Electronic Transactions on Numerical Analysis.
Volume 60, pp. 405–420, 2024.

ISSN 1068–9613.
DOI: 10.1553/etna_vol60s405

ETNA
Kent State University and
Johann Radon Institute (RICAM)

ANALYSIS OF A ONE-DIMENSIONAL NONLOCAL THERMOELASTIC PROBLEM*

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Abstract. In this paper we study a one-dimensional dynamic thermoelastic problem assuming that the elastic coefficient is negative. Following the ideas proposed by Eringen in the 80s, a nonlocal term is introduced into the constitutive equation for the displacements, leading to a hyperbolic problem. Then, we consider the numerical approximation of the problem by using the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property is proved, and an a priori error analysis is done, from which we can conclude the linear convergence of the approximations under suitable regularity for the continuous solution. Finally, some numerical simulations are presented, including the demonstration of the numerical convergence and the behavior of the discrete energy for several choices of the constitutive parameters.

Key words. nonlocal thermoelasticity, finite elements, discrete stability, a priori error estimates, numerical simulations

AMS subject classifications. 65M15, 65M60, 65M12

1. Introduction.

Thermoelasticity is a topic which has received a huge quantity of contributions in the literature. We can find a large amount of studies focused on the quantitative and qualitative properties of the solutions. Among the qualitative ones, we can cite some issues related to the existence, uniqueness, and the energy decay of the solutions; however, these properties are only obtained if the tensors that define this problem satisfy certain conditions. For example, in order to guarantee that this problem is well posed, it is needed that the elasticity tensor is positive definite (among other conditions; see, for instance, [12, 13]), but the axioms of thermomechanics do not imply this property. At the same time, it is well known that this condition does not hold when the solid is initially prestressed or if there exists a flux of initial heat [5, 6, 7, 8, 9, 10]. Therefore, it is rather natural to study our thermoelastic problem even in the case that the elasticity tensor is not positive definite.

In this work, we study a one-dimensional thermoelastic problem (see the equations (2.1)–(2.4) below) when the elastic coefficient is negative. Thus, in order to guarantee the existence of solutions provided in Appendix A, we assume that the mechanical problem is nonlocal in the sense of Eringen [3, 4]. In fact, this type of mechanism would allow us to obtain the existence and the continuous dependence on the data of the solutions.

Keeping in mind the above comments, it will be convenient to recall how we can extend the nonlocality. Since we are going to work in the one-dimensional homogeneous case, we restrict ourselves to this setting. In this case, we already know that the system of equations can be written in the following form:

$$t_x = \rho\ddot{u},$$
$$t = \int_0^\ell \alpha(|x-x'|)\sigma(x')\,dx',$$
$$\sigma = Cu_x,$$

*Received January 24, 2024. Accepted June 11, 2024. Published online on August 26, 2024. Recommended by Christian Vergara.

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where u is the displacement, σ represents the stress, t is the nonlocal stress, ρ denotes the mass density, and C is the elastic coefficient.

Usually, the kernel of the nonlocality $\alpha(|x-x'|)$ satisfies several properties [11]. One of them is that it must correspond to a Green's function of a certain differential operator, that is,

$$L\alpha(|x-x'|)=\delta(x-x'),$$

where δ is the Dirac symbol. When we apply the differential operator to the relation between the stresses, we find that

$$Lt=\sigma,$$

and so we have

$$\sigma_x=L(\rho\ddot{u}).$$

In this work, we study the thermoelastic problem when the solid is homogeneous and isotropic, and we also have

$$L=I-\varepsilon^2\partial_{xx},$$

where ∂_{xx} represents the second-order spatial derivative.

Finally, it is worth to note that we impose that

$$\sigma=-\mu^* u_x-\beta\theta, \qquad q=m\theta_x, \qquad \rho\eta=c\theta+\beta u_x,$$

where q is the heat flux, η is the entropy, and θ is the relative temperature. We also recall the energy equation

$$\rho\dot{\eta}=q_x.$$

This article is organized in the following way. The thermomechanical problem is presented in the next section. We note that, for the sake of simplicity in the writing of this work, we have sketched the analytical results in Appendix A at the end of the manuscript. Then, in Section 3 we focus on the main part of this contribution, the numerical analysis of this problem. By using the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives, the fully discrete scheme is introduced. A discrete stability property is obtained, and an a priori error analysis is performed applying some technical estimates and a discrete version of Gronwall's lemma. Linear convergence of the approximations is derived if we assume additional regularity for the continuous solution. Some numerical simulations are presented in Section 4 to demonstrate the accuracy of the approximations in an academical example and the behavior of the discrete energy for some particular choices of the parameters, obtaining stable and unstable solutions. Finally, the existence of a unique solution and its regularity are sketched in Appendix A.

