}\mathcal{M}^{-1}\begin{bmatrix}\mathcal{H}\boldsymbol{\Phi}_{\mathcal{L}}\mathfrak{D}_{\mathcal{L}} & -I_{m+n}\end{bmatrix},$$

where $\boldsymbol{\Phi}_{\mathcal{L}} = \begin{bmatrix}\boldsymbol{\Phi}_{\mathcal{L}_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & I_{2mn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi}_{\mathcal{L}_2}\end{bmatrix}$ *and* $\mathcal{H}$ *and* $\mathfrak{D}_{\mathcal{L}}$ *are defined as in* (4.3) *and* (4.5)*, respectively.*

Proof. The proof follows in a similar way as the proof method of Lemma 3.10. ∎

We now present compact formulae of the structured *NCN*, *MCN*, and *CCN* introduced in Definition 2.1. We use Lemmas 2.3 and (4.4) to prove the following theorem.

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THEOREM 4.5. *The structured NCN, MCN, and CCN for the linear function* $\mathbf{L}[x^\top,\, y^\top]^\top$ *of the solution of the GSPP* (1.1) *with the structure* $\mathcal{L}$, *respectively, are given by*

$$\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{L}) = \frac{\left\|\mathbf{L}\mathcal{M}^{-1}\left[\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}\mathfrak{D}_{\mathcal{L}}^{-1} \quad -I_{m+n}\right]\right\|_2 \left\|\left[\mathcal{M} \quad \mathbf{d}\right]\right\|_F}{\|\mathbf{L}[x^\top,\, y^\top]^\top\|_2},$$

$$\mathscr{M}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{L}) = \frac{\left\| |\mathbf{L}\mathcal{M}^{-1}\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}| \begin{bmatrix} \mathrm{vec}_{\mathcal{L}_1}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|C|) \\ \mathrm{vec}_{\mathcal{L}_2}(|D|) \end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty}{\left\|\mathbf{L}[x^\top,\, y^\top]^\top\right\|_\infty},$$

$$\begin{aligned}&\mathscr{C}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{L})\\ &\quad = \left\| \mathbf{D}^{\dagger}_{\mathbf{L}[x^\top, y^\top]^\top} |\mathbf{L}\mathcal{M}^{-1}\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}| \begin{bmatrix} \mathrm{vec}_{\mathcal{L}_1}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|C|) \\ \mathrm{vec}_{\mathcal{L}_2}(|D|) \end{bmatrix} + \mathbf{D}^{\dagger}_{\mathbf{L}[x^\top, y^\top]^\top} |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty .\end{aligned}$$

Proof. Similarly to the proof method of Proposition 3.2, we have

$$\begin{aligned}\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{L}) &= \mathscr{K}(\Upsilon, \left[\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top, f^\top, g^\top\right]^\top),\\ \mathscr{M}(\mathbf{L}[x^\top, y^\top]^\top;\, \mathcal{L}) &= \mathscr{M}(\Upsilon, \left[\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top, f^\top, g^\top\right]^\top),\\ \mathscr{C}(\mathbf{L}[x^\top,\, y^\top]^\top;\, \mathcal{L}) &= \mathscr{C}(\Upsilon, \left[\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top, f^\top, g^\top\right]^\top).\end{aligned}$$

Using Lemma 4.4 and the *NCN* formula provided in Lemma 2.3, we have

$$\begin{aligned}\mathscr{K}(\mathbf{L}[x^\top,\, y^\top]^\top;\mathcal{L}) &= \frac{\left\|\mathbf{d}\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top,\, f^\top,\, g^\top]^\top\right)\right\|_2 \left\|\begin{bmatrix}\mathfrak{D}_{\mathcal{L}}\boldsymbol{v} \\ f \\ g\end{bmatrix}\right\|_2}{\|\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top,\, f^\top,\, g^\top]^\top\right)\|_2}\\ &= \frac{\left\|\mathbf{L}\mathcal{M}^{-1}\left[\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}\mathfrak{D}_{\mathcal{L}}^{-1} \quad -I_{m+n}\right]\right\|_2 \left\|\left[\mathcal{M} \quad \mathbf{d}\right]\right\|_F}{\|\mathbf{L}[x^\top,\, y^\top]^\top\|_2}.\end{aligned}$$

For the structured *MCN*, again using Lemmas 2.3 and 4.4, we obtain

$$\begin{aligned}\mathscr{M}(\mathbf{L}[x^\top,\, y^\top]^\top;\mathcal{L}) &= \frac{\left\| \left|\mathbf{d}\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top,\, f^\top,\, g^\top]^\top\right)\right| \left|\begin{bmatrix}\mathfrak{D}_{\mathcal{L}}\boldsymbol{v} \\ f \\ g\end{bmatrix}\right| \right\|_\infty}{\|\Upsilon\left([\mathfrak{D}_{\mathcal{L}}\boldsymbol{v}^\top,\, f^\top,\, g^\top]^\top\right)\|_\infty}\\ &= \frac{\left\| \left|\mathbf{L}\mathcal{M}^{-1}\left[\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}\mathfrak{D}_{\mathcal{L}}^{-1} \quad -I_{m+n}\right]\right| \left|\begin{bmatrix}\mathfrak{D}_{\mathcal{L}}\boldsymbol{v} \\ f \\ g\end{bmatrix}\right| \right\|_\infty}{\|\mathbf{L}[x^\top,\, y^\top]^\top\|_\infty}\\ &= \frac{\left\| |\mathbf{L}\mathcal{M}^{-1}\mathcal{H}\mathbf{\Phi}_{\mathcal{L}}| \begin{bmatrix} \mathrm{vec}_{\mathcal{L}_1}(|A|) \\ \mathrm{vec}(|B|) \\ \mathrm{vec}(|C|) \\ \mathrm{vec}_{\mathcal{L}_2}(|D|) \end{bmatrix} + |\mathbf{L}\mathcal{M}^{-1}| \begin{bmatrix} |f| \\ |g| \end{bmatrix} \right\|_\infty}{\left\|\mathbf{L}[x^\top,\, y^\top]^\top\right\|_\infty}.\end{aligned}$$

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The rest of the proof follows similarly. □

REMARK 4.6. To compute the inverses of $\mathcal{M}$ and $\mathfrak{D}_{\mathcal{L}}$, one can follow a similar procedure as discussed in Remark 3.12.

REMARK 4.7. Considering $\mathbf{L} = I_{m+n}, \begin{bmatrix} I_m & \mathbf{0} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}$ in Theorem 4.5, we obtain the structured *NCN*, *MCN*, and *CCN* for the solutions $[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}$, x, and y, respectively.

REMARK 4.8. For $A \in \mathcal{S}_m$ and $D \in \mathcal{S}_n$, set

$$\mathbf{\Phi}_{\mathcal{S}} = \begin{bmatrix} \mathbf{\Phi}_{\mathcal{S}_m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2mn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{\mathcal{S}_n} \end{bmatrix} \quad \text{and} \quad \mathfrak{D}_{\mathcal{S}} = \begin{bmatrix} \mathfrak{D}_{\mathcal{S}_m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2mn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathfrak{D}_{\mathcal{S}_n} \end{bmatrix},$$

where $\mathbf{\Phi}_{\mathcal{S}_m}, \mathbf{\Phi}_{\mathcal{S}_n}, \mathfrak{D}_{\mathcal{S}_m}$, and $\mathfrak{D}_{\mathcal{S}_n}$ are defined as in Section 3.2. Then the structured *NCN*, *MCN*, and *CCN* when $\mathcal{L}_1 = \mathcal{S}_m$ and $\mathcal{L}_2 = \mathcal{S}_n$ are obtained by substituting $\mathbf{\Phi}_{\mathcal{L}} = \mathbf{\Phi}_{\mathcal{S}}$, $\mathfrak{D}_{\mathcal{L}} = \mathfrak{D}_{\mathcal{S}}$, $\mathrm{vec}_{\mathcal{L}_1}(A) = \mathrm{vec}_{\mathcal{S}_m}(A)$, and $\mathrm{vec}_{\mathcal{L}_2}(D) = \mathrm{vec}_{\mathcal{S}_n}(D)$ in Theorem 4.5.

