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ERROR ANALYSIS OF A JACOBI MODIFIED PROJECTION-TYPE METHOD FOR WEAKLY SINGULAR VOLTERRA–HAMMERSTEIN INTEGRAL EQUATIONS*

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Abstract. The paper proposes polynomial-based projection-type and modified projection-type methods to solve weakly singular Volterra–Hammerstein integral equations of the second kind. Jacobi polynomials are used as basis functions. This type of equations often exhibits singular behavior at the left endpoint of the integration interval, and the exact solutions are typically nonsmooth. In the method under consideration, the independent variable is first transformed to provide a new integral equation with a smoother solution, allowing the Jacobi spectral method to be easily applied to the transformed equation and a full convergence analysis of the method to be performed. In different numerical tests, the effectiveness of the proposed approach is demonstrated.

Key words. Volterra–Hammerstein integral equations, Jacobi polynomials, weakly singular kernels, orthogonal projection, interpolatory projection, superconvergence

AMS subject classifications. 45B05, 45G10, 65R20

1. Introduction.

Consider the nonlinear weakly singular Volterra integral equation of Hammerstein type defined by

$$Y(z) = F(z) + \int_0^z \frac{1}{(z-x)^\gamma} K(z,x)\Psi(x,Y(x))dx, \qquad z \in [0,T], \quad 0 < \gamma < 1, \tag{1.1}$$

where the kernel function K and the source function F are given smooth functions, with Ψ the known nonlinear function and Y the unknown function to be found in a Banach space $\mathbb{X}$. The use of this type of integral equations can be found in many areas, for example, gas absorption, heat conduction, heat transfer, boundary layer problems, and many other scientific and technological fields [11, 21, 22]. The numerical analysis of (1.1) is not simple, this is due to the fact that the solutions usually exhibit a weak singularity at the point of integration $z = 0$ even when the inhomogeneous term F is regular. In general, this type of integral equations have nonsmooth solutions. Several authors have established numerical methods for approximating the nonlinear weakly singular Volterra integral equations in a number of works. A brief review of some of these techniques will follow.

In [8], Brunner proposed the collocation approach for nonlinear singular Volterra integral equations on a graded mesh, whereas the fully discrete collocation method is analyzed in [13]. In [23], the authors addressed the use of a hybrid collocation approach for nonlinear Volterra integral equations with weakly singular kernels of type (1.1). In [4], several analytical and computational approaches are presented for a class of nonlinear singular Volterra–Hammerstein integral equations whose exact solutions are typically nonsmooth. A study of extrapolation methods for the numerical solution of weakly singular nonlinear Volterra integral equations is considered in [24], whereas a Nyström-type method was studied in [7] after a smoothing transformation. In a study by Allouch et al. [6], Galerkin-type and modified Galerkin-type methods were proposed to solve weakly singular Hammerstein integral equations with logarithmic kernels using graded-mesh methods. We also refer to [16], where Galerkin and multi-Galerkin methods were applied obtaining superconvergence results. However, all of these piecewise

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polynomial-based projection methods indirectly lead to a large system of nonlinear equations, which increases the cost of numerical computations. To reduce computational complexity, in this article, we propose to use finite-dimensional approximation subspaces $\mathbb{X}_n$ given as global polynomial subspaces of $\mathbb{X}$. Specifically, Jacobi or Legendre polynomial subspaces can be employed for this purpose, effectively reducing the computational complexity compared to piecewise polynomial bases. Additionally, Jacobi and Legendre spectral methods offer high accuracy of the approximate solutions.

In the past decade, researchers have paid more attention on spectral methods for the numerical solution of Volterra integral equations. In [3], S. S. Allaei et al. discussed the convergence analysis of Jacobi spectral collocation methods for integral equations of type (1.1), where nonsmooth solutions were considered. In the method under consideration, an independent variable transformation is added to achieve a new equation with a smoother solution. Inspired by the work of [3], the authors introduced in [14] Galerkin and multi-Galerkin methods based on a sequence of orthogonal projectors and obtained superconvergence results. These methods have also been analyzed in [17] to find the approximate solutions of an integral equation of type (1.1), and the authors obtained a convergence analysis in two different cases where the exact solution was sufficiently smooth and nonsmooth. In [1], Nili Ahmadabadi et al. discussed the error analysis for weakly singular Volterra–Hammerstein integral equations based on the tau-approximation method.

In [20], Mandal and Nelakanti discussed Galerkin and multi-Galerkin methods using Legendre polynomials to obtain superconvergence for the numerical solution of weakly Fredholm–Hammerstein integral equations with kernels of algebraic and logarithmic type. Note that the discrete version of the projection and multi-projection methods are analyzed by the same authors in [19].

The aim of this work is to investigate the projection-type and modified projection-type methods for solving equation (1.1) using global polynomial basis functions. Here, we use Jacobi polynomials, which possess the property of orthogonality while being easily generated recursively. As a first step, a variable transformation is performed on the original equation to obtain a new equation with a smoother solution. We use variable and function transformations to change the integration domain from $[0, z]$ to $[-1, 1]$ so that the Jacobi polynomials theory can be used and superconvergence results may be obtained.

The motivation to consider Jacobi spectral methods in this article is to obtain better superconvergence results as in the case of piecewise polynomial subspaces while solving a smaller nonlinear system in comparison to the piecewise polynomial case. Another motivation to use this approach is its ability to incorporate the singularity of the kernel of equation (1.1) into the weight function, thus improving the superconvergence results.

The layout of the paper is as follows. In Section 2, we introduce the spectral approaches for the Volterra–Hammerstein integral equations (1.1). As a result, a set of algebraic equations is obtained, and the solution of the considered problem is presented. The convergence analysis will be carried out in Section 3, and implementation details of the proposed methods are provided in Section 4. The numerical results in Section 5 will be used to validate the theoretical findings obtained in Section 3. Finally, in Section 6, we give a conclusion.

2. Preliminaries and notations.

It is widely recognized that spectral methods are efficient tools to find the numerical approximate solution of differential equations with smooth solutions. Therefore, in order to make this approach practical, the following variable transformations should be used as in [3]:

$$x = t^r \quad \text{and} \quad z = s^r, \qquad r > 1, \quad r \in \mathbb{N}.$$

The integral equation of (1.1) is then transformed into the following equation:

$$\overline{Y}(s) = \overline{F}(s) + r\int_0^s \frac{t^{r-1}}{(s^r - t^r)^\gamma}\overline{K}(s,t)\overline{\Psi}(t,\overline{Y}(t))dt, \quad s \in [0, T^{1/r}], \quad 0 < \gamma < 1, \tag{2.1}$$

where

$$\overline{Y}(s) = Y(s^r), \qquad \overline{F}(s) = F(s^r),$$
$$\overline{K}(s,t) = K(s^r, t^r), \qquad \overline{\Psi}(t, \overline{Y}(t)) = \Psi(t^r, Y(t^r)).$$

Based on the analysis in [3], equation (2.1) has a unique solution on the interval $[0, T^{1/r}]$ and can be expressed as

$$\overline{Y}(s) = \overline{F}(s) + \int_0^s s^{-1}\phi(s^{-1}t)g(s,t,\overline{Y}(t))dt,$$

where

$$\phi(\eta) = \eta^{r-1}(1-\eta^r)^{-\gamma} \in L^1(0, T^{1/r}), \qquad g(s,t,\overline{Y}(t)) = rs^{r(1-\gamma)}\overline{K}(s,t)\overline{\Psi}(t,\overline{Y}(t)).$$

We define $\Delta_T = \{(s,t),\ \ 0 \le s \le t \le T\}$. Let $\overline{F} \in \mathcal{C}^m([0, T^{1/r}])$, $\overline{K} \in \mathcal{C}^m(\Delta_{T^{1/r}})$, and $\overline{\Psi} \in \mathcal{C}^m([0, T^{1/r}] \times D)$, where r is an integer such that $r > \frac{m}{1-\gamma}$ and m is a positive integer, with $D \subset \mathbb{R}$. Then $g \in \mathcal{C}^m(\Delta_{T^{1/r}} \times D)$, and as a result, by [3, Lemma 1], $\overline{Y} \in \mathcal{C}^m([0, T^{1/r}])$.

To apply the orthogonal polynomials theory, we can use the following linear transformation as described in [3]:

$$s = T^{1/r}\frac{x+1}{2}, \quad t = T^{1/r}\frac{\sigma+1}{2}, \qquad x, \sigma \in [-1, 1]. \tag{2.2}$$

Then (2.1) becomes

$$y(x) = f(x) + r\left(\frac{T^{1/r}}{2}\right)^{r(1-\gamma)}\int_{-1}^x \frac{(\sigma+1)^{r-1}}{((x+1)^r - (\sigma+1)^r)^\gamma}\widetilde{K}(x,\sigma)\psi(\eta, y(\sigma))d\sigma, \qquad x \in [-1, 1], \tag{2.3}$$

where

$$y(x) = \overline{Y}\left(T^{1/r}\frac{x+1}{2}\right), \qquad f(x) = \overline{F}\left(T^{1/r}\frac{x+1}{2}\right),$$
$$\widetilde{K}(x,\sigma) = \overline{K}\left(T^{1/r}\frac{x+1}{2}, T^{1/r}\frac{\sigma+1}{2}\right),$$
$$\psi(\sigma, y(\sigma)) = \overline{\Psi}\left(T^{1/r}\frac{\sigma+1}{2}, \overline{Y}\left(T^{1/r}\frac{\sigma+1}{2}\right)\right).$$

Using the formula

$$u^n - v^n = (u-v)(u^{n-1} + u^{n-2}v + \ldots + v^{n-1}),$$

the Volterra integral equation (2.3) can be written as

$$y(x) = f(x) + \int_{-1}^x (x-\sigma)^{-\gamma}\widehat{K}(x,\sigma)\psi(\sigma, y(\sigma))d\sigma, \qquad x \in [-1, 1], \tag{2.4}$$

with the kernel $\widehat{K}$ being given by

$$\widehat{K}(x,\sigma) = r\left(\frac{T^{1/r}}{2}\right)^{r(1-\gamma)} \frac{(\sigma+1)^{r-1}}{(P_{r-1}(x,\sigma))^{\gamma}}\widetilde{K}(x,\sigma), \qquad r \geq 2,$$

and

$$P_{r-1}(x,\sigma) = (x+1)^{r-1} + (x+1)^{r-2}(\sigma+1) + \ldots + (\sigma+1)^{r-1}.$$

Then, using the linear transformation $\eta : [-1,1]\times[-1,1] \to [-1,1]$

$$\eta(x,\theta) = \frac{x+1}{2}\theta + \frac{x-1}{2}, \qquad \theta \in [-1,1], \tag{2.5}$$

equation (2.4) becomes

$$y(x) = f(x) + \int_{-1}^{1}(1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))\psi(\eta(x,\theta),y(\eta(x,\theta)))d\theta, \quad x \in [-1,1], \tag{2.6}$$

where

$$\kappa(x,\eta(x,\theta)) = r\left(\frac{(x+1)T^{1/r}}{4}\right)^{r(1-\gamma)} \frac{(1+\theta)^{r+\gamma-1}\widetilde{K}(x,\eta(x,\theta))}{[2^{r-1}+2^{r-2}(1+\theta)+\ldots+(1+\theta)^{r-1}]^{\gamma}}, \quad r \geq 2.$$

We define the Hammerstein integral operator $\mathcal{K} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{K}(y)(x) = \int_{-1}^{1}(1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))\psi(\eta(x,\theta),y(\eta(x,\theta)))d\theta, \qquad x \in [-1,1].$$

In operator form, equation (2.6) can be expressed as

$$y = f + \mathcal{K}(y). \tag{2.7}$$

Define $\Omega = [-1,1]\times\mathbb{R}$. Throughout this article, the following assumptions are made about κ, f, and ψ:

(i). $\kappa \in \mathcal{C}([-1,1]\times[-1,1])$, $f \in \mathcal{C}([-1,1])$, and $\psi \in \mathcal{C}(\Omega)$.