Next, consider the linear system $\mathcal{M}z = \mathbf{d}$, where $\mathcal{M} \in \mathbb{R}^{l\times l}$ is any nonsingular matrix and $\mathbf{d} \in \mathbb{R}^l$. Then this system can be partitioned as a *GSPP* (1.1) by setting $l = m+n$. Let $\Delta\mathcal{M} \in \mathbb{R}^{l\times l}$ and $\Delta\mathbf{d} \in \mathbb{R}^l$. Then the perturbed system is given by

$$(\mathcal{M} + \Delta\mathcal{M})(z + \Delta z) = (\mathbf{d} + \Delta\mathbf{d}).$$

In [35] and [32], the following formulae for the unstructured *MCN* and *CCN* for the solution of the above linear system are proposed:

$$\widetilde{\mathscr{M}}(z) := \lim_{\eta\to 0} \sup \left\{ \frac{\|\Delta z\|_\infty}{\eta\|z\|_\infty} : |\Delta\mathcal{M}| \le \eta|\mathcal{M}|, |\Delta\mathbf{d}| \le \eta|\mathbf{d}| \right\}$$

$$= \frac{\big\| |\mathcal{M}^{-1}||\mathcal{M}||z| + |\mathcal{M}^{-1}||\mathbf{d}| \big\|_\infty}{\|z\|_\infty}, \tag{4.6}$$

$$\widetilde{\mathscr{C}}(z) := \lim_{\eta\to 0} \sup \left\{ \frac{1}{\eta} \left\| \frac{\Delta z}{z} \right\|_\infty : |\Delta\mathcal{M}| \le \eta|\mathcal{M}|, |\Delta\mathbf{d}| \le \eta|\mathbf{d}| \right\}$$

$$= \left\| \frac{|\mathcal{M}^{-1}||\mathcal{M}||z| + |\mathcal{M}^{-1}||\mathbf{d}|}{|z|} \right\|_\infty. \tag{4.7}$$

REMARK 4.9. Considering $\mathbf{\Phi}_{\mathcal{S}_m} = I_{m^2}$ and $\mathbf{\Phi}_{\mathcal{S}_n} = I_{n^2}$ in the formula for $\mathscr{K}([x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}; \mathcal{L})$, we obtain the unstructured *NCN* for $\mathcal{M}z = \mathbf{d}$, where $\mathcal{M} \in \mathbb{R}^{l\times l}$, $\mathbf{d} \in \mathbb{R}^l$, and $l = (m+n)$, which is given by

$$\widetilde{\mathscr{K}}(z) := \lim_{\eta\to 0} \sup \left\{ \frac{\|\Delta z\|_2}{\eta\|z\|_2} : \left\| \begin{bmatrix} \Delta\mathcal{M} & \Delta\mathbf{d} \end{bmatrix} \right\|_F \le \eta \left\| \begin{bmatrix} \mathcal{M} & \mathbf{d} \end{bmatrix} \right\|_F \right\}$$

$$= \frac{\left\| \mathcal{M}^{-1} \begin{bmatrix} \mathcal{H} & -I_{m+n} \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} \mathcal{M} & \mathbf{d} \end{bmatrix} \right\|_F}{\|z\|_2}.$$

The next theorem compares the structured *NCN*, *MCN*, and *CCN* obtained in Theorem 4.5 and the unstructured counterparts defined above.

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THEOREM 4.10. *Let* $z = [x^\top,\, y^\top]^\top$ *and* $\mathbf{L} = I_{m+n}$. *Then, for the GSPP* (1.1) *with the structure* $\mathcal{L}$, *the following relations holds:*

$$\mathscr{K}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{K}}([x^\top,\, y^\top]^\top),$$
$$\mathscr{M}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{M}}([x^\top,\, y^\top]^\top),$$
$$\mathscr{C}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{C}}([x^\top,\, y^\top]^\top).$$

Proof. Since $\|\mathbf{\Phi}_\mathcal{L}\mathfrak{D}_\mathcal{L}^{-1}\|_2 = 1$, the proof follows similar to the proof method of Theorem 3.14. Hence, from Theorem 4.5 and Remark 4.9, we have

$$\mathscr{K}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{K}}([x^\top,\, y^\top]^\top).$$

Now, using the property that the matrices $\mathbf{\Phi}_{\mathcal{L}_i}$, $i = 1, 2$, have at most one nonzero entry in each row [24], and similar to Theorem 3.14, we obtain

$$|\mathbf{\Phi}_{\mathcal{L}_1}\mathrm{vec}_{\mathcal{L}_1}(A)| = |\mathbf{\Phi}_{\mathcal{L}_1}|\mathrm{vec}_{\mathcal{L}_1}(|A|) \qquad \text{and} \qquad |\mathbf{\Phi}_{\mathcal{L}_2}\mathrm{vec}_{\mathcal{L}_2}(D)| = |\mathbf{\Phi}_{\mathcal{L}_2}|\mathrm{vec}_{\mathcal{L}_2}(|D|).$$

Then

$$|\mathcal{M}^{-1}\mathcal{H}\mathbf{\Phi}_\mathcal{L}|\begin{bmatrix}\mathrm{vec}_{\mathcal{L}_1}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|C|)\\ \mathrm{vec}_{\mathcal{L}_2}(|D|)\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix} \leq |\mathcal{M}^{-1}||\mathcal{H}|\begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|C|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}$$
$$\leq |\mathcal{M}^{-1}|\begin{bmatrix}|x^\top|\otimes I_m & I_m \otimes |y^\top| & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & |x^\top|\otimes I_n & |y^\top|\otimes I_n\end{bmatrix}\begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\\ \mathrm{vec}(|C|)\\ \mathrm{vec}(|D|)\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}$$
$$= |\mathcal{M}^{-1}||\mathcal{M}|\begin{bmatrix}|x|\\ |y|\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}.$$

Consequently, by Theorem 4.5, we have

$$\mathscr{M}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \frac{\left\| |\mathcal{M}^{-1}||\mathcal{M}|\begin{bmatrix}|x|\\ |y|\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}\right\|_\infty}{\|[x^\top,\, y^\top]^\top\|_\infty},$$
$$\mathscr{C}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \left\| \frac{|\mathcal{M}^{-1}||\mathcal{M}|\begin{bmatrix}|x|\\ |y|\end{bmatrix} + |\mathcal{M}^{-1}|\begin{bmatrix}|f|\\ |g|\end{bmatrix}}{|\,[x^\top,\, y^\top]^\top\,|}\right\|_\infty.$$

Now, consider $l = m + n$, $z = [x^\top,\, y^\top]^\top$ and $\mathbf{d} = [f^\top,\, g^\top]^\top$ in (4.6) and (4.7). Then, from the above results, we obtain the following relations:

$$\mathscr{M}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{M}}([x^\top,\, y^\top]^\top) \quad \text{and} \quad \mathscr{C}([x^\top,\, y^\top]^\top;\, \mathcal{L}) \leq \widetilde{\mathscr{C}}([x^\top,\, y^\top]^\top).$$