(ii). $M_1 = \sup_{x\in[-1,1]}\int_{-1}^{1}(1-\theta^2)^{-\gamma}\,|\kappa(x,\eta(x,\theta))|\,d\theta$.

(iii). The nonlinear function $\psi(t,u)$ is Lipschitz continuous in $u \in \mathbb{R}$, i.e., there exists a constant $q_1 > 0$, for which $|\psi(t,u_1) - \psi(t,u_2)| \leq q_1|u_1-u_2|$, for all $u_1,u_2 \in \mathbb{R}$.

(iv). The partial derivative $\partial\psi/\partial u$ of ψ with respect to the second variable exists and is Lipschitz continuous, that is, there exists a $q_2 > 0$ such that

$$\left|\frac{\partial\psi}{\partial u}(t,u_1) - \frac{\partial\psi}{\partial u}(t,u_2)\right| \leq q_2|u_1-u_2|, \qquad \text{for all } u_1,u_2 \in \mathbb{R}.$$

If $M_1q_1 < 1$, then equation (2.7) has a unique solution, say $y_0 \in \mathcal{C}[-1,1]$.

Using assumption (iv), the operator $\mathcal{K}$ at $y_0 \in \mathcal{C}[-1,1]$ is Fréchet-differentiable, and $\mathcal{K}'(y_0)$ is M_1q_2-Lipschitz. The Fréchet derivative is given by

$$(\mathcal{K}'(y_0)g)(x) = \int_{-1}^{1}(1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))\frac{\partial\psi}{\partial u}(\eta(x,\theta),y_0(\eta(x,\theta)))g(\eta(x,\theta))d\theta.$$

Then $\mathcal{K}'(y_0)$ is a compact operator on $\mathcal{C}[-1,1]$. The uniformly boundedness of $\mathcal{K}'(y_0)$ follows from

$$\begin{aligned}
&\|\mathcal{K}'(y_0)g\|_\infty \\
&\quad= \sup_{x\in[-1,1]}\left|\int_{-1}^{1}(1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))\frac{\partial\psi}{\partial u}(\eta(x,\theta),y_0(\eta(x,\theta)))g(\eta(x,\theta))d\theta\right| \\
&\quad\le \sup_{x\in[-1,1]}\int_{-1}^{1}(1-\theta^2)^{-\gamma}\left|\kappa(x,\eta(x,\theta))\right|\left|\frac{\partial\psi}{\partial u}(\eta(x,\theta),y_0(\eta(x,\theta)))\right|\left|g(\eta(x,\theta))\right|d\theta \\
&\quad\le M_1M_2\sup_{x\in[-1,1]}\|g(\eta(x,.))\|_\infty.
\end{aligned}$$

This implies,

$$\|\mathcal{K}'(y_0)\|_\infty \le M_1M_2, \tag{2.8}$$

where

$$M_2=\sup_{x,\theta\in[-1,1]}\left|\frac{\partial\psi}{\partial u}(\eta(x,\theta),y_0(\eta(x,\theta)))\right|.$$

Next we study the Jacobi spectral method for the equation (2.7). We begin by recalling some properties of the Jacobi polynomials that will be useful later. Let $\mathbb{X}_n$ represent the set of all polynomials of degree $\le n$ defined on the interval $[-1,1]$. As in [10], let us denote by $J_n^{\alpha,\beta}(x)$ the Jacobi polynomial of degree n with weight function

$$\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta,\qquad \alpha,\beta>-1. \tag{2.9}$$

The following three-term recurrence relation can be used to construct the Jacobi polynomials:

$$\begin{aligned}
&J_0^{\alpha,\beta}(x)=1,\quad J_1^{\alpha,\beta}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta), && x\in[-1,1],\\
&J_{i+1}^{\alpha,\beta}(x)=\left(a_i^{\alpha,\beta}-b_i^{\alpha,\beta}\right)J_i^{\alpha,\beta}(x)-c_i^{\alpha,\beta}J_{i-1}^{\alpha,\beta}(x), && i=1,2,\dots,n-1,
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
a_i^{\alpha,\beta}&=\frac{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}{2(i+1)(i+\alpha+\beta+1)},\\
b_i^{\alpha,\beta}&=\frac{(\beta^2-\alpha^2)(2i+\alpha+\beta+2)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)},\\
c_i^{\alpha,\beta}&=\frac{(i+\alpha)(i+\beta)(2i+\alpha+\beta+2)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}.
\end{aligned}$$

From the theory of Jacobi polynomials, the following orthogonality relation holds:

$$\int_{-1}^{1}\omega^{\alpha,\beta}(t)J_i^{\alpha,\beta}(t)J_j^{\alpha,\beta}(t)dt=\gamma_i^{\alpha,\beta}\delta_{ij},$$

where

$$\gamma_i^{\alpha,\beta}=\frac{2^{\alpha+\beta+1}\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\Gamma(i+1)\Gamma(i+\alpha+\beta+1)},\qquad i\ge 1.$$

Then $\{\varphi_i^{\alpha,\beta}(s)=\frac{1}{\sqrt{\gamma_i^{\alpha,\beta}}}J_i^{\alpha,\beta}(s): i=0,1,\dots,n\}$ form an orthonormal basis for $\mathbb{X}_n$.

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Orthogonal projection operator. Define a weighted space as

$$L^2_{\omega^{\alpha,\beta}} = \{y : y \text{ is mesurable and } \|y\|_{\omega^{\alpha,\beta}} < +\infty\}.$$

For $y_1, y_2 \in L^2_{\omega^{\alpha,\beta}}$, the inner product is given by

$$\langle y_1, y_2\rangle_{\omega^{\alpha,\beta}} = \int_{-1}^{1} \omega^{\alpha,\beta}(t) y_1(t) y_2(t) dt,$$

and the norm is

$$\|y_1\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} \omega^{\alpha,\beta}(t)(y_1(t))^2 dt\right)^{\frac{1}{2}}.$$

Let $\Pi_n^{\alpha,\beta}$ be the orthogonal projection operator defined from $L^2_{\omega^{\alpha,\beta}}$ to $\mathbb{X}_n$. Then for all $y \in L^2_{\omega^{\alpha,\beta}}$, we have

$$\begin{aligned} (\Pi_n^{\alpha,\beta} y)(s) &= \sum_{i=0}^{n} \langle y, \varphi_i^{\alpha,\beta}\rangle_{\omega^{\alpha,\beta}} \varphi_i^{\alpha,\beta}(s), \\ \left\langle \Pi_n^{\alpha,\beta} y, \varphi_i^{\alpha,\beta}\right\rangle_{\omega^{\alpha,\beta}} &= \left\langle y, \varphi_i^{\alpha,\beta}\right\rangle_{\omega^{\alpha,\beta}}, \qquad i = 0, 1, \dots, n. \end{aligned} \tag{2.11}$$

Interpolatory projection operator. Let the interpolatory projection $I_n^{\alpha,\beta} : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be defined by

$$(I_n^{\alpha,\beta} y)(\tau_i^{\alpha,\beta}) = y(\tau_i^{\alpha,\beta}), \qquad i = 0, 1, \dots, n, \quad y \in \mathcal{C}[-1,1], \tag{2.12}$$

where $\left\{\tau_0^{\alpha,\beta}, \tau_1^{\alpha,\beta}, \dots, \tau_n^{\alpha,\beta}\right\}$ are the zeros of the Jacobi polynomial $J_{n+1}^{\alpha,\beta}$. In the Lagrange form, $I_n^{\alpha,\beta} y$ is given by

$$(I_n^{\alpha,\beta} y)(s) = \sum_{j=0}^{n} y(\tau_j^{\alpha,\beta}) \ell_j^{\alpha,\beta}(s), \qquad s \in [-1,1],$$

where $\ell_j^{\alpha,\beta}$ is the unique polynomial of degree n that satisfies $\ell_j^{\alpha,\beta}(\tau_i^{\alpha,\beta}) = \delta_{ij}$. Next, we recall some crucial properties of $\Pi_n^{\alpha,\beta}$ and $I_n^{\alpha,\beta}$ that we will use in the next section. Firstly, we introduce some weighted Hilbert spaces. For simplicity, denote $D^k y = \frac{\partial^k y}{\partial x^k}$. For a non-negative integer r, define

$$H^r_{\omega^{\alpha,\beta}} = \left\{y : D^k y \in L^2_{\omega^{\alpha,\beta}}, \ 0 \le k \le r\right\},$$

with the following norm and semi-norm

$$\|y\|_{H^r_{\omega^{\alpha,\beta}}} = \left(\sum_{k=0}^{r} \|D^k y\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}, \qquad |y|_{H^{r,n}_{\omega^{\alpha,\beta}}} = \left(\sum_{k=\min\{r,n+1\}}^{r} \|D^k y\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}},$$

respectively. A generic constant C that is independent of n will be used throughout the work.

LEMMA 2.1 ([18, 26]). *Let $\pi_n^{\alpha,\beta} : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be either the orthogonal projection $\Pi_n^{\alpha,\beta}$ or the interpolatory projection operator $I_n^{\alpha,\beta}$ defined as above. There is a constant $p > 0$ independent of n such that for $y \in \mathcal{C}[-1,1]$,*

$$\|\pi_n^{\alpha,\beta} y\|_{\omega^{\alpha,\beta}} \le p\|y\|_\infty, \tag{2.13}$$

$$\|\pi_n^{\alpha,\beta} y\|_{\omega^{\alpha,\beta}} \le \|y\|_{\omega^{\alpha,\beta}}. \tag{2.14}$$

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Moreover, for any $y \in \mathcal{C}^r[-1,1]$, $r \geq 1$,

$$\|y - \pi_n^{\alpha,\beta} y\|_{\omega^{\alpha,\beta}} \leq C n^{-r} |y|_{H^{r,n}_{\omega^{\alpha,\beta}}}. \tag{2.15}$$

In the infinity norm, the operator $\pi_n^{\alpha,\beta}$ is unbounded. More specifically,

$$\|\Pi_n^{\alpha,\beta}\|_\infty \leq C \log n \tag{2.16}$$

and

$$\|I_n^{\alpha,\beta}\|_\infty = \begin{cases} \mathcal{O}(\log(n)), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ \mathcal{O}(n^{\eta+\frac{1}{2}}),\ \eta = \max\{\alpha,\beta\}, & \text{otherwise}; \end{cases} \tag{2.17}$$

see [10].

Throughout this article, we restrict ourselves to $\alpha = -\gamma$, $\beta = -\gamma$ in the weight function (2.9), where $0 < \gamma < 1$. As a result, the corresponding weight function and inner product become, respectively,

$$\omega^{-\gamma,-\gamma}(x) = (1-x^2)^{-\gamma}, \qquad 0 < \gamma < 1,$$
$$\langle y_1, y_2 \rangle_{\omega^{-\gamma,-\gamma}} = \int_{-1}^{1} (1-t^2)^{-\gamma} y_1(t) y_2(t) dt.$$

Note that

$$\begin{aligned} \|y(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} &= \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} y(\eta(x,\theta))^2 d\theta \right|^{\frac{1}{2}} \\ &\leq \|y(\eta(x,.))\|_\infty \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} d\theta \right|^{\frac{1}{2}} = \Lambda \|y(\eta(x,.))\|_\infty, \end{aligned}$$

where

$$\Lambda = \sqrt{\frac{\sqrt{\pi}\,\Gamma(1-\gamma)}{\Gamma(\frac{3}{2}-\gamma)}} < \infty.$$

Here $\Gamma(z)$ is the usual gamma function.