Hence, the proof follows. ☐

5. Application to *WTRLS* problems. Consider the *WTRLS* problem (1.2) and let $\mathbf{r} = W(f - Qy)$. Then the minimization problem (1.2) can be expressed as the following augmented linear system

$$\widehat{\mathcal{M}}\begin{bmatrix}\mathbf{r}\\ y\end{bmatrix} := \begin{bmatrix}W^{-1} & Q\\ Q^\top & -\lambda I_n\end{bmatrix}\begin{bmatrix}\mathbf{r}\\ y\end{bmatrix} = \begin{bmatrix}f\\ \mathbf{0}\end{bmatrix}. \tag{5.1}$$

Identifying $A = W^{-1}$, $B = Q^{\mathsf{T}}$, $D = -\lambda I_n$, $x = \mathbf{r}$, and $g = \mathbf{0}$, we can see that the augmented system (5.1) is in the form of the *GSPP* (2.3). Therefore, finding the *CNs* of the *WTRLS* problem (1.2) is equivalent to the *CNs* of the *GSPP* (2.3) for y with $g = \mathbf{0}$. This can be accomplished by Theorem 3.4. Before that, we reformulate (2.2) (with $B = C$) as

$$\mathcal{M}^{-1} = \begin{bmatrix} M & N \\ K & S^{-1} \end{bmatrix},$$

where

$$M = A^{-1} + A^{-1}B^{\mathsf{T}}S^{-1}BA^{-1}, \qquad N = -A^{-1}B^{\mathsf{T}}S^{-1},$$
$$K = -S^{-1}BA^{-1}, \qquad S = D - BA^{-1}B^{\mathsf{T}}.$$

THEOREM 5.1. *Let y be the unique solution of the problem (1.2) and $\mathbf{r} = W(f - Qy)$. Then the structured NCN, MCN, and CCN for y, respectively, are given by*

$$\mathscr{K}^{rls}(y;\, \mathcal{E}) = \frac{\left\| \left[(\mathbf{r}^{\mathsf{T}} \otimes \widetilde{K})\boldsymbol{\Phi}_{\mathcal{S}_m}\mathfrak{D}_{\mathcal{S}_m}^{-1} \quad (K \otimes y^{\mathsf{T}} + \mathbf{r}^{\mathsf{T}} \otimes \widetilde{S}^{-1})\boldsymbol{\Phi}_{\mathcal{T}_{nm}}\mathfrak{D}_{\mathcal{E}}^{-1} \quad y^{\mathsf{T}} \otimes \widetilde{S}^{-1} \quad -\widetilde{K} \quad -\widetilde{S}^{-1} \right] \right\|_2}{\|y\|_2 / \left\| \left[\widehat{\mathcal{M}} \quad \mathbf{d} \right] \right\|_F},$$

$$\mathscr{M}^{rls}(y;\, \mathcal{E}) = \frac{\|\mathcal{N}_y\|_\infty}{\|y\|_\infty},$$

$$\mathscr{C}^{rls}(y;\, \mathcal{E}) = \left\| \mathbf{D}_y^{\dagger} \mathcal{N}_y \right\|_\infty,$$

where

$$\mathcal{N}_y = \left| (\mathbf{r}^{\mathsf{T}} \otimes \widetilde{K})\boldsymbol{\Phi}_{\mathcal{S}_m} \right| \mathrm{vec}_{\mathcal{S}}(|A|) + \left| ((K \otimes y^{\mathsf{T}}) + (\mathbf{r}^{\mathsf{T}} \otimes \widetilde{S}^{-1}))\boldsymbol{\Phi}_{\mathcal{T}_{nm}} \right| \mathrm{vec}_{\mathcal{T}}(|Q^{\mathsf{T}}|)$$
$$+ |\widetilde{S}^{-1}||D||y| + |K||f|,$$

$$\widetilde{K} = -\widetilde{S}^{-1}Q^{\mathsf{T}}W, \qquad \textit{and}$$

$$\widetilde{S} = -(\lambda I_n + Q^{\mathsf{T}}WQ).$$

Proof. Let $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix} \in \mathbb{R}^{n \times (m+n)}$, $A = W^{-1}, B = Q^{\mathsf{T}}, D = -\lambda I_n$, $x = \mathbf{r}$, and $g = \mathbf{0}$. Then, from Theorem 3.4, we have

$$\begin{aligned}
&\mathbf{L}\mathcal{M}^{-1} \left[\mathcal{R}\boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} \quad -I_{m+n} \right] \\
&= \begin{bmatrix} \widetilde{K} & \widetilde{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{R} & -I_{m+n} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{\mathcal{E}}\mathfrak{D}_{\mathcal{E}}^{-1} & \mathbf{0} \\ \mathbf{0} & I_{m+n} \end{bmatrix} \\
&= \left[(\mathbf{r}^{\mathsf{T}} \otimes K)\boldsymbol{\Phi}_{\mathcal{S}_m}\mathfrak{D}_{\mathcal{S}_m}^{-1} \quad (\widetilde{K} \otimes y^{\mathsf{T}} + \mathbf{r}^{\mathsf{T}} \otimes \widetilde{S}^{-1})\boldsymbol{\Phi}_{\mathcal{T}_{nm}}\mathfrak{D}_{\mathcal{T}_{nm}}^{-1} \quad y^{\mathsf{T}} \otimes \widetilde{S}^{-1} \quad -\widetilde{K} \quad -\widetilde{S}^{-1} \right].
\end{aligned}$$

Hence, the expression for $\mathscr{K}^{rls}(y;\, \mathcal{E})$ is obtained from Theorem 3.4. The rest of the proof follows in a similar manner. □

Since, in most cases of the *WTRLS* problem, the weighting matrix W and the regularization matrix $D = -\lambda I_n$ has no perturbation, we consider $\Delta A = \mathbf{0}$ and $\Delta D = \mathbf{0}$. Moreover, as $g = \mathbf{0}$, we assume $\Delta g = \mathbf{0}$. Then, the perturbation expansion in Lemma 2.4 can be reformulated as

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\mathcal{M}^{-1} \begin{bmatrix} I_m \otimes y^{\mathsf{T}} & -I_m \\ x^{\mathsf{T}} \otimes I_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta B) \\ \Delta f \end{bmatrix} = -\begin{bmatrix} \mathcal{R}_{rls} & -\begin{bmatrix} M \\ K \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta B) \\ \Delta f \end{bmatrix},$$

where

$$\mathcal{R}_{rls} = \begin{bmatrix} M \otimes y^{\mathsf{T}} + x^{\mathsf{T}} \otimes N \\ K \otimes y^{\mathsf{T}} + x^{\mathsf{T}} \otimes S^{-1} \end{bmatrix}.$$

Now, applying a similar method as in Section 3.2, we obtain the following expressions for the *NCN*, *MCN*, and *CCN* for $\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}$ when $B = C$ and $g = \mathbf{0}$.