In the projection-type approach, for the numerical solution of (2.7), the function $z(t) = \psi(t, y(t))$ is approximated by the polynomial $z_n = \pi_n^{-\gamma,-\gamma} z$ of degree $\leq n$. The intended approximation y_n of the solution y_0 is specified as

$$y_n = f + \mathcal{K}_n(y_n), \tag{2.18}$$

where $\mathcal{K}_n$ is the nonlinear operator given by

$$\mathcal{K}_n(y)(x) = \int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta)) z_n(\eta(x,\theta)) d\theta, \qquad x \in [-1,1]. \tag{2.19}$$

To obtain an approximate solution that is more accurate than y_n, the following modified projection type-method is suggested in [5]:

$$y_n^M - \mathcal{K}_n^M(y_n^M) = f, \tag{2.20}$$

where

$$\mathcal{K}_n^M(y) = \pi_n^{-\gamma,-\gamma}\mathcal{K}(y) + \mathcal{K}_n(y) - \pi_n^{-\gamma,-\gamma}\mathcal{K}_n(y). \tag{2.21}$$

In this case, the iterated modified projection-type solution is given by

$$\widetilde{y}_n^M = \mathcal{K}(y_n^M) + f. \tag{2.22}$$

Throughout the paper, this method will be referred to as modified Galerkin-type or as modified collocation-type method when the orthogonal projection or the interpolatory projection is used, respectively.

The existence and uniqueness of the approximate solutions can be discussed by recalling the following lemma from [2].

LEMMA 2.2. *Consider $\mathbb{X}$ a Banach space and A, A_n bounded linear operators on $\mathbb{X}$. Suppose that $\|A_n - A\| \to 0$, as $n \to \infty$, and that the operator $\mathcal{I} - A$ is invertible. Then for a sufficiently large n, the operator $(\mathcal{I} - A_n)^{-1}$ exists and is uniformly bounded on $\mathbb{X}$.*

3. Convergence rates. The purpose of this section is to prove convergence rates in the orthogonal and interpolatory projections cases. It is necessary to prove the following lemma for the main results of this section.

LEMMA 3.1. *Let $y_0 \in \mathcal{C}[-1,1]$ be an isolated solution of (2.7). Assume that 1 is not an eigenvalue of $\mathcal{K}'(y_0)$. Then for sufficiently large n, the operators $\mathcal{I} - (\mathcal{K}_n)'(y_0)$ are invertible, and there exists a constant $C_1 > 0$ independent of n such that*

$$\|(\mathcal{I} - (\mathcal{K}_n)'(y_0))^{-1}\|_\infty \le C_1.$$

Proof. Let $z_1(t) = \frac{\partial\psi}{\partial u}(t, y_0(t))$. By the Cauchy-Schwarz inequality, we can deduce that for each $g \in \mathcal{C}[-1,1]$ and each $x \in [-1,1]$,

$$\begin{aligned}
&|[\mathcal{K}'(y_0) - (\mathcal{K}_n)'(y_0)]g(x)| \\
&\quad = \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta))(\mathcal{I} - \pi_n^{-\gamma,-\gamma}) z_1(\eta(x,\theta)) g(\eta(x,\theta)) d\theta \right| \\
&\quad \le \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} |\kappa(x,\eta(x,\theta))|^2 d\theta \right]^{\frac{1}{2}} \\
&\qquad \times \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} \left|(\mathcal{I} - \pi_n^{-\gamma,-\gamma}) z_1(\eta(x,\theta)) g(\eta(x,\theta))\right|^2 d\theta \right]^{\frac{1}{2}}.
\end{aligned}$$

Hence, taking the supremum we deduce that

$$\|[\mathcal{K}'(y_0) - (\mathcal{K}_n)'(y_0)]g\|_\infty \le M_3 \sup_{x\in[-1,1]} \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma}) z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \|g(\eta(x,.))\|_\infty,$$

which in turn gives

$$\|\mathcal{K}'(y_0) - (\mathcal{K}_n)'(y_0)\|_\infty \le M_3 \sup_{x\in[-1,1]} \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma}) z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}}, \tag{3.1}$$

where

$$M_3 = \sup_{-1\le x\le 1} \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} |\kappa(x,\eta(x,\theta))|^2 d\theta \right]^{\frac{1}{2}}.$$

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Since by assumption $z_1 \in \mathcal{C}[-1,1]$, we have $\|z_1 - \pi_n^{-\gamma,-\gamma} z_1\|_{\omega^{-\gamma,-\gamma}} \to 0$ as $n \to \infty$, which implies that

$$\|\mathcal{K}'(y_0) - (\mathcal{K}_n)'(y_0)\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{3.2}$$

Based on (2.13), we can show that $\mathcal{K}_n'(y_0)$ is uniformly bounded by

$$\begin{aligned}
\|\mathcal{K}_n'(y_0)g\|_\infty &= \sup_{x\in[-1,1]} \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x,\eta(x,\theta)) \pi_n^{-\gamma,-\gamma} z_1(\eta(x,\theta)) g(\eta(x,\theta)) d\theta \right| \\
&\le \sup_{x\in[-1,1]} \|\kappa(x,\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \|\pi_n^{-\gamma,-\gamma} z_1(\eta(x,.)) g(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \\
&\le pM_3 \sup_{x\in[-1,1]} \|z_1(\eta(x,.))\|_\infty \|g(\eta(x,.))\|_\infty.
\end{aligned}$$

As a result,

$$\|\mathcal{K}_n'(y_0)\|_\infty \le pM_2M_3.$$

This shows that $\|\mathcal{K}_n'(y_0)\|_\infty$ is collectively compact. By Lemma 2.2, the operators $(\mathcal{I} - \mathcal{K}_n'(y_0))^{-1}$ exist and are uniformly bounded in the infinity norm. This ends the proof. ∎

Theorem 2 in [25] can be used to prove the following bound:

$$\begin{aligned}
\|y_0 - y_n\|_\infty &\le \big\|(\mathcal{I} - (\mathcal{K}_n)'(y_0))^{-1}\big\|_\infty \big\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\big\|_\infty \\
&\le C_1 \big\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\big\|_\infty.
\end{aligned} \tag{3.3}$$

LEMMA 3.2. *Let $y_0 \in \mathcal{C}[-1,1]$ be an isolated solution of (2.7) and $\frac{\partial \psi}{\partial u} \in \mathcal{C}^r(\Omega)$. Assume that 1 is not an eigenvalue of $\mathcal{K}'(y_0)$. Then for n large enough, the operators $(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}$ exist and are uniformly bounded, i.e., there exists a constant $C_2 > 0$ independent of n such that*

$$\|(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}\| \le C_2. \tag{3.4}$$

Proof. Firstly, we present the proof for the weighted L_2-norm. Note that

$$\begin{aligned}
\|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_{\omega^{-\gamma,-\gamma}} &= \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma})(\mathcal{K}'(x_0) - \mathcal{K}_n'(x))\|_{\omega^{-\gamma,-\gamma}} \\
&\le (1 + \|\pi_n^{-\gamma,-\gamma}\|_{\omega^{-\gamma,-\gamma}}) \|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\|_{\omega^{-\gamma,-\gamma}} \\
&\le (1+p)\Lambda \|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\|_\infty.
\end{aligned}$$

Hence using the estimate (3.2), we see that

$$\|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_{\omega^{-\gamma,-\gamma}} \le (1+p)\Lambda \|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty \to 0 \qquad \text{as } n \to \infty,$$

and by applying Lemma 2.2 it can be shown that the operators $(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}$ exist and are uniformly bounded for some large enough n.

Now we prove (3.4) for the infinity norm. By using (2.15) and (3.1), we can write

$$\begin{aligned}
&\|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_\infty \\
&\quad = \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma})(\mathcal{K}'(x_0) - \mathcal{K}_n'(x))\|_\infty \\
&\quad \le (1 + \|\pi_n^{-\gamma,-\gamma}\|_\infty) \|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\|_\infty \\
&\quad \le M_3 (1 + \|\pi_n^{-\gamma,-\gamma}\|_\infty) \sup_{x\in[-1,1]} \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma}) z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \\
&\quad \le M_3 C (1 + \|\pi_n^{-\gamma,-\gamma}\|_\infty) n^{-r} \sup_{x\in[-1,1]} |z_1(\eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}}.
\end{aligned}$$

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Let $\Pi_n^{-\gamma,-\gamma}$ be the orthogonal projection defined by (2.11). Thus, we conclude from (2.16) that

$$\begin{aligned}\|\mathcal{K}'(y_0)-(\mathcal{K}_n^M)'(y_0)\|_\infty &\le M_3C(1+\log(n))n^{-r}\sup_{x\in[-1,1]}|z_1(\eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}}\\ &=\mathcal{O}(\log(n)n^{-r})\to 0 \qquad \text{as } n\to\infty.\end{aligned}\tag{3.5}$$

Similarly, for the interpolatory projection, the bound (2.17) implies that

$$\begin{aligned}\|\mathcal{K}'(y_0)-(\mathcal{K}_n^M)'(y_0)\|_\infty &\le M_3C(1+\|I_n^{-\gamma,-\gamma}\|_\infty)n^{-r}\sup_{x\in[-1,1]}|z_1(\eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}}\\ &=\begin{cases}\mathcal{O}(\log(n)n^{-r}), & \frac{1}{2}\le\gamma<1,\\ \mathcal{O}(n^{\frac{1}{2}-\gamma-r}), & \text{otherwise}\end{cases}\to 0 \qquad \text{as } n\to\infty.\end{aligned}$$

This together with (3.5) and Lemma 2.2 gives (3.4). The proof is complete. ☐

It should be mentioned that the condition $\frac{\partial\psi}{\partial u}\in\mathcal{C}^r(\Omega)$ is not necessary to prove (3.4) for the L_2-norm. In the following proposition, we summarize our error estimates for the solutions y_n^M and $\widetilde{y}_n^M$.

PROPOSITION 3.3. *Let $y_0\in\mathcal{C}[-1,1]$ be the unique solution of (2.7) and $\frac{\partial\psi}{\partial u}\in\mathcal{C}^r(\Omega)$. Suppose 1 is not an eigenvalue of $\mathcal{K}'(y_0)$. Then, for n large enough, we get*

$$\|y_0-y_n^M\|\le C_2(1+\|\pi_n^{-\gamma,-\gamma}\|)\|\mathcal{K}(y_0)-\mathcal{K}_n(y_0)\|,\tag{3.6}$$

$$\begin{aligned}\|y_0-\widetilde{y}_n^M\|_\infty\le C_3(1+\|\pi_n^{-\gamma,-\gamma}\|_\infty)\|y_0-y_n^M\|^2_{\omega^{-\gamma,-\gamma}}\\ +(1+C_4)\|\mathcal{K}'(y_0)[\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)]\|_\infty\\ +C_4\|\mathcal{K}'(y_0)-(\mathcal{K}_n^M)'(y_0)\|_\infty\|\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)\|_\infty,\end{aligned}\tag{3.7}$$

where C_3 and C_4 are constants independent of n.

Proof. Again from the work of Vainikko [25] we can conclude that

$$\begin{aligned}\|y_0-y_n^M\|&\le\|(\mathcal{I}-(\mathcal{K}_n^M)'(y_0))^{-1}(\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0))\|\\ &\le C_2\|\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)\|.\end{aligned}\tag{3.8}$$

Note that from (2.21), we have

$$\begin{aligned}\|\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)\|&=\|(\mathcal{I}-\pi_n^{-\gamma,-\gamma})(\mathcal{K}(y_0)-\mathcal{K}_n(y_0))\|\\ &\le(1+\|\pi_n^{-\gamma,-\gamma}\|)\|\mathcal{K}(y_0)-\mathcal{K}_n(y_0)\|,\end{aligned}\tag{3.9}$$

which together with (3.8) leads to (3.6).

Next, using the estimates (2.7) and (2.22), we can verify that

$$y_0-\widetilde{y}_n^M=\mathcal{K}y_0-\mathcal{K}y_n^M.$$

When we apply the mean value theorem to $\mathcal{K}$, we obtain

$$\begin{aligned}\mathcal{K}y_0-\mathcal{K}y_n^M&=\mathcal{K}'(y_0+\theta(y_0-y_n^M))(y_0-y_n^M)\\ &=[\mathcal{K}'(y_0+\theta(y_0-y_n^M))-\mathcal{K}'(y_0)+\mathcal{K}'(y_0)](y_0-y_n^M),\end{aligned}$$

for some $0<\theta<1$. Denote $\zeta_n=y_0+\theta(y_0-y_n^M)$. By taking the norms on both sides of the above equation, the following is revealed:

$$\|y_0-\widetilde{y}_n^M\|_\infty\le\|[\mathcal{K}'(\zeta_n)-\mathcal{K}'(y_0)](y_0-y_n^M)\|_\infty+\|\mathcal{K}'(y_0)(y_0-y_n^M)\|_\infty.\tag{3.10}$$

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We proceed now in the same manner as in the proof of Theorem 4.3 of Kant et al. [14] to estimate the first term of the above estimate. As a result of the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&|[\mathcal{K}'(\zeta_n) - \mathcal{K}'(y_0)](y_0 - y_n^M)(x)| \\
&\quad = \left| \int_{-1}^{1} \Big[(1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta)) \right. \\
&\qquad\qquad \left. \times \big(\frac{\partial \psi}{\partial u}(., \zeta_n) - \frac{\partial \psi}{\partial u}(., y_0) \big)(y_0 - y_n^M)(\eta(x,\theta)) \Big] d\theta \right| \\
&\quad \le \sup_{x,\theta \in [-1,1]} |\kappa(x, \eta(x,\theta))| \\
&\qquad \times \int_{-1}^{1} (1-\theta^2)^{-\gamma} \left| \big(\frac{\partial \psi}{\partial u}(., \zeta_n) - \frac{\partial \psi}{\partial u}(., y_0) \big)(y_0 - y_n^M)(\eta(x,\theta)) \right| d\theta \\
&\quad \le M_4 \left\| \big(\frac{\partial \psi}{\partial u}(., \zeta_n) - \frac{\partial \psi}{\partial u}(., y_0) \big)(\eta(x,.)) \right\|_{\omega^{-\gamma,-\gamma}} \|y_0 - y_n^M\|_{\omega^{-\gamma,-\gamma}},
\end{aligned}
\tag{3.11}
$$

where

$$
M_4 = \sup_{x,\theta \in [-1,1]} |\kappa(x, \eta(x,\theta))|.
$$

Now applying the Lipschitz continuity of $\frac{\partial \psi}{\partial u}$ with respect to the second variable, we obtain

$$
\begin{aligned}
&\left\| \left(\frac{\partial \psi}{\partial u}(., \zeta_n) - \frac{\partial \psi}{\partial u}(., y_0) \right) (\eta(x,.)) \right\|^2_{\omega^{-\gamma,-\gamma}} \\
&\quad = \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} \left[\left(\frac{\partial \psi}{\partial u}(., \zeta_n) - \frac{\partial \psi}{\partial u}(., y_0) \right) (\eta(x,\theta)) \right]^2 d\theta \right| \\
&\quad \le q_2^2 \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} \big[\zeta_n(\eta(x,\theta)) - y_0(\eta(x,\theta)) \big]^2 d\theta \right| \\
&\quad = q_2^2 \theta^2 \|\zeta_n - y_0\|^2_{\omega^{-\gamma,-\gamma}} \le q_2^2 \theta^2 \|y_0 - y_n^M\|^2_{\omega^{-\gamma,-\gamma}}.
\end{aligned}
$$

Then, by combining the above estimate with (3.11), we get

$$
\|[\mathcal{K}'(\zeta_n) - \mathcal{K}'(y_0)](y_0 - y_n^M)\|_\infty \le M_4 q_2 \theta \|y_0 - y_n^M\|^2_{\omega^{-\gamma,-\gamma}}.
\tag{3.12}
$$

In order to estimate the second term of (3.10), we must consider the following:

$$
\begin{aligned}
&(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))(y_0 - y_n^M) \\
&\quad = \mathcal{K}(y_0) - \mathcal{K}_n^M(y_0) - (\mathcal{K}_n^M)'(y_0)(y_0 - y_n^M) + \mathcal{K}_n^M(y_0) - \mathcal{K}_n^M(y_n^M).
\end{aligned}
$$

Applying $\mathcal{K}'(y_0)$ to both sides of the above equation and using the mean value theorem, we deduce that

$$
\begin{aligned}
\mathcal{K}'(y_0)(y_0 - y_n^M) &= \mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0) \\
&\qquad - (\mathcal{K}_n^M)'(y_0)(y_0 - y_n^M) + \mathcal{K}_n^M(y_0) - \mathcal{K}_n^M(y_n^M)] \\
&= \mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)] \\
&\qquad + \mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[(\mathcal{K}_n^M)'(\zeta_n)) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M).
\end{aligned}
$$

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As a result of taking the norm on both sides of the above equation and using the formulas in (2.8) and (3.4), we find

$$
\begin{aligned}
&\|\mathcal{K}'(y_0)(y_0 - y_n^M)\|_\infty \\
&\quad\le \|\mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\qquad + \|\mathcal{K}'(y_0)\|_\infty \|(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}\|_\infty \\
&\qquad\quad \times \|[(\mathcal{K}_n^M)'(\zeta_n) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M)\|_\infty \\
&\quad\le \|\mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\qquad + M_1 M_2 C_2 \|[(\mathcal{K}_n^M)'(\zeta_n) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M)\|_\infty.
\end{aligned}
\tag{3.13}
$$

By the formula

$$
(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1} = \mathcal{I} + (\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}(\mathcal{K}_n^M)'(y_0),
$$

it holds that

$$
\begin{aligned}
&\|\mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\quad = \|\mathcal{K}'(y_0)[\mathcal{I} + (\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}(\mathcal{K}_n^M)'(y_0)][\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\quad \le \|\mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\qquad + \|\mathcal{K}'(y_0)(\mathcal{I} - (\mathcal{K}_n^M)'(y_0))^{-1}(\mathcal{K}_n^M)'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\quad \le \|\mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty + M_1 M_2 C_2 \|(\mathcal{K}_n^M)'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty.
\end{aligned}
$$

This is combined with the estimate (3.13), resulting in

$$
\begin{aligned}
&\|\mathcal{K}'(y_0)(y_0 - y_n^M)\|_\infty \\
&\quad \le \|\mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\
&\qquad + M_1 M_2 C_2 \|[(\mathcal{K}_n^M)'(\zeta_n) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M)\|_\infty \\
&\qquad + M_1 M_2 C_2 \|(\mathcal{K}_n^M)'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty.
\end{aligned}
\tag{3.14}
$$

From equation (2.21), we have

$$
(\mathcal{K}_n^M)'(y) = \pi_n^{-\gamma,-\gamma}\mathcal{K}'(y) + (\mathcal{I} - \pi_n^{-\gamma,-\gamma})\mathcal{K}_n'(y), \qquad y \in \mathcal{C}[-1,1].
$$

Using the above result, we obtain

$$
\begin{aligned}
&\|[(\mathcal{K}_n^M)'(\zeta_n) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M)\|_\infty \\
&\quad = \|\pi_n^{-\gamma,-\gamma}[\mathcal{K}'(\zeta_n) - \mathcal{K}'(y_0)](y_0 - y_n^M)\|_\infty \\
&\qquad + \|(\mathcal{I} - \pi_n^{-\gamma,-\gamma})[(\mathcal{K}_n)'(\zeta_n) - (\mathcal{K}_n)'(y_0)](y_0 - y_n^M)\|_\infty \\
&\quad \le \|\pi_n^{-\gamma,-\gamma}\|_\infty \|[\mathcal{K}'(\zeta_n) - \mathcal{K}'(y_0)](y_0 - y_n^M)\|_\infty \\
&\qquad + (1 + \|\pi_n^{-\gamma,-\gamma}\|_\infty)\|[(\mathcal{K}_n)'(\zeta_n) - (\mathcal{K}_n)'(y_0)](y_0 - y_n^M)\|_\infty.
\end{aligned}
\tag{3.15}
$$

After completing the analogous steps of (3.11) to (3.12) and using (2.14), we can say that

$$
\|[\mathcal{K}_n'(\zeta_n) - \mathcal{K}_n'(y_0)](y_0 - y_n^M)\|_\infty \le M_4 q_2 \theta \|y_0 - y_n^M\|^2_{\omega^{-\gamma,-\gamma}}.
\tag{3.16}
$$

If we combine (3.16) and (3.12) with (3.15), we deduce that

$$
\begin{aligned}
&\|[(\mathcal{K}_n^M)'(\zeta_n) - (\mathcal{K}_n^M)'(y_0)](y_0 - y_n^M)\|_\infty \\
&\quad \le M_4 q_2 \theta (1 + 2\|\pi_n^{-\gamma,-\gamma}\|_\infty)\|y_0 - y_n^M\|^2_{\omega^{-\gamma,-\gamma}}.
\end{aligned}
\tag{3.17}
$$

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Now consider

$$\begin{aligned}(\mathcal{K}_n^M)'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\\ &= [\mathcal{K}'(y_0) - (\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0))][\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\\ &= \mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)] - [\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)][\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)].\end{aligned}$$

We thus obtain

$$\begin{aligned}&\|(\mathcal{K}_n^M)'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty\\ &\quad= \|\mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty\\ &\qquad + \|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_\infty \|\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)\|_\infty.\end{aligned} \tag{3.18}$$

Now using the estimates (3.17) and (3.18) in (3.14), one has

$$\begin{aligned}&\|\mathcal{K}'(y_0)(y_0 - y_n^M)\|_\infty\\ &\quad\leq M_4 q_2 \theta (1 + 2\|\pi_n^{-\gamma,-\gamma}\|_\infty)\|y_0 - y_n^M\|^2_{\omega^{-\gamma,-\gamma}}\\ &\qquad + (1 + M_1 M_2 C_2)\|\mathcal{K}'(y_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty\\ &\qquad + M_1 M_2 C_2 \|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_\infty \|\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)\|_\infty.\end{aligned} \tag{3.19}$$

Finally, combining (3.12) and (3.19) with (3.10) ends the proof of (3.7) with $C_3 = 2M_4 q_2 \theta$ and $C_4 = M_1 M_2 C_2$. ☐