THEOREM 5.2. *Let $\Delta B \in \mathcal{T}^{n\times m}$. With the above notations, the structured NCN, MCN, and CCN for the GSPP (2.3), respectively, are given by*

$$\begin{aligned}
\widehat{\mathscr{K}}(\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}) &:= \lim_{\eta\to 0} \sup \left\{ \frac{\|\mathbf{L}[\Delta x^{\mathsf{T}}, \Delta y^{\mathsf{T}}]^{\mathsf{T}}\|_2}{\eta \|\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}\|_2} : \left\| \begin{bmatrix} \Delta B & \Delta f \end{bmatrix} \right\|_F \leq \eta \left\| \begin{bmatrix} B & f \end{bmatrix} \right\|_F \right\} \\
&= \frac{\left\| \mathbf{L} \begin{bmatrix} \mathcal{R}_{rls} \mathbf{\Phi}_{\mathcal{T}_{nm}} \mathfrak{D}_{\mathcal{T}_{nm}}^{-1} & -\begin{bmatrix} M \\ K \end{bmatrix} \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} B & f \end{bmatrix} \right\|_F}{\|\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}\|_2}, \\
\widehat{\mathscr{M}}(\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}) &:= \lim_{\eta\to 0} \sup \left\{ \frac{\|\mathbf{L}[\Delta x^{\mathsf{T}}, \Delta y^{\mathsf{T}}]^{\mathsf{T}}\|_\infty}{\eta \|\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}\|_\infty} : \left| \begin{bmatrix} \Delta B & \Delta f \end{bmatrix} \right| \leq \eta \left| \begin{bmatrix} B & f \end{bmatrix} \right| \right\} \\
&= \frac{\left\| |\mathbf{L}\mathcal{R}_{rls} \mathbf{\Phi}_{\mathcal{T}_{nm}}| \mathrm{vec}_{\mathcal{T}}(|B|) + \left| \mathbf{L} \begin{bmatrix} M \\ K \end{bmatrix} \right| |f| \right\|_\infty}{\|\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}\|_\infty}, \\
\widehat{\mathscr{C}}(\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}) &:= \lim_{\eta\to 0} \sup \left\{ \frac{1}{\eta} \left\| \frac{\mathbf{L}[\Delta x^{\mathsf{T}}, \Delta y^{\mathsf{T}}]^{\mathsf{T}}}{\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}} \right\|_\infty : \left| \begin{bmatrix} \Delta B & \Delta f \end{bmatrix} \right| \leq \eta \left| \begin{bmatrix} B & f \end{bmatrix} \right| \right\} \\
&= \left\| \mathbf{D}^{\dagger}_{\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}} |\mathbf{L}\mathcal{R}_{rls} \mathbf{\Phi}_{\mathcal{T}_{nm}}| \mathrm{vec}_{\mathcal{T}}(|B|) + \mathbf{D}^{\dagger}_{\mathbf{L}[x^{\mathsf{T}}, y^{\mathsf{T}}]^{\mathsf{T}}} \left| \mathbf{L} \begin{bmatrix} M \\ K \end{bmatrix} \right| |f| \right\|_\infty.
\end{aligned}$$

Proof. For applying Lemma 2.3, we define

$$\begin{aligned}
&\widehat{\zeta} : \mathbb{R}^{m+n-1} \times \mathbb{R}^m \mapsto \mathbb{R}^{m+n} \\
&\widehat{\zeta}\left([\mathfrak{D}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(B)^{\mathsf{T}}, f^{\mathsf{T}}]^{\mathsf{T}} \right) := \mathbf{L} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{L}\mathcal{M}^{-1} \begin{bmatrix} f \\ \mathbf{0} \end{bmatrix}.
\end{aligned}$$

Then, the map $\widehat{\zeta}$ is continuously Fréchet differentiable at $[\mathfrak{D}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(B)^{\mathsf{T}}, f^{\mathsf{T}}]^{\mathsf{T}}$ with

$$\mathbf{d}\widehat{\zeta}\left([\mathfrak{D}_{\mathcal{T}_{nm}} \mathrm{vec}_{\mathcal{T}}(B)^{\mathsf{T}}, f^{\mathsf{T}}]^{\mathsf{T}} \right) = -\mathbf{L} \begin{bmatrix} \mathcal{R}_{rls} \mathbf{\Phi}_{\mathcal{T}_{nm}} \mathfrak{D}_{\mathcal{T}_{nm}}^{-1} & -\begin{bmatrix} M \\ K \end{bmatrix} \end{bmatrix}.$$

The rest of the proof follows similarly to Theorem 3.11. ∎

Using the above result, we can derive the following structured *CNs* for the problem (1.2), when the weighting matrix and the regularization matrix have no perturbation.

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COROLLARY 5.3. *The structured NCN, MCM, and CCN for the solution y of the WTRLS problem* (1.2)*, respectively, are given by*

$$\widehat{\mathscr{K}}^{rls}(y) = \frac{\left\| \left[(\widetilde{K} \otimes y^{\top} + \mathbf{r}^{\top} \otimes \widetilde{S}^{-1}) \mathbf{\Phi}_{\mathcal{T}_{nm}} \mathfrak{D}^{-1}_{\mathcal{T}_{nm}} \quad -\widetilde{K} \right] \right\|_2 \| [Q \quad f] \|_F}{\|y\|_2},$$

$$\widehat{\mathscr{M}}^{rls}(y) = \frac{\left\| |(\widetilde{K} \otimes y^{\top} + \mathbf{r}^{\top} \otimes \widetilde{S}^{-1}) \mathbf{\Phi}_{\mathcal{T}_{nm}} | \mathrm{vec}_{\mathcal{T}}(|Q^{\top}|) + |\widetilde{K}||f| \right\|_{\infty}}{\|y\|_{\infty}},$$

$$\widehat{\mathscr{C}}^{rls}(y) = \|\mathbf{D}_y^{\dagger} |(\widetilde{K} \otimes y^{\top} + \mathbf{r}^{\top} \otimes \widetilde{S}^{-1}) \mathbf{\Phi}_{\mathcal{T}_{nm}} | \mathrm{vec}_{\mathcal{T}}(|Q^{\top}|) + \mathbf{D}_y^{\dagger} |\widetilde{K}||f| \|_{\infty},$$

where $\widetilde{K} = \widetilde{S}^{-1} Q^{\top} W$ and $\widetilde{S} = -(\lambda I_n + Q^{\top} W Q)$.

Proof. Substituting $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix} \in \mathbb{R}^{n \times (m+n)}$, $B = Q^{\top}$, $A = W^{-1}$, $D = -\lambda I_n$, and $x = W(f - Qy)$ in Theorem 5.2, the proof follows. ☐

REMARK 5.4. We consider the Tikhonov regularization problem

$$\min_{w \in \mathbb{R}^n} \left\{ \|B^{\top} w - f\|_2^2 + \lambda \|Rw\|_2^2 \right\},$$

where R is the regularization matrix and $\lambda > 0$ a regularization parameter. Then, substituting $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix} \in \mathbb{R}^{n \times (m+n)}$, $A = I_m$, $D = -\lambda R^{\top} R$, $x = (f - B^{\top} w)$, and $y = w$ in Theorem 5.2, we can recover the structured *NCN*, *MCN*, and *CCN* formulae for a Toeplitz structure discussed in [29].

6. Numerical experiments. In order to verify the reliability of the proposed structured *CNs*, we present several numerical experiments in this section. All numerical tests are conducted using MATLAB R2023b on an Intel Core i7-10700 CPU, 2.90GHz, 16GB memory, with machine precision $\mu = 2.2 \times 10^{-16}$.

We construct the perturbations to the input data as follows:

$$\Delta A = 10^{-q} \cdot \Delta A_1 \odot A, \quad \Delta B = 10^{-q} \cdot \Delta B_1 \odot B, \quad \Delta C = 10^{-q} \cdot \Delta C_1 \odot C, \tag{6.1}$$

$$\Delta D = 10^{-q} \cdot \Delta D_1 \odot D, \quad \Delta f = 10^{-q} \cdot \Delta f_1 \odot f, \quad \Delta g = 10^{-q} \cdot \Delta g_1 \odot g, \tag{6.2}$$

where $\Delta A_1 \in \mathbb{R}^{m \times m}$, $\Delta B_1, \Delta C_1 \in \mathbb{R}^{n \times m}$, and $\Delta D_1 \in \mathbb{R}^{n \times n}$ are random matrices preserving the structures of original matrices. Here, $\odot$ represents the entrywise multiplication of two matrices of the same dimensions.