3.1. Galerkin-type and modified Galerkin-type methods.

THEOREM 3.4. *We assume that* $f \in \mathcal{C}^r[-1,1]$, $\psi \in \mathcal{C}^r(\Omega)$, *and* $\kappa \in \mathcal{C}^r[-1,1]^2$. *Let* $\Pi_n^{-\gamma,-\gamma} : \mathcal{C}[-1,1] \to \mathbb{X}_n$ *be the orthogonal projection defined by* (2.11) *(for* $\alpha = -\gamma$ *and* $\beta = -\gamma$*) and* y_0 *be the exact solution defined by* (2.7). *Let* $y_n^G, y_n^{M,G}, \widetilde{y}_n^{M,G}$ *be the approximate solutions of* (2.18), (2.20), *and* (2.22), *respectively. Then there holds*

$$\|y_0 - y_n^G\|_\infty = \mathcal{O}(n^{-2r}). \tag{3.20}$$

In addition, if $\frac{\partial \psi}{\partial u} \in \mathcal{C}^r(\Omega)$, *then*

$$\|y_0 - y_n^{M,G}\|_\infty = \mathcal{O}(\log(n) n^{-2r}), \tag{3.21}$$

$$\|y_0 - \widetilde{y}_n^{M,G}\|_\infty = \mathcal{O}(n^{-3r}). \tag{3.22}$$

Proof. Let $z_0(t) = \psi(t, x_0(t))$. As a result of the Cauchy-Schwarz inequality and the estimate (2.15),

$$\begin{aligned}&\big\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\big\|_\infty\\ &\quad= \sup_{x\in[-1,1]} \left| \int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta))(\mathcal{I} - \Pi_n^{-\gamma,-\gamma}) z_0(\eta(x,\theta)) d\theta \right|\\ &\quad= \sup_{x\in[-1,1]} \left| \langle \kappa(x, \eta(x,.)), (\mathcal{I} - \Pi_n^{-\gamma,-\gamma}) z_0(\eta(x,.)) \rangle_{\omega^{-\gamma,-\gamma}} \right|\\ &\quad= \sup_{x\in[-1,1]} \left| \langle (\mathcal{I} - \Pi_n^{-\gamma,-\gamma})\kappa(x, \eta(x,.)), (\mathcal{I} - \Pi_n^{-\gamma,-\gamma}) z_0(\eta(x,.)) \rangle_{\omega^{-\gamma,-\gamma}} \right|\\ &\quad\leq \sup_{x\in[-1,1]} \Big[\|(\mathcal{I} - \Pi_n^{-\gamma,-\gamma})\kappa(x, \eta(x,.))\|_{\omega^{-\gamma,-\gamma}}\\ &\qquad\qquad \times \|(\mathcal{I} - \Pi_n^{-\gamma,-\gamma}) z_0(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \Big]\\ &\quad\leq C^2 n^{-2r} \sup_{x\in[-1,1]} |\kappa(x, \eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}} |z_0(\eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}}.\end{aligned} \tag{3.23}$$

As a consequence,

$$\big\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\big\|_\infty = \mathcal{O}(n^{-2r}). \tag{3.24}$$

Then, combining the above estimate with (3.3), the bound (3.20) follows.

Now using the estimates (2.16) in (3.6), we have

$$\begin{aligned} \|y_0 - y_n^{M,G}\|_\infty &\le C_2(1 + \|\Pi_n^{-\gamma,-\gamma}\|_\infty)\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty \\ &\le C_2(1 + C\log n)\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty. \end{aligned} \tag{3.25}$$

Hence, (3.21) is obtained by combining (3.24) and (3.25). In addition, by referring to the bounds (2.13) and (3.24), we get the corresponding estimate in the L_2-norm given by

$$\begin{aligned} \|y_0 - y_n^{M,G}\|_{\omega^{-\gamma,-\gamma}} &\le C_2(1 + \|\Pi_n^{-\gamma,-\gamma}\|_{\omega^{-\gamma,-\gamma}})\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_{\omega^{-\gamma,-\gamma}} \\ &\le \Lambda C_2(1+p)\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty \\ &= \mathcal{O}(n^{-2r}). \end{aligned} \tag{3.26}$$

For the iterated solution, the second estimate in (3.7) can be written as

$$\begin{aligned} &\|\mathcal{K}'(x_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty \\ &= \sup_{x\in[-1,1]} \Big| \int_{-1}^{1} (1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))z_1(\eta(x,\theta))(\mathcal{I} - \Pi_n^{-\gamma,-\gamma})[\mathcal{K}(y_0) - \mathcal{K}_n(y_0)](\theta)d\theta\Big| \\ &= \sup_{x\in[-1,1]} \big|\langle \kappa(x,\eta(x,.))z_1(\eta(x,.)), (\mathcal{I} - \Pi_n^{-\gamma,-\gamma})[\mathcal{K}(y_0) - \mathcal{K}_n(y_0)]\rangle_{\omega^{-\gamma,-\gamma}}\big| \\ &= \sup_{x\in[-1,1]} \big|\langle (\mathcal{I}-\Pi_n^{-\gamma,-\gamma})\kappa(x,\eta(x,.))z_1(\eta(x,.)), (\mathcal{I}-\Pi_n^{-\gamma,-\gamma})[\mathcal{K}(y_0)-\mathcal{K}_n(y_0)]\rangle_{\omega^{-\gamma,-\gamma}}\big| \\ &\le \sup_{x\in[-1,1]} \|(\mathcal{I} - \Pi_n^{-\gamma,-\gamma})\kappa(x,\eta(x,.))z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \\ &\qquad\qquad \times \|(\mathcal{I} - \Pi_n^{-\gamma,-\gamma})[\mathcal{K}(y_0) - \mathcal{K}_n(y_0)]\|_{\omega^{-\gamma,-\gamma}} \\ &\le \Lambda(1+p) \sup_{x\in[-1,1]} \|(\mathcal{I} - \Pi_n^{-\gamma,-\gamma})\kappa(x,\eta(x,.))z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}}\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty. \end{aligned}$$

Thus, by using (2.15) and (3.23), we get

$$\|\mathcal{K}'(x_0)[\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)]\|_\infty = \mathcal{O}(n^{-3r}). \tag{3.27}$$

Now following the analogue steps of (3.23), we can show that

$$\big\|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\big\|_\infty = \mathcal{O}(n^{-2r}).$$

This implies

$$\begin{aligned} \|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_\infty &\le (1 + \|\Pi_n^{-\gamma,-\gamma}\|_\infty)\|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\|_\infty \\ &\le (1 + C\log n)\|\mathcal{K}'(y_0) - \mathcal{K}_n'(y_0)\|_\infty \\ &= \mathcal{O}(\log(n)n^{-2r}). \end{aligned} \tag{3.28}$$

On the other hand, using the estimates (2.16) and (3.24) in (3.9), we get

$$\|\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)\|_\infty = \mathcal{O}(\log(n)n^{-2r}). \tag{3.29}$$

By (3.28) and (3.29), the resulting global order for the third term in (3.7) is

$$\|\mathcal{K}'(y_0) - (\mathcal{K}_n^M)'(y_0)\|_\infty\|\mathcal{K}(y_0) - \mathcal{K}_n^M(y_0)\|_\infty = \mathcal{O}((\log(n))^2 n^{-4r}). \tag{3.30}$$

Finally, combining the estimates (3.26), (3.27), and (3.30) with (3.7), we deduce (3.22). ∎

3.2. Collocation-type and modified collocation-type methods.

THEOREM 3.5. *We assume that $f \in \mathcal{C}^r[-1,1]$, $\psi \in \mathcal{C}^r(\Omega)$, and $\kappa \in \mathcal{C}^r[-1,1]^2$. Let $I_n^{-\gamma,-\gamma} : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be the interpolatory projection defined by (2.12) (for $\alpha = -\gamma$ and $\beta = -\gamma$) and y_0 be the exact solution defined by (2.7). Let $y_n^C, y_n^{M,C}, \widetilde{y}_n^{M,C}$ be the approximate solutions of (2.18), (2.20), and (2.22) respectively. Then there holds*

$$\|y_0 - y_n^C\|_\infty = \mathcal{O}(n^{-r}). \tag{3.31}$$

In addition, if $\frac{\partial \psi}{\partial u} \in \mathcal{C}^r(\Omega)$, then

$$\|y_0 - y_n^{M,C}\|_\infty = \begin{cases} \mathcal{O}(\log(n) n^{-r}), & \frac{1}{2} \le \gamma < 1, \\ \mathcal{O}(n^{\frac{1}{2}-\gamma-r}), & \text{otherwise}, \end{cases} \tag{3.32}$$

$$\|y_0 - \widetilde{y}_n^{M,C}\|_\infty = \mathcal{O}(n^{-r}). \tag{3.33}$$

Proof. As a result of (2.15) and the Cauchy-Schwarz inequality, we can show that

$$\begin{aligned} &\big\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\big\|_\infty \\ &\quad = \sup_{x\in[-1,1]} \left| \int_{-1}^{1} (\mathcal{I} - \theta^2)^{-\gamma} \kappa(x, \eta(x,\theta))(\mathcal{I} - I_n^{-\gamma,-\gamma}) z_0(\eta(x,\theta)) d\theta \right| \\ &\quad \le \sup_{x\in[-1,1]} \|\kappa(x,\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \|(\mathcal{I} - I_n^{-\gamma,-\gamma}) z_0(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}} \\ &\quad \le M_3 n^{-r} \sup_{x\in[-1,1]} |z_0(\eta(x,.))|_{H^{r,n}_{\omega^{-\gamma,-\gamma}}} . \end{aligned} \tag{3.34}$$

This allows us to conclude that

$$\|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty = \mathcal{O}(n^{-r}). \tag{3.35}$$

Then, combining the above estimate with (3.3), the bound (3.31) follows.

Next, by (2.17), (3.6), and (3.35), we deduce that

$$\|y_0 - y_n^{M,C}\|_\infty = \begin{cases} \mathcal{O}(\log(n) n^{-r}), & \frac{1}{2} \le \gamma < 1, \\ \mathcal{O}(n^{\frac{1}{2}-\gamma-r}), & \text{otherwise}, \end{cases}$$

and (3.32) is then immediate. Similarly to (3.26), an enhancement in the rate of convergence for $y_n^{M,C}$ in the L_2-norm is given by

$$\begin{aligned} \|y_0 - y_n^{M,C}\|_{\omega^{-\gamma,-\gamma}} &\le C_2(1 + \|I_n^{-\gamma,-\gamma}\|_{\omega^{-\gamma,-\gamma}}) \|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_{\omega^{-\gamma,-\gamma}} \\ &\le \Lambda C_2(1+p) \|\mathcal{K}(y_0) - \mathcal{K}_n(y_0)\|_\infty \\ &= \mathcal{O}(n^{-r}). \end{aligned} \tag{3.36}$$

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The last estimate is a consequence of (2.13) and (3.35). Taking into consideration the second term of (3.7) yields

$$
\begin{aligned}
&\|\mathcal{K}'(y_0)(\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0))\|_\infty \\
&\quad= \sup_{x\in[-1,1]}\left|\int_{-1}^{1}(1-\theta^2)^{-\gamma}\kappa(x,\eta(x,\theta))z_1(\eta(x,\theta))\right.\\
&\qquad\qquad\left.\times(\mathcal{I}-I_n^{-\gamma,-\gamma})[\mathcal{K}(y_0)-\mathcal{K}_n(y_0)](\theta)d\theta\right|\\
&\quad\le \sup_{x\in[-1,1]}\Big[\|\kappa(x,\eta(x,.))z_1(\eta(x,.))\|_{\omega^{-\gamma,-\gamma}}\\
&\qquad\qquad\times\|(\mathcal{I}-I_n^{-\gamma,-\gamma})[\mathcal{K}(y_0)-\mathcal{K}_n(y_0)]\|_{\omega^{-\gamma,-\gamma}}\Big]\\
&\quad\le M_5(1+\|I_n^{-\gamma,-\gamma}\|_{\omega^{-\gamma,-\gamma}})\|\mathcal{K}(y_0)-\mathcal{K}_n(y_0)\|_{\omega^{-\gamma,-\gamma}}\\
&\quad\le \Lambda M_5(1+p)\|\mathcal{K}(y_0)-\mathcal{K}_n(y_0)\|_\infty.
\end{aligned}
\tag{3.37}
$$

where

$$
M_5=\sup_{-1\le x\le 1}\left[\int_{-1}^{1}(1-\theta^2)^{-\gamma}|\kappa(x,\eta(x,\theta))z_1(\eta(x,\theta))|^2d\theta\right]^{\frac12}.
$$

Thus, the estimate (3.35) shows that

$$
\|\mathcal{K}'(x_0)[\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)]\|_\infty=\mathcal{O}(n^{-r}).
$$

Lastly, similarly to (3.34), it can be shown that

$$
\left\|\mathcal{K}'(y_0)-\mathcal{K}_n'(y_0)\right\|_\infty=\mathcal{O}(n^{-r}).
$$

From this, it follows

$$
\begin{aligned}
\|\mathcal{K}'(y_0)-(\mathcal{K}_n^M)'(y_0)\|_\infty &\le (1+\|I_n^{-\gamma,-\gamma}\|_\infty)\|\mathcal{K}'(y_0)-\mathcal{K}_n'(y_0)\|_\infty\\
&=\begin{cases}\mathcal{O}(\log(n)n^{-r}), & \frac12\le\gamma<1,\\ \mathcal{O}(n^{\frac12-\gamma-r}), & \text{otherwise.}\end{cases}
\end{aligned}
\tag{3.38}
$$

On the other hand, it follows from (2.17), (3.35), and (3.9) that

$$
\|\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)\|_\infty=\begin{cases}\mathcal{O}(\log(n)n^{-r}), & \frac12\le\gamma<1,\\ \mathcal{O}(n^{\frac12-\gamma-r}), & \text{otherwise.}\end{cases}
\tag{3.39}
$$

By (3.38) and (3.39), the third term in (3.7) is given by

$$
\begin{aligned}
&\|\mathcal{K}'(y_0)-(\mathcal{K}_n^M)'(y_0)\|_\infty\|\mathcal{K}(y_0)-\mathcal{K}_n^M(y_0)\|_\infty\\
&\qquad=\begin{cases}\mathcal{O}((\log(n))^2n^{-2r}), & \frac12\le\gamma<1,\\ \mathcal{O}(n^{1-2\gamma-2r}), & \text{otherwise.}\end{cases}
\end{aligned}
\tag{3.40}
$$

Combining (3.36), (3.37), and (3.40) with (3.7) ends the proof of (3.33). □

4. Implementation note. For a given positive integer n, we denote the Jacobi polynomials of degree $\le n$ by $\{\varphi_i\}_{i=0}^n=\{\varphi_i^{-\gamma,-\gamma}\}_{i=0}^n$ and by $\{\ell_i\}_{i=0}^n=\{\ell_i^{-\gamma,-\gamma}\}_{i=0}^n$ the Lagrange interpolation basis function associated with $\{\tau_i\}_{i=0}^n=\{\tau_i^{-\gamma,-\gamma}\}_{i=0}^n$, which are the set of $n+1$ Jacobi–Gauss points.

4.1. Jacobi spectral projection-type method. Let $\Pi_n^{-\gamma,-\gamma}$ be the orthogonal projection defined by (2.11) and $\kappa_j(s) := \langle \kappa(s,.), \varphi_j \rangle_{\omega^{-\gamma,-\gamma}}$. In order to give more details about the implementation of the Galerkin-type solution y_n^G, it is easy to derive from (2.18) that y_n^G has the form

$$y_n^G = f + \sum_{j=0}^{n} a_j \kappa_j,$$

where the coefficients a_i are solution of the nonlinear system of equations

$$a_i - \left\langle \psi \left(., f + \sum_{j=0}^{n} a_j \kappa_j \right), \varphi_i \right\rangle_{\omega^{-\gamma,-\gamma}} = 0, \qquad i = 0, 1, \dots, n.$$

Similarly, for the interpolatory projection, the collocation-type solution y_n^C of equation (2.18) is given by

$$y_n^C = f + \sum_{j=0}^{n} b_j \bar{\kappa}_j,$$

where $\bar{\kappa}_j(s) := \langle \kappa(s,.), \ell_j \rangle_{\omega^{-\gamma,-\gamma}}$ and the coefficients b_i are solution of the nonlinear system of equations

$$b_i - \psi \left(., f + \sum_{j=0}^{n} b_j \bar{\kappa}_j \right) = 0, \qquad i = 0, 1, \dots, n.$$

REMARK 4.1. In the actual computations, the integral operator and the inner product based on the Jacobi weight cannot be evaluated exactly. We replace these integrals with the Gauss–Jacobi quadrature formula

$$\int_{-1}^{1} \omega^{-\gamma,-\gamma}(t) f(t) dt = \sum_{i=1}^{M(n)} \omega_i f(t_i),$$

where the weights ω_i and nodes t_i are computed as described in [12, p. 705] and the number of nodes is simply written as $M(n)$, which depends on n.

4.2. Jacobi spectral modified projection-type method. Let $\Pi_n^{-\gamma,-\gamma}$ be the orthogonal projection defined by (2.11). From equation (2.20) we can easily show that the approximate solution $y_n^{M,G}$ has the form

$$y_n^{M,G} = f + \sum_{p=0}^{n} a_p \varphi_p + \sum_{q=0}^{n} b_q \kappa_q, \tag{4.1}$$

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where the coefficients $\{a_i, b_i, i = 0, 1, \ldots, n\}$ are obtained by substituting $y_n^{M,G}$ from equation (4.1) into equation (2.20). Then, we successively have

$$\begin{aligned}\Pi_n^{-\gamma,-\gamma}\mathcal{K}y_n^{M,G} &= \sum_{i=0}^{n} \left\langle \mathcal{K}y_n^{M,G}, \varphi_i \right\rangle_{\omega^{-\gamma,-\gamma}} \varphi_i \\ &= \sum_{i=0}^{n} \left\{ \int_{-1}^{1} (1-x^2)^{-\gamma} \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta)) \bar{z}(\eta(x,\theta)) d\theta \right] \varphi_i(x) dx \right\} \varphi_i, \\ \mathcal{K}_n y_n^{M,G} &= \sum_{j=0}^{n} \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} \bar{z}(\eta(.,\theta)) \varphi_j(\eta(.,\theta)) d\theta \right] \kappa_j, \\ \Pi_n^{-\gamma,-\gamma}\mathcal{K}_n y_n^{M,G} &= \sum_{i=0}^{n} \left\langle \mathcal{K}_n y_n^{M,G}, \varphi_i \right\rangle_{\omega^{-\gamma,-\gamma}} \varphi_i \\ &= \sum_{i=0}^{n} \left\{ \sum_{j=0}^{n} \left[\int_{-1}^{1} \bar{z}(\eta(.,\theta)) \varphi_j(\eta(.,\theta)) d\theta \right] \langle \kappa_j, \varphi_i \rangle_{\omega^{-\gamma,-\gamma}} \right\} \varphi_i,\end{aligned}$$

where

$$\bar{z}(t) = \psi\left(t, f(t) + \sum_{p=0}^{n} a_p \varphi_p(t) + \sum_{q=0}^{n} b_q \kappa_q(t)\right).$$

By identifying the coefficients of φ_i and κ_j, respectively, we obtain the nonlinear system

$$\begin{aligned} a_i &= \int_{-1}^{1} (1-x^2)^{-\gamma} \left[\int_{-1}^{1} (1-\theta^2)^{-\gamma} \kappa(x, \eta(x,\theta)) \bar{z}(\eta(x,\theta)) d\theta \right] \varphi_i(x) dx \\ &\qquad\qquad - \sum_{j=0}^{n} \langle \kappa_j, \varphi_i \rangle_{\omega^{-\gamma,-\gamma}}, \\ b_j &= \int_{-1}^{1} (1-\theta^2)^{-\gamma} \bar{z}(\eta(.,\theta)) \varphi_j(\eta(.,\theta)) d\theta. \end{aligned}$$

For the interpolatory projection given by (2.12), we apply $I_n^{-\gamma,-\gamma}$ and $(\mathcal{I} - I_n^{-\gamma,-\gamma})$ to equation (2.20) and obtain

$$I_n^{-\gamma,-\gamma} y_n^{M,C} - I_n^{-\gamma,-\gamma}\mathcal{K}y_n^{M,C} = I_n^{-\gamma,-\gamma} f, \tag{4.2}$$

$$(\mathcal{I} - I_n^{-\gamma,-\gamma}) y_n^{M,C} - (\mathcal{I} - I_n^{-\gamma,-\gamma})\mathcal{K}_n y_n^{M,C} = (\mathcal{I} - I_n^{-\gamma,-\gamma}) f. \tag{4.3}$$

By writing

$$\mathcal{K}y_n^{M,C} = \mathcal{K}(\mathcal{I} - I_n^{-\gamma,-\gamma}) y_n^{M,C} + \mathcal{K} I_n^{-\gamma,-\gamma} y_n^{M,C},$$

and replacing $(\mathcal{I} - I_n^{-\gamma,-\gamma}) y_n^{M,C}$ by its expression from equation (4.3), $\mathcal{K}y_n^{M,C}$ becomes

$$\mathcal{K}y_n^{M,C} = \mathcal{K}\left((\mathcal{I} - I_n^{-\gamma,-\gamma})\mathcal{K}_n y_n^{M,C} + I_n^{-\gamma,-\gamma} y_n^{M,C} + (\mathcal{I} - I_n^{-\gamma,-\gamma}) f \right).$$

Now, by replacing $\mathcal{K}y_n^{M,C}$ in equation (4.2), we obtain

$$\begin{aligned} I_n^{-\gamma,-\gamma} y_n^{M,C} - I_n^{-\gamma,-\gamma}\mathcal{K}\left((\mathcal{I} - I_n^{-\gamma,-\gamma})\mathcal{K}_n y_n^{M,C} + I_n^{-\gamma,-\gamma} y_n^{M,C} + (\mathcal{I} - I_n^{-\gamma,-\gamma}) f \right) \\ = I_n^{-\gamma,-\gamma} f, \end{aligned}$$

and then, for $i = 0, 1, \ldots, n$, we have

$$y_n^{M,C}(\tau_i) - \mathcal{K}\Big((\mathcal{I} - I_n^{-\gamma,-\gamma})\mathcal{K}_n y_n^{M,C} + I_n^{-\gamma,-\gamma} y_n^{M,C} + (\mathcal{I} - I_n^{-\gamma,-\gamma})f\Big)(\tau_i) = f(\tau_i).$$

From (4.3), the approximate solution is given by

$$\begin{aligned} y_n^{M,C} &= I_n^{-\gamma,-\gamma} y_n^{M,C} + (\mathcal{I} - I_n^{-\gamma,-\gamma})\mathcal{K}_n y_n^{M,C} + (\mathcal{I} - I_n^{-\gamma,-\gamma})f \\ &= f + \sum_{i=0}^{n}(a_i - f_i)\ell_i + \sum_{i=0}^{n}\psi(\tau_i, a_i)\left[\int_{-1}^{1}\ell_i(\eta(.,\theta))\kappa(.,\eta(.,\theta))d\theta\right] \\ &\quad - \sum_{j=0}^{n}\left\{\int_{-1}^{1}\left[\sum_{i=0}^{n}\psi(\tau_i, a_i)\ell_i(\eta(.,\theta))\right]\kappa(\tau_j, \eta(.,\theta))d\theta\right\}\ell_j, \end{aligned} \tag{4.4}$$

where $f_j := f(\tau_j)$.