Suppose that $[x^{\top}, y^{\top}]^{\top}$ and $[\tilde{x}^{\top}, \tilde{y}^{\top}]^{\top}$ are the unique solutions of the original *GSPP* and the perturbed *GSPP*, respectively. To estimate an upper bound for the forward error in the solution, the normwise, mixed, and componentwise relative errors in $\mathbf{L}[x^{\top}, y^{\top}]^{\top}$, respectively, are defined by:

$$relk = \frac{\|\mathbf{L}[\tilde{x}^{\top}, \tilde{y}^{\top}]^{\top} - \mathbf{L}[x^{\top}, y^{\top}]^{\top}\|_2}{\|\mathbf{L}[x^{\top}, y^{\top}]^{\top}\|_2}, \quad relm = \frac{\|\mathbf{L}[\tilde{x}^{\top}, \tilde{y}^{\top}]^{\top} - \mathbf{L}[x^{\top}, y^{\top}]^{\top}\|_{\infty}}{\|\mathbf{L}[x^{\top}, y^{\top}]^{\top}\|_{\infty}},$$

$$relc = \left\| \frac{\mathbf{L}[\tilde{x}^{\top}, \tilde{y}^{\top}]^{\top} - \mathbf{L}[x^{\top}, y^{\top}]^{\top}}{\mathbf{L}[x^{\top}, y^{\top}]^{\top}} \right\|_{\infty}.$$

The following quantities

$$\eta_1 \cdot \mathscr{K}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}), \quad \eta_2 \cdot \mathscr{M}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}), \quad \eta_2 \cdot \mathscr{C}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}) \quad \text{and}$$
$$\eta_1 \cdot \mathscr{K}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}; \mathbb{S}), \quad \eta_2 \cdot \mathscr{M}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}; \mathbb{S}), \quad \eta_2 \cdot \mathscr{C}(\mathbf{L}[x^{\top}, y^{\top}]^{\top}; \mathbb{S}),$$

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with $\mathbb{S} = \{\mathcal{E}, \mathcal{L}\}$, are the estimated upper bounds of *relk*, *relm*, and *relc* obtained by the *CNs* in the unstructured and structured cases, respectively. Here, the quantities η_1 and η_2 are defined as [27]:

$$\eta_1 = \begin{cases} \dfrac{\left\|\begin{bmatrix}\Delta H & \Delta\mathbf{d}\end{bmatrix}\right\|_F}{\left\|\begin{bmatrix}H & \mathbf{d}\end{bmatrix}\right\|_F} & \text{when } \mathbb{S} = \mathcal{E}, \\[2ex] \dfrac{\left\|\begin{bmatrix}\Delta \mathcal{M} & \Delta\mathbf{d}\end{bmatrix}\right\|_F}{\left\|\begin{bmatrix}\mathcal{M} & \mathbf{d}\end{bmatrix}\right\|_F} & \text{when } \mathbb{S} = \mathcal{L}, \end{cases}$$

and

$$\eta_2 = \min\{\eta : \left|\begin{bmatrix}\Delta\mathcal{M} & \Delta\mathbf{d}\end{bmatrix}\right| \leq \eta \left|\begin{bmatrix}\mathcal{M} & \mathbf{d}\end{bmatrix}\right|\}.$$

We choose the matrix $\mathbf{L}$ as I_{m+n}, $\begin{bmatrix}I_m & \mathbf{0}\end{bmatrix}$, and $\begin{bmatrix}\mathbf{0} & I_n\end{bmatrix}$, so that the *CNs* for $[x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}$, x, and y, respectively, are obtained.

EXAMPLE 6.1. Consider the *GSPP* (2.3), where the data matrices A, B, D, f, and g are given as follows:

$$A = \begin{bmatrix} \epsilon_1 & \epsilon_1 & -0.01 & 10 & 10 & -30 & 30 & 10 & 30 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 & 10 & 10 & -30 & 30 & 10 \\ -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 & 10 & 10 & -30 & 30 \\ 10 & -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 & 10 & 10 & -30 \\ 10 & 10 & -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 & 10 & 10 \\ -30 & 10 & 10 & -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 & 10 \\ 30 & -30 & 10 & 10 & -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 & -0.01 \\ 10 & 30 & -30 & 10 & 10 & -0.01 & \epsilon_1 & \epsilon_1 & \epsilon_1 \\ 30 & 10 & 30 & -30 & 10 & 10 & -0.01 & \epsilon_1 & \epsilon_1 \end{bmatrix} \in \mathcal{S}_9,$$

$$B = \begin{bmatrix} 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -0.02 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -0.03 & -0.002 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\ -0.04 & -0.03 & -0.002 & 8 & 1 & 2 & 3 & 4 & 5 \\ -0.05 & -0.04 & -0.03 & -0.002 & 8 & 1 & 2 & 3 & 4 \end{bmatrix} \in \mathcal{T}^{4\times 9},$$

$$D = 0.05 * \text{std}(B^\mathsf{T}). * \texttt{randn}(n, n),$$

$f = [1, \dots, 1, m-1, 1]^\mathsf{T} \in \mathbb{R}^9$, and $g = \texttt{randn}(n, 1) \in \mathbb{R}^4$, where $\texttt{randn}(m, n)$ denotes the $m \times n$ random matrix generated by the MATLAB command $\texttt{randn}$ and std(B^T) denotes the standard deviation of B^T. Here, $m = 9$ and $n = 4$. For the perturbations to the input data constructed as in (6.1)–(6.2) with $q = 7$, $\Delta B_1 \in \mathcal{T}^{n\times m}$ is a randomly generated Toeplitz matrix, $\Delta A_1 = \frac{1}{2}(\widehat{A} + \widehat{A}^\mathsf{T})$, and $\widehat{A} \in \mathbb{R}^{m\times m}$, $\Delta D_1 \in \mathbb{R}^{n\times n}$ are random matrices.

The numerical results for different choices of ϵ_1 are reported in Tables 6.1–6.3 using the formulae presented in Theorems 3.4 and 3.11. The sizes of η_1 and η_2 are about 10^{-8} and 10^{-7}, respectively, for all cases. It can be observed that the structured *CNs*, in all cases, are much smaller (almost one order less) than their unstructured counterparts. Moreover, the estimated upper bounds proposed by the *CNs* for *relk*, *relm*, and *relc* are sharper in the structured case than in the unstructured ones. Notably, the structured *MCN* and *CCN* give sharper bounds than the *NCN* for the relative error as they are of the same order or one order larger than *relm*

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TABLE 6.1
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = I_{13}$ for Example 6.1.

ϵ_1	*relk*	$\mathscr{K}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}})$	$\mathscr{K}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}};\, \mathcal{E})$	*relm*	$\mathscr{M}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}})$	$\mathscr{M}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}};\, \mathcal{E})$	*relc*	$\mathscr{C}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}})$	$\mathscr{C}([x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}};\, \mathcal{E})$
10	7.2266e-07	2.6621e+03	2.5899e+03	6.7684e-07	1.3407e+02	7.4069e+01	7.5571e-06	1.9545e+03	9.1963e+02
10^{0}	1.1562e-06	2.5310e+03	2.4517e+03	8.2825e-07	1.2620e+02	6.1160e+01	7.5375e-06	1.1485e+03	5.5659e+02
10^{-1}	1.7604e-06	2.9961e+03	2.9210e+03	1.7656e-06	1.4518e+02	7.4370e+01	1.1800e-05	1.1229e+03	5.0019e+02
10^{-2}	5.2202e-07	1.8120e+03	1.7515e+03	5.1969e-07	1.1842e+02	5.5675e+01	5.1819e-06	1.4140e+03	8.8193e+02
10^{-2}	6.1932e-07	3.1643e+03	3.0814e+03	5.5990e-07	1.6201e+02	8.4378e+01	3.5558e-06	1.2309e+03	7.3506e+02
10^{-3}	2.4807e-06	1.6465e+03	1.5744e+03	2.0678e-06	1.1802e+02	7.9648e+01	1.3033e-05	1.2165e+03	7.7459e+02
10^{-4}	1.2229e-06	3.0661e+03	2.9578e+03	1.3129e-06	1.5980e+02	9.0334e+01	6.3654e-06	1.2789e+03	9.3070e+02

TABLE 6.2
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = \begin{bmatrix} I_9 & \mathbf{0} \end{bmatrix}$ for Example 6.1.