REMARK 4.2. In general, $\widetilde{y}_n^{M,C}$ is an improvement over $y_n^{M,C}$, obtained by substituting (4.4) into the definition (2.22). Now, applying $I_n^{-\gamma,-\gamma}$ to both sides of equations (2.20) and (2.22), we obtain

$$I_n^{-\gamma,-\gamma} y_n^{M,C} = I_n^{-\gamma,-\gamma}\mathcal{K} y_n^{M,C} + I_n^{-\gamma,-\gamma} f = I_n^{-\gamma,-\gamma}\widetilde{y}_n^{M,C},$$

and this yields

$$y_n^{M,C}(\tau_j) = \widetilde{y}_n^{M,C}(\tau_j), \qquad j = 0, 1, \ldots, n.$$

Using the above formula, we can prove that at the collocation node points, the convergence of $y_n^{M,C}$ to y_0 is as rapid as that of $\widetilde{y}_n^{M,C}$ to y_0. Hence, the estimate (3.33) gives the following superconvergence result for $y_n^{M,C}$ at the collocation points:

$$\max_{0 \le j \le n} |y(\tau_j) - y_n^{M,C}(\tau_j)| = \mathcal{O}(n^{-r}).$$

REMARK 4.3.

(i). By comparing the aspect of the methods from the above theoretical results we observe that the Jacobi Galerkin-type method provides better results with a faster convergence rate; however it is more expensive in terms of the computational cost than the Jacobi collocation-type method. This is due to the calculation of the double integration term to obtain the approximate solution.

(ii). Again, the size of the system of equations to be solved in the implementation of the modified Galerkin-type method is twice that of the modified collocation-type method. Moreover, the iterated modified Galerkin-type solution $\widetilde{y}_n^{M}$ converges faster than the modified Galerkin-type solution y_n^{M} and even faster than the solutions obtained by the proposed method using the interpolation projection.

5. Numerical results. In this section, we present the numerical results obtained by the projection-type and modified projection-type methods to verify our theoretical results. This will be achieved by using the Jacobi polynomials as basis functions of the subspace $\mathbb{X}_n$, which are generated by the recurrence relations as described in (2.10). As a result, we present the errors of the approximation solutions under the proposed methods in the infinity norm. Moreover, we give the maximum of the error of the solution $y_n^{M,C}$ at the collocation points, defined as

$$\max_{0 \le j \le n} |y_0(\tau_j) - y_n^{M,C}(\tau_j)| = \max_j |y_{0,j} - y_{n,j}^{M,C}|.$$

For the error calculation we consider a fine partition of the interval $[-1, 1]$ formed by the points $s_i = (2i-1)/m$, $i = 1, 2, \dots, m$. Here m is chosen to be a large number, for example, 100. The computations have been carried out for values n between 1 and 6. Note that all required integrals were calculated by the Gauss–Jacobi quadrature formula, and the Newton–Raphson method was used to solve the nonlinear systems. The numerical algorithms were implemented using Wolfram Mathematica on a computer with an Intel(R) Core(TM) i7-8550U CPU@1.80GHz (with a maximum speed of 1.99 GHz), 16.00GB RAM, and a 64-bit operating system.

The numerical results are presented in Tables 5.1 through 5.5. For a comprehensive illustration, they are graphically displayed in Figures 5.1 and 5.3 for two examples.

EXAMPLE 5.1 (Brunner et al. [9]). In this example, we consider the following Volterra–Hammerstein integral equation with a weakly singular kernel:

$$Y(z) = F(z) + \int_0^z \frac{1}{(z-x)^\gamma} K(z,x)\Psi(x, Y(x))dx, \qquad z \in [0,1], \;\; 0 < \gamma < 1, \tag{5.1}$$

where $K(x,z)\Psi(x,Y(x)) = (Y(x))^2$, $\gamma = \frac{1}{2}$, $F(z) = \sqrt{z} - \frac{4}{3} z^{\frac{3}{2}}$, and the exact solution is given by $Y(x) = \sqrt{x}$. Using the transformations $x = t^2$ and $z = s^2$, we obtain the transformed equation

$$\overline{Y}(s) = s - \frac{4}{3}s^3 + 2\int_0^s \frac{t\overline{Y}^2(t)}{(s^2-t^2)^{\frac{1}{2}}} dt, \qquad s \in [0,1],$$

with $\overline{Y}(s) = Y(s^2) = s$, and the following integral equation can be obtained by applying the transformations (2.2) and (2.5):

$$y(x) = f(x) + \int_{-1}^1 (1-\theta^2)^{-\frac{1}{2}} \kappa(x, \eta(x,\theta))\psi(\eta(x,\theta), y(\eta(x,\theta)))d\theta, \qquad x \in [-1,1],$$

where

$$f(x) = \frac{x+1}{2} - \frac{1}{6}(x+1)^3,$$

$$\kappa(x, \eta(x,\theta)) = \left(\frac{x+1}{2}\right)(\theta+3)^{-\frac{1}{2}}(\theta+1)^{\frac{3}{2}},$$

$$\psi(\eta(x,\theta), y(\eta(x,\theta))) = y(\eta(x,\theta))^2.$$

TABLE 5.1

Error norms and computation times using the Jacobi spectral Galerkin-type, the modified Galerkin-type, and the iterated modified Galerkin-type methods for Example 5.1.

n	$\|y_0 - y_n^G\|_\infty$	$CPU(s)$	$\|y_0 - y_n^{M,G}\|_\infty$	$CPU(s)$	$\|y_0 - \widetilde{y}_n^{M,G}\|_\infty$	$CPU(s)$
1	3.44528×10^{-1}	0.055	5.27355×10^{-3}	0.087	5.46058×10^{-3}	0.155
2	1.45011×10^{-3}	0.101	2.31953×10^{-5}	0.330	1.50777×10^{-6}	0.935
3	1.45958×10^{-4}	0.156	1.50882×10^{-6}	0.951	8.75691×10^{-8}	2.496
4	2.31936×10^{-5}	0.273	2.02481×10^{-7}	2.337	1.09059×10^{-8}	7.043
5	5.23218×10^{-6}	0.459	4.10667×10^{-8}	5.453	2.10034×10^{-9}	12.99
6	1.51092×10^{-6}	0.759	1.09059×10^{-8}	10.71	5.37177×10^{-10}	26.93

TABLE 5.2
Error norms and computation times using the Jacobi spectral collocation-type and the modified collocation-type methods for Example 5.1.

n	$\|y_0 - y_n^C\|_\infty$	$CPU(s)$	$\|y_0 - y_n^{M,C}\|_\infty$	$CPU(s)$
1	3.50131×10^{-1}	0.036	3.44528×10^{-1}	0.152
2	1.47370×10^{-3}	0.062	1.45955×10^{-4}	0.255
3	1.48332×10^{-4}	0.117	5.23218×10^{-6}	0.583
4	2.35709×10^{-5}	0.205	5.17953×10^{-7}	1.210
5	5.31728×10^{-6}	0.321	8.75691×10^{-8}	2.883
6	1.53336×10^{-6}	0.494	2.05826×10^{-8}	5.595

TABLE 5.3
Error norms using the Jacobi spectral iterated modified collocation-type method and the maximum errors at the collocation points for Example 5.1, along with their computation times.

n	$\|y_0 - \widetilde{y}_n^{M,C}\|_\infty$	$CPU(s)$	$\max_i \|y_{0,i} - y_{n,i}^{M,C}\|$	$CPU(s)$
1	1.36791×10^{-2}	0.165	6.72045×10^{-3}	0.499
2	5.23223×10^{-6}	0.585	3.73597×10^{-6}	0.137
3	2.02481×10^{-7}	1.732	1.66419×10^{-7}	0.385
4	2.05826×10^{-8}	4.374	1.81047×10^{-8}	0.978
5	3.50461×10^{-9}	9.466	3.20166×10^{-9}	2.230
6	8.25028×10^{-10}	18.97	7.71453×10^{-10}	4.517

EXAMPLE 5.2. Consider the following Volterra–Hammerstein integral equation with a weakly singular kernel:

$$Y(z) = F(z) + \int_0^z \frac{1}{(z-x)^\gamma} K(x,z)\Psi(x,Y(x))dx, \qquad z\in[0,1],\ \ 0<\gamma<1, \tag{5.2}$$

where $K(x,z) = (xz)^{\frac{1}{24}}$, $\Psi(x,Y(x))) = (Y(x))^5$, $\gamma = \frac{3}{4}$, and $F(z)$ is selected so that $Y(z) = 2z^{\frac{1}{3}} - 2z^{\frac{1}{6}} + \frac{1}{2}$. We apply a variable transformation $x = t^6$ and $z = s^6$, which reduces to

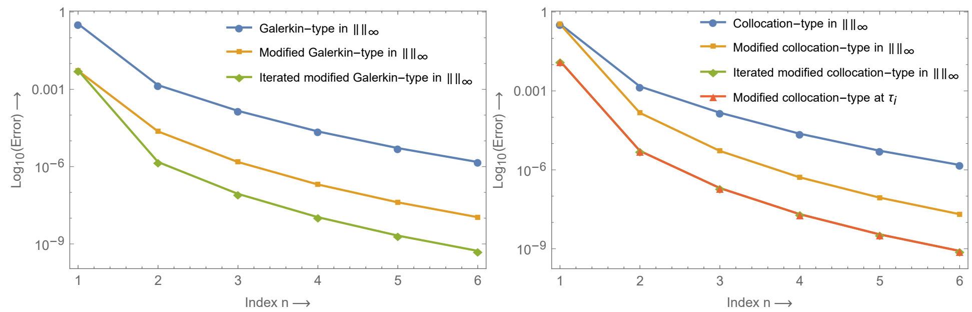
$$\overline{Y}(s) = \overline{F}(s) + 6\int_0^s \frac{t^5(st)^{\frac{1}{4}}\,\overline{Y}^5(t)}{(s^6-t^6)^{\frac{3}{4}}}dt, \qquad s\in[0,1],$$

with $\overline{F}(s)$ selected so that $\overline{Y}(s) = Y(s^6) = 2s^2 - 2s + \frac{1}{2}$. When the transformations (2.2) and (2.5) are applied to the integral equation (5.2), the result is

$$y(x) = f(x) + \int_{-1}^1 (1-\theta^2)^{-\frac{3}{4}}\kappa(x,\eta(x,\theta))\psi(\eta(x,\theta),y(\eta(x,\theta)))d\theta, \qquad x\in[-1,1],$$

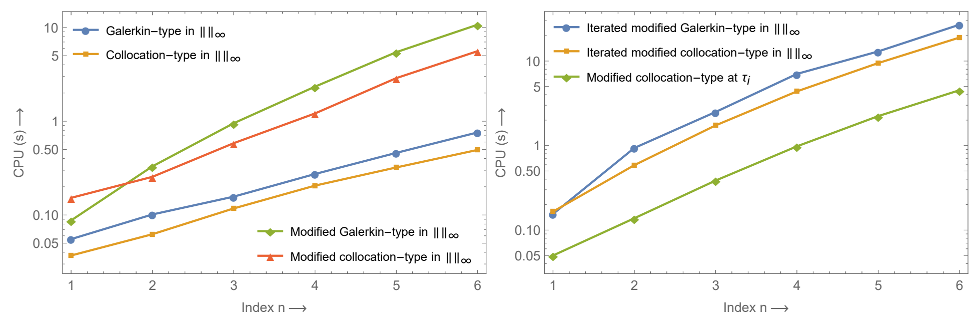
where

$$\kappa(x,\eta(x,\theta)) = \frac{3}{2^{\frac{3}{4}}}\left(\frac{x+1}{2}\right)^2 \frac{(1+\theta)^6}{[2^5+2^4(1+\theta)+2^3(1+\theta)^2+2^2(1+\theta)^3+2(1+\theta)^4+(1+\theta)^5]^{\frac{3}{4}}},$$
$$\psi(\eta(x,\theta),y(\eta(x,\theta))) = y(\eta(x,\theta))^5.$$



(a) Methods based on orthogonal projection. (b) Methods based on interpolatory projection.