ϵ_1	*relk*	$\mathscr{K}(x)$	$\mathscr{K}(x;\, \mathcal{E})$	*relm*	$\mathscr{M}(x)$	$\mathscr{M}(x;\, \mathcal{E})$	*relc*	$\mathscr{C}(x)$	$\mathscr{C}(x;\, \mathcal{E})$
10	7.9185e-07	2.9377e+03	2.8591e+03	5.8903e-07	1.1477e+02	6.7631e+01	7.5571e-06	4.9310e+02	2.6889e+02
10^{0}	9.0322e-07	2.0236e+03	1.9605e+03	9.7497e-07	1.3501e+02	6.2699e+01	2.7439e-06	3.3927e+02	1.5957e+02
10^{-1}	1.9559e-06	3.4895e+03	3.4010e+03	1.8283e-06	1.4073e+02	7.9523e+01	1.1800e-05	1.1229e+03	5.0019e+02
10^{-2}	3.6024e-07	1.3749e+03	1.3294e+03	4.2107e-07	9.5080e+01	4.5826e+01	5.9700e-07	1.4140e+03	8.8193e+02
10^{-2}	6.4001e-07	2.9172e+03	2.8412e+03	7.1460e-07	1.5484e+02	8.2478e+01	2.9604e-06	1.1599e+03	6.2412e+02
10^{-3}	1.4527e-06	7.9932e+02	7.6489e+02	2.1781e-06	9.3409e+01	6.0897e+01	6.8343e-06	2.9309e+02	1.9108e+02
10^{-4}	8.9097e-07	3.1005e+03	2.9895e+03	1.0788e-06	1.3093e+02	6.4355e+01	4.5008e-06	1.2789e+03	9.3070e+02

TABLE 6.3
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_4 \end{bmatrix}$ for Example 6.1.

ϵ_1	*relk*	$\mathscr{K}(y)$	$\mathscr{K}(y;\, \mathcal{E})$	*relm*	$\mathscr{M}(y)$	$\mathscr{M}(y;\, \mathcal{E})$	*relc*	$\mathscr{C}(y)$	$\mathscr{C}(y;\, \mathcal{E})$
10	7.2081e-07	2.6548e+03	4.5585e+02	6.7684e-07	1.3407e+02	7.4069e+01	5.3298e-06	1.9545e+03	9.1963e+02
10^{0}	1.1662e-06	2.5513e+03	4.1391e+02	8.2825e-07	1.2620e+02	6.1160e+01	7.5375e-06	1.1485e+03	5.5659e+02
10^{-1}	1.7531e-06	2.9773e+03	6.4241e+02	1.7656e-06	1.4518e+02	7.4370e+01	1.9676e-06	1.9721e+02	1.0083e+02
10^{-2}	5.3000e-07	1.8351e+03	3.2251e+02	5.1969e-07	1.1842e+02	5.5675e+01	5.1819e-06	8.0176e+02	4.6153e+02
10^{-2}	6.1861e-07	3.1725e+03	5.2490e+02	5.5990e-07	1.6201e+02	8.4378e+01	3.5558e-06	1.2309e+03	7.3506e+02
10^{-3}	2.5575e-06	1.7068e+03	2.3662e+02	2.0678e-06	1.1802e+02	7.9648e+01	1.3033e-05	1.2165e+03	7.7459e+02
10^{-4}	1.2378e-06	3.0646e+03	6.8237e+02	1.3129e-06	1.5980e+02	9.0334e+01	6.3654e-06	2.9429e+02	1.8874e+02

and *relc*, respectively. This indicates that it is more preferable to apply $\mathscr{M}(\mathbf{L}[x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}};\, \mathcal{E})$ and $\mathscr{C}(\mathbf{L}[x^{\mathsf{T}},\, y^{\mathsf{T}}]^{\mathsf{T}};\, \mathcal{E})$ to measure the true conditioning of the *GSPP* (2.3).

EXAMPLE 6.2. In this example, we consider the *GSPP* (2.3) arising from the *WTRLS* problem [11]. Here $m = n$, and the Toeplitz matrix B is given as

$$B = [b_{ij}] \in \mathcal{T}^{n\times n} \qquad \text{with} \quad b_{ij} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(i-j)^2}{2\sigma^2}},$$

$A \in \mathbb{R}^{n\times n}$ is set to be a positive diagonal random matrix, and $D = -\nu I_n$ ($\nu > 0$). The right-hand side vector is taken as $\mathbf{d} = \texttt{randn}(2n, 1) \in \mathbb{R}^{2n}$.

We select $\sigma = 2$ and $\nu = 0.001$ as in [10]. We set $q = 8$ and construct perturbation matrices as in Example 6.1. In all cases, we observed $\eta_1 \approx \mathcal{O}(10^{-9})$ and $\eta_2 \approx \mathcal{O}(10^{-8})$. The numerical results for the structured and unstructured *NCN*, *MCN*, and *CCN* and the exact relative errors are reported in Tables 6.4–6.6 for different values of n.

We use Theorem 3.11 and Remark 3.13 to compute the structured *CNs* and Theorem 3.4 to compute the unstructured *CNs*. The results presented in Tables 6.4–6.6 reveal that the structured *NCN*, *MCM*, and *CCN* are much smaller than the unstructured ones for all values n. Specifically, for large matrices (with dimensions of $\mathcal{M}$ taken up to 400), the structured *CNs* are approximately an order of magnitude smaller than the unstructured ones, showcasing the superiority of the proposed structured *CNs*.

TABLE 6.4
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = I_{2n}$ for Example 6.2.

$n = m$	*relk*	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relm*	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relc*	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$
50	4.1808e-07	2.8177e+04	2.4798e+04	4.6643e-07	1.4438e+03	5.3588e+02	3.3769e-05	5.7501e+04	2.0978e+04
100	2.4188e-07	4.8911e+03	4.6982e+03	2.5583e-07	1.4305e+02	4.1253e+01	1.4497e-05	1.1440e+04	2.4661e+03
150	5.3749e-07	1.9378e+04	1.7985e+04	6.1184e-07	5.5108e+02	1.3986e+02	2.4998e-04	3.6099e+05	8.1050e+04
200	7.5206e-07	3.2373e+04	9.4706e+03	8.8297e-07	1.0386e+03	4.5302e+02	9.7741e-05	2.0373e+05	4.4730e+04

TABLE 6.5
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = \begin{bmatrix} I_n & \mathbf{0} \end{bmatrix}$ for Example 6.2.

$n = m$	*relk*	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relm*	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relc*	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$
50	3.8496e-07	2.7536e+04	2.4281e+04	4.1735e-07	1.2042e+03	4.6611e+02	3.2289e-06	1.0158e+04	3.1020e+03
100	3.0293e-07	7.0491e+03	6.7372e+03	3.7151e-07	2.7486e+02	7.2328e+01	6.2591e-06	6.5226e+03	1.2953e+03
150	7.2376e-07	2.7944e+04	2.5692e+04	8.4422e-07	7.5098e+02	2.0082e+02	6.3056e-05	3.6099e+05	8.1050E+04
200	8.0141e-07	3.6034e+04	3.2041e+04	7.4664e-07	1.0283e+03	4.1526e+02	9.7741e-05	2.0067e+05	4.4730e+04

TABLE 6.6
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}$ for Example 6.2.