FIG. 5.1. *Numerical errors for Example 5.1. Here (a) and (b) represent the graphs of the errors obtained in Tables 5.1 through 5.3.*



(a) Projection-type and modified projection-type methods (b) Iterated modified projection-type methods

FIG. 5.2. *CPU time results for Example 5.1. Here (a) and (b) represent the graphs of the CPU times obtained in Tables 5.1 through 5.3.*

REMARK 5.3. Brunner et al. [9] solved Example 5.1 using the collocation method with piecewise polynomials. In their approach, achieving an error of order 10^{-6} requires solving a system of equations of size 128×128. In contrast, our Jacobi spectral collocation-type method, as shown in Table 5.2, achieves a comparable level of accuracy by solving a smaller system of equations of size 6×6. Therefore, our method necessitates solving a significantly smaller system of equations compared to Brunner et al. [9].

The tables above illustrate the performance of the Jacobi projection-type and the modified projection-type methods applied to equations (5.1) and (5.2), respectively. As expected, it can be observed that the obtained maximum absolute errors show a great improvement when the proposed method is applied to the transformed equations. A smooth solution makes the approximation more accurate and will improve the convergence rates. In addition to being computationally efficient, Jacobi polynomials also provide a high degree of accuracy, making them ideal for spectral methods.

Comparing the aspects of the methods, we observe that the Jacobi modified projection-type method achieves better convergence rates than the Jacobi projection-type method, while the iterated Jacobi modified projection-type method converges the fastest among them. For example, in Tables 5.1, 5.2, and 5.3 of Example 5.1, to achieve an error of order 10^{-6}, the Jacobi spectral projection method requires solving a system of equations of size 6×6, whereas the Jacobi spectral modified method needs a system of equations of size 3×3. In contrast, the

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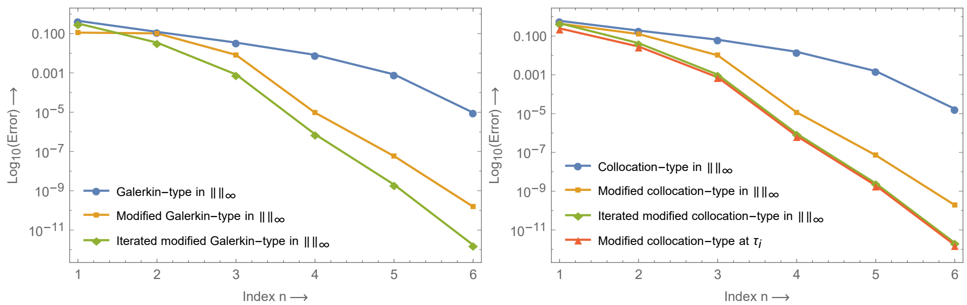

(a) Methods based on orthogonal projection. (b) Methods based on interpolatory projection.

FIG. 5.3. *Numerical errors of Example 5.2. Here (a) and (b) represent the graphs of the errors obtained in Tables 5.4 and 5.5.*

TABLE 5.4
Error norms using the Jacobi spectral Galerkin-type, the modified Galerkin-type, and the iterated modified Galerkin-type methods for Example 5.2.

n	$\|y_0 - y_n^G\|_\infty$	$\|y_0 - y_n^{M,G}\|_\infty$	$\|y_0 - \widetilde{y}_n^{M,G}\|_\infty$
1	4.48904×10^{-1}	1.12607×10^{-1}	3.28904×10^{-1}
2	1.23015×10^{-1}	1.03015×10^{-1}	3.49220×10^{-2}
3	3.49220×10^{-2}	8.29061×10^{-3}	7.97763×10^{-4}
4	8.90061×10^{-3}	9.48552×10^{-6}	7.54757×10^{-7}
5	8.36663×10^{-4}	5.83087×10^{-8}	2.05953×10^{-9}
6	9.55752×10^{-6}	1.55961×10^{-10}	1.75129×10^{-12}

TABLE 5.5
Error norms using the Jacobi spectral collocation-type, the modified collocation-type, and the iterated modified collocation-type methods and the maximum errors at the collocation points for Example 5.2.

n	$\|y_0 - y_n^C\|_\infty$	$\|y_0 - y_n^{M,C}\|_\infty$	$\|y_0 - \widetilde{y}_n^{M,C}\|_\infty$	$\max_i \lvert y_{0,i} - y_{n,i}^{M,C}\rvert$
1	6.11415×10^{-1}	4.38442×10^{-1}	4.58792×10^{-1}	2.48590×10^{-1}
2	1.91500×10^{-1}	1.26665×10^{-1}	4.29393×10^{-2}	2.88077×10^{-2}
3	6.49181×10^{-2}	1.01939×10^{-2}	1.02874×10^{-3}	7.47953×10^{-4}
4	1.54118×10^{-2}	1.16632×10^{-5}	9.28031×10^{-7}	7.01412×10^{-7}
5	1.55531×10^{-3}	7.16950×10^{-8}	2.53235×10^{-9}	1.95584×10^{-9}
6	1.76331×10^{-5}	1.91763×10^{-10}	2.15512×10^{-12}	1.68665×10^{-12}

iterated Jacobi spectral modified projection-type method only requires solving a system of equations of size 2×2. A similar observation can be made for Example 5.2.

In Figure 5.2, we investigate the required CPU time (in seconds) to emphasize the differences between the various approaches. The results displayed in the figure show that methods based on interpolatory projection require less CPU time compared to those based on orthogonal projection to achieve high-precision calculations. This disparity can be attributed to the fewer number of arithmetic operations related to the integration calculation involved in the interpolatory projection methods compared to orthogonal projection methods.

6. Conclusion. In this study, efficient Jacobi spectral methods are described for the numerical solution of weakly singular Volterra–Hammerstein integral equations. Theoretically, error bounds and convergence rates of the presented methods are obtained. To confirm our

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theoretical findings, we have provided two examples, demonstrating the accordance of our results with the earlier theoretical analysis. Furthermore, the proposed approaches exhibit good accuracy even with low-degree polynomials. It is noteworthy that achieving a comparable level of accuracy with piecewise polynomials would require solving much larger nonlinear systems. We believe that sharper estimates, particularly for the interpolatory projection, could be provided, requiring fewer arithmetic operations than the orthogonal projection.

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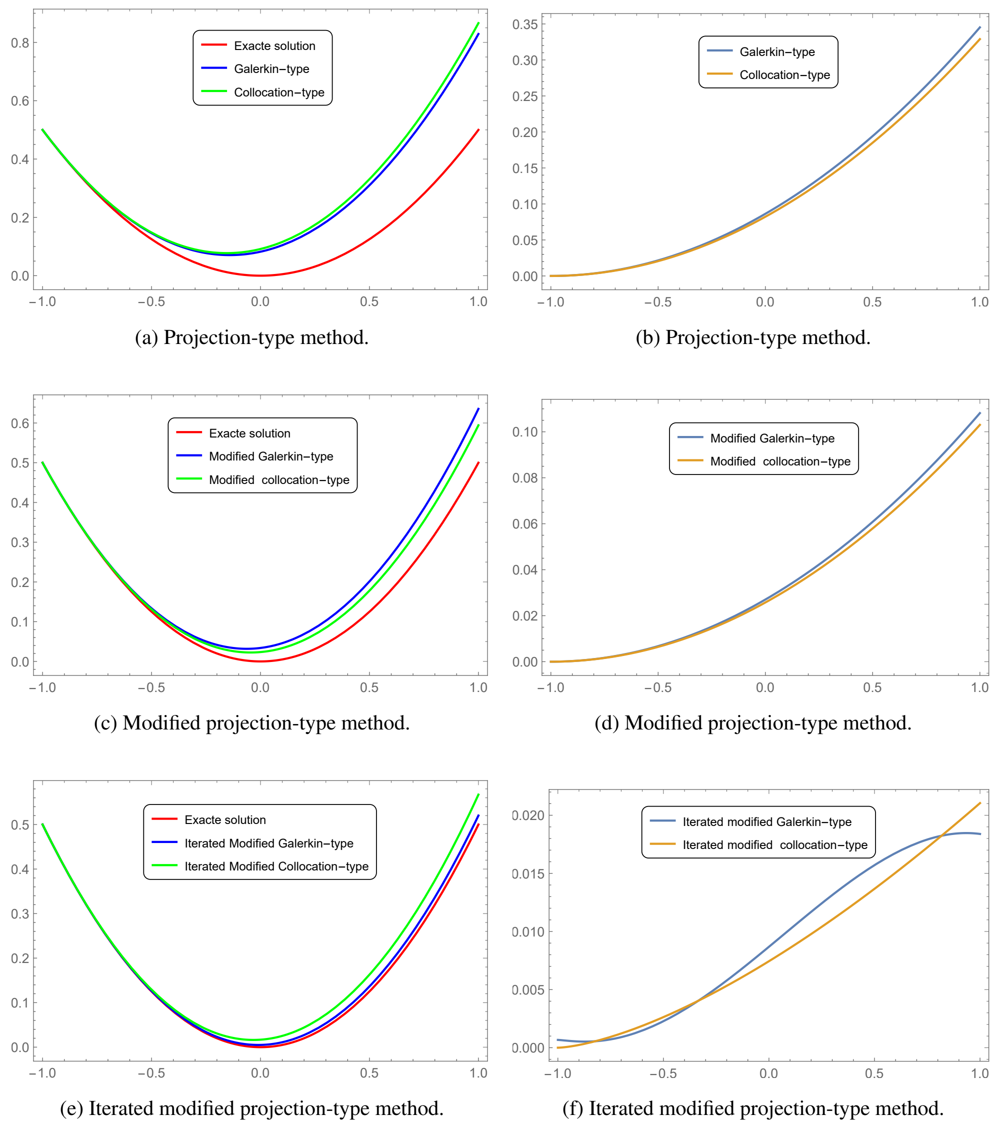

(a) Projection-type method. (b) Projection-type method.

(c) Modified projection-type method. (d) Modified projection-type method.

(e) Iterated modified projection-type method. (f) Iterated modified projection-type method.

FIG. 6.1. *For $n = 1$, we compare on the left the exact solution of Example 5.2 with the approximations produced by the various methods, while on the right, we give the corresponding errors in absolute values.*

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