$n = m$	*relk*	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{K}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relm*	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{M}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$	*relc*	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T})$	$\mathscr{C}([x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T}; \mathcal{E})$
50	4.2087e-07	2.8235e+04	7.2137e+03	4.6643e-07	1.4438e+03	5.3588e+02	3.3769e-05	5.7501e+04	2.0978e+04
100	2.3999e-07	4.8471e+03	1.1173e+03	2.5583e-07	1.4305e+02	4.1253e+01	1.4497e-05	1.1440e+04	2.4661e+03
150	5.2664e-07	1.8878e+04	5.6962e+03	6.1184e-07	5.5108e+02	1.3986e+02	2.4998e-04	9.0978e+04	3.1977e+04
200	7.4779e-07	3.2089e+04	9.2643e+03	8.8297e-07	1.0386e+03	4.5302e+02	5.4363e-05	2.0373e+05	3.4820e+04

EXAMPLE 6.3. In this example, we consider the *GSPP* arising from the discretization of the Stokes equation by an upwind scheme [5]:

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \tilde{f} & \text{in } \mathbf{\Omega}, \\ \nabla \cdot \mathbf{u} = \tilde{g} & \text{in } \mathbf{\Omega}, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\mathbf{\Omega}, \\ \int_{\mathbf{\Omega}} p(x)dx = 0, \end{cases} \tag{6.3}$$

where $\mathbf{\Omega} = (0,1) \times (0,1) \in \mathbb{R}^2$, $\partial\mathbf{\Omega}$ is the boundary of $\mathbf{\Omega}$, μ is the viscosity parameter, Δ is the Laplace operator, ∇ represents the gradient, $\nabla\cdot$ is the divergence, $\mathbf{u}$ is the velocity vector, and p is the scalar function representing the pressure. By discretizing (6.3), we obtain the

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TABLE 6.7

Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = I_{m+n}$ for Example 6.3.

r	*relk*	$\widetilde{\mathscr{K}}(z)$	$\mathscr{K}([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T};\, \mathcal{L})$	*relm*	$\widetilde{\mathscr{M}}(z)$	$\mathscr{M}([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T};\, \mathcal{L})$	*relc*	$\widetilde{\mathscr{C}}(z)$	$\mathscr{C}([x^\mathsf{T},\, y^\mathsf{T}]^\mathsf{T};\, \mathcal{L})$
3	4.6396e-08	1.0866e+02	1.0325e+02	9.2530e-08	1.0315e+02	8.3160e+01	9.2530e-08	1.0315e+02	8.3160e+01
4	1.0295e-07	1.2567e+03	1.1946e+03	4.0283e-07	1.1158e+03	8.9754e+02	4.0283e-07	1.1158e+03	8.9754e+02
5	1.3490e-07	1.1905e+03	1.1256e+03	5.3926e-07	1.2062e+03	9.4423e+02	5.3926e-07	1.2062e+03	9.4423e+02
6	1.1442e-07	1.4744e+03	1.3738e+03	3.8692e-07	1.1110e+03	8.5833e+02	3.8692e-07	1.1110e+03	8.5833e+02
7	1.4617e-07	2.5853e+03	2.5366e+03	5.2901e-07	1.1384e+03	9.1026e+02	5.2901e-07	1.1384e+03	8.1026e+02
8	5.1493e-08	2.6605e+03	2.1679e+03	2.0993e-07	1.0634e+03	8.8527e+02	2.0993e-07	1.0634e+03	8.8527e+02
9	7.7302e-08	1.2791e+03	1.0043e+03	2.5382e-07	1.0339e+03	8.2775e+02	2.5382e-07	1.0339e+03	8.2775e+02
10	1.2621e-07	1.5807e+04	1.5205e+04	4.5006e-07	1.0406e+04	8.3004e+03	4.5006e-07	1.0406e+04	8.3004e+03

GSPP (1.1) with

$$A = \begin{bmatrix} I_r \otimes T + T \otimes I_r & \mathbf{0} \\ \mathbf{0} & I_r \otimes T + T \otimes I_r \end{bmatrix} \in \mathbb{R}^{2r^2 \times 2r^2},$$
$$B^\mathsf{T} = \begin{bmatrix} I_r \otimes G \\ G \otimes I_r \end{bmatrix} \in \mathbb{R}^{2r^2 \times r^2},$$
$$C = -B, \qquad D = \mathbf{0},$$

where

$$T = \frac{\mu}{h^2}\,\mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{r\times r} \qquad \text{and} \qquad G = \frac{1}{h}\,\mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{r\times r}.$$

Here, $\mathrm{tridiag}(a, b, c)$ denotes the tridiagonal matrix with diagonal entries b, sub-diagonal entries a and super-diagonal entries c. Note that for this test problem $\mu = 0.1$, $m = 2r^2$, and $n = r^2$, and we choose $\mathbf{d} = [f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T}$ so that the exact solution is given by $z = [1, 1, \dots, 1]^\mathsf{T} \in \mathbb{R}^{m+n}$. To avoid making A too sparse, we add $X = 0.5(X_1 + X_1^\mathsf{T})$ to A, where $X_1 = \mathtt{sprandn}(m, n, 0.1)$. Here, $\mathtt{sprandn}(m, n, 0.1)$ denotes an $m \times n$ sparse random matrix with a density of 0.1.

The perturbations in the input data are constructed as in (6.1)–(6.2) with $q = 8$, $\Delta A_1 = \frac{1}{2}(\widehat{A} + \widehat{A}^\mathsf{T})$, where $\widehat{A} \in \mathbb{R}^{m\times m}$ is a random matrix. The numerical result for the structured and unstructured *NCN*, *MCN*, and *CCN* with $\mathbf{L} = I_{m+n}$ are presented in Table 6.7 for $r = 3, 4, \dots, 10$. Since the block matrix A is symmetric, we compute the structured *NCN*, *MCN* and *CCN* using Theorem 4.5 and Remark 4.8 with $D = \mathbf{0}$. The unstructured *CNs* are computed using (4.6), (4.7), and Remark 4.9.

We observe $\eta_1 \approx \mathcal{O}(10^{-9})$ and $\eta_2 \approx \mathcal{O}(10^{-8})$ in all cases. The results reported in Table 6.7 demonstrate that for all values of r, the structured *MCN* and *CCN* are almost one order smaller than the unstructured *MCN* and *CCN*. Moreover, the estimated upper bounds of the relative error of the solution produced by the structured *CNs* are sharper than those obtained by the unstructured *CNs* irrespective of the increasing size of $\mathcal{M}$ (taken up to 300).

EXAMPLE 6.4. Consider the following second-order ODE [4]:

$$\begin{cases} u_1^{''}(t) - u_2^{'}(t) - \dfrac{1}{t} u_2(t) = \mathbf{0}, \\ u_1^{'}(t) - \dfrac{1}{t} u_1(t) + u_2^{''}(t) - \dfrac{2}{t^2} u_2(t) = \sigma(t), \qquad 0 < t < 1, \\ u_1(0) = 0, \quad u_1(1) = 0, \quad u_1^{'}(0) = 0, \quad \text{and} \quad u_2(0) = 0, \end{cases} \tag{6.4}$$

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where $\sigma(t) = 3t^3 - 4t^2 + 13t - 2/t$ and the exact solutions are $u_1(t) = t^2(1-t)^2$ and $u_2(t) = 3t^3 - 4t^2 + t$.

The sinc discretization [4] of the second-order ODE system (6.4) yields the *GSPP* (1.1) (with $m = n$). The block matrices A, B, C, and D are computed using the formulae provided in [4, pp. 114–115]. To exploit the Toeplitz structures for A and D, we define them as follows:

$$A = D = T_1 + T_2 \in \mathbb{R}^{n\times n}.$$

The other block matrices are given by

$$B^{\mathsf{T}} = \frac{1}{2}(K_1T_1 + T_1K_1) + K_2 \in \mathbb{R}^{n\times n} \quad \text{and} \quad C = -\frac{1}{2}(K_1T_1 + T_1K_1) + K_3 \in \mathbb{R}^{n\times n},$$

where $T_1, T_2 \in \mathbb{R}^{n\times n}$ are defined by

$$T_1 = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \dots & \frac{(-1)^{n-1}}{n-1} \\ 1 & 0 & \ddots & \ddots & \vdots \\ -\frac{1}{2} & 1 & \vdots & -1 & \frac{1}{2} \\ \vdots & \ddots & \ddots & 0 & -1 \\ -\frac{(-1)^{n-1}}{n-1} & \dots & -\frac{1}{2} & 1 & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} \frac{\pi^2}{3} & -2 & \frac{2}{2^2} & \dots & \frac{2(-1)^{n-1}}{(n-1)^2} \\ -2 & \frac{\pi^2}{3} & \ddots & \ddots & \vdots \\ \frac{2}{2^2} & -2 & \vdots & -2 & \frac{2}{2^2} \\ \vdots & \ddots & \ddots & \frac{\pi^2}{3} & -2 \\ \frac{2(-1)^{n-1}}{(n-1)^2} & \dots & \frac{2}{2^2} & -2 & \frac{\pi^2}{3} \end{bmatrix},$$

and

$$\begin{aligned}
&K_1 := \mathbf{D}_{\chi_1}, \quad K_C^i := \mathbf{D}_{\chi_C^i}, \quad K_G^i := \mathbf{D}_{\chi_G^i}, \quad K_i = \frac{1}{2}(K_C^i + K_G^i), \\
&\chi_1 = [g_1(t_{-N}), g_1(t_{-N+1}), \dots, g_1(t_N)]^{\mathsf{T}}, \\
&\chi_C^i := [g_C^i(t_{-N}), g_C^i(t_{-N+1}), \dots, g_C^i(t_N)]^{\mathsf{T}}, \qquad i = 2, 3, \\
&\chi_G^i := [g_G^i(t_{-N}), g_G^i(t_{-N+1}), \dots, g_G^i(t_N)]^{\mathsf{T}}, \qquad i = 2, 3, \\
&g_1 = h\frac{\mu_1}{\phi'}, \\
&g_C^2 = g_C^3 = -h^2\frac{\mu_0}{(\phi')^2}, \qquad g_G^2 = -h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' + \frac{\mu_0}{(\phi')^2}\right), \\
&g_G^3 = h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' - \frac{\mu_0}{(\phi')^2}\right).
\end{aligned}$$

The functions $\mu_1(t), \mu_0(t)$, and $\phi(t)$ are given by $\mu_1(t) = 1$, $\mu_0(t) = -\frac{1}{t}$, and $\phi(t) = \ln(t/(1-t))$. Moreover, $n = 2N+1$, $h = \pi/\sqrt{2N}$, and $t_k = \phi^{-1}(kh)$. Further, we select $f = \mathbf{0} \in \mathbb{R}^n$ and $g = \texttt{randn}(n,1) \in \mathbb{R}^n$ using the MATLAB command `randn`.

We take $q = 7$ and generate the perturbation matrices as in (6.1)–(6.2) with ΔA and ΔD being Toeplitz. The numerical results for the structured and unstructured *CNs* and the

TABLE 6.8
Comparison of the unstructured and structured NCN, MCN, and CCN with their corresponding relative errors when $\mathbf{L} = I_{2n}$ for Example 6.4.

n	*relk*	$\widetilde{\mathscr{K}}(z)$	$\mathscr{K}([x^\mathsf{T},\ y^\mathsf{T}]^\mathsf{T};\ \mathcal{L})$	*relm*	$\widetilde{\mathscr{M}}(z)$	$\mathscr{M}([x^\mathsf{T},\ y^\mathsf{T}]^\mathsf{T};\ \mathcal{L})$	*relc*	$\widetilde{\mathscr{C}}(z)$	$\mathscr{C}([x^\mathsf{T},\ y^\mathsf{T}]^\mathsf{T};\ \mathcal{L})$
41	2.1668e-07	8.4401e+02	8.3043e+02	1.3862e-06	1.4301e+02	5.2023e+01	9.6646e-05	1.1794e+04	5.2908e+03
81	1.6126e-07	1.5963e+03	1.2802e+03	1.2767e-07	2.0050e+02	3.0540e+01	2.5905e-05	4.8140e+04	1.0074e+03
121	4.4422e-07	3.2151e+03	3.2054e+03	3.3468e-07	4.0321e+02	9.9676e+01	5.6058e-06	1.7998e+04	8.6974e+03
161	4.6746e-07	6.4824e+03	4.4279e+03	4.2745e-07	8.7713e+02	9.7876e+01	1.8302e-05	1.1962e+05	9.1779e+04
201	1.1583e-06	1.0882e+04	1.0725e+04	9.4135e-07	1.0197e+03	7.0819e+02	1.2433e-05	1.2847e+05	8.9705e+04

relative errors for $N = 20, 40, 60, 80, 100$ are reported in Table 6.8. We find that for all cases, η_1 and η_2 are approximately of order 10^{-8} and 10^{-7}, respectively. We compute the structured *NCN*, *MCN*, and *CCN* using Theorem 4.5, where the block matrices A and D have Toeplitz structures. The unstructured *CNs* are computed using (4.6), (4.7), and Remark 4.9. The solution $z = [x^\mathsf{T}, y^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2n}$ is computed using the MATLAB function $\mathcal{M}\backslash\mathbf{d}$, where $\mathbf{d} = [f^\mathsf{T}, g^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2n}$.

The results in Table 6.8 illustrate that, for all values of N, the structured *CNs* provide sharper upper bounds for the relative error in the solution. Furthermore, the structured *MCN* and *CCN* are nearly an order of magnitude smaller than the unstructured ones, even as the size of the matrix $\mathcal{M}$ increases (taken up to 402).

7. Conclusions. In this paper, by considering structure-preserving perturbations of the block matrices, we have investigated the structured *NCN*, *MCN*, and *CCN* for the linear function $\mathbf{L}[x^\mathsf{T},\ y^\mathsf{T}]^\mathsf{T}$ of the solution of *GSPPs*. We present compact formulae of structured *CNs* for $\mathbf{L}[x^\mathsf{T},\ y^\mathsf{T}]^\mathsf{T}$ in two cases. First, when $B = C$ is Toeplitz and A is symmetric. Second, when $B \neq C$ and the matrices A and D possess linear structures. Furthermore, we have obtained unstructured *CN* formulae for $B = C$, which generalize previous results for the *CNs* of *GSPPs* when $\mathbf{L}$ is I_{m+n}, $\begin{bmatrix} I_m & \mathbf{0} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}$. Additionally, relations between structured and unstructured *CNs* are obtained. It is found that the structured *CNs* are always smaller than their unstructured counterparts. An application of the obtained structured *CN* formulae is provided for finding the structured *CNs* for *WTRLS* problems, and they are also used to retrieve some prior found results for Tikhonov regularization problems. Numerical experiments are performed to validate the theoretical findings pertaining to the proposed structured *CNs*. Moreover, empirical investigations indicate that the proposed structured *MCN* and *CCN* give much more accurate error estimations of the solution of *GSPPs* compared to unstructured *CNs*.

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