2. The thermomechanical model. Let us denote by $(0,\ell)$, with $\ell>0$ being the length of the bar, the one-dimensional domain occupied by the thermoelastic material. As usual, let $x\in(0,\ell)$ be the spatial and $t\in[0,T]$ the time variable, where $[0,T]$ is the time interval and $T>0$ is the final time.

If we denote by u and θ the displacement field and the temperature, respectively, taking into account that

$$\begin{aligned}&\sigma_x=-\mu^* u_{xx}-\beta\theta_x=L(\rho\ddot{u})=\rho\ddot{u}-\rho\varepsilon^2\ddot{u}_{xx},\\&\rho\dot{\eta}=c\dot{\theta}+\beta\dot{u}_x=q_x=m\theta_{xx},\end{aligned}$$

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we study the following thermoelastic problem:

Find the displacement $u : [0,\ell] \times [0,T] \to \mathbb{R}$ *and the temperature* $\theta : [0,\ell] \times [0,T] \to \mathbb{R}$ *such that*

$$\rho\ddot{u} - \rho\varepsilon^2\ddot{u}_{xx} = -\mu^* u_{xx} - \beta\theta_x \quad \text{in } (0,\ell)\times(0,T), \tag{2.1}$$

$$c\dot{\theta} = m\theta_{xx} - \beta\dot{u}_x \quad \text{in } (0,\ell)\times(0,T), \tag{2.2}$$

$$u(x,0) = u^0(x), \quad \dot{u}(x,0) = v^0(x), \quad \theta(x,0) = \theta^0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.3}$$

$$u(0,t) = u(\ell,t) = \theta(0,t) = \theta(\ell,t) = 0 \quad \text{for a.e. } t \in (0,T). \tag{2.4}$$

In the previous system of equations, $\rho > 0$ is the mass density, $\mu^* > 0$ is the elasticity coefficient, β is the thermal coupling, $c > 0$ is the heat capacity, and $m > 0$ is the thermal diffusion. Moreover, $\varepsilon > 0$ is a regularization parameter which inserts the non-locality into the model. It is worth noting that μ^* defines a non-positive definite elasticity.

For the sake of clarity in the presentation of the results in this paper, the existence of a unique solution is sketched at the end of the manuscript in Appendix A. As can be seen there, the above problem admits a unique solution with the following regularity:

$$u \in C^2([0,T]; H_0^1(0,\ell)) \cap C([0,T]; H^2(0,\ell)),$$
$$\theta \in C([0,T]; H^2(0,\ell)) \cap C^1([0,T]; L^2(0,\ell)).$$

We note that, from the above results, we can conclude that the initial data u^0, v^0, and θ^0 must satisfy the regularity conditions:

$$u^0,\, \theta^0 \in H_0^1(0,\ell) \cap H^2(0,\ell), \quad v^0 \in H_0^1(0,\ell).$$

Now, we derive the weak formulation of the problem (2.1)–(2.4). Denoting $Y = L^2(0,\ell)$ and $V = H_0^1(0,\ell)$, multiplying equations (2.1) and (2.2) by adequate test functions, and taking into account the boundary conditions (2.4), we obtain the following variational problem written in terms of the velocity field $v = \dot{u}$ and the temperature θ:

Find the velocity field $v : [0,T] \to V$ *and the temperature* $\theta : [0,T] \to V$ *such that* $v(0) = v^0$, $\theta(0) = \theta^0$, *and, for a.e.* $t \in (0,T)$ *and for all* $w, r \in V$,

$$\rho(\dot{v}(t), w)_Y + \rho\varepsilon^2(\dot{v}_x(t), w_x)_Y - \mu^*(u_x(t), w_x)_Y + \beta(\theta_x(t), w)_Y = 0, \tag{2.5}$$

$$c(\dot{\theta}(t), r)_Y + m(\theta_x(t), r_x)_Y + \beta(v_x(t), r)_Y = 0, \tag{2.6}$$

where the displacement field u is recovered from the equation:

$$u(t) = \int_0^t v(s)\, ds + u^0. \tag{2.7}$$

In the variational equations (2.5) and (2.6), we have used the notation $(\cdot,\cdot)_X$ for the inner product in the Hilbert space X. Moreover, we will represent by $\|\cdot\|_X$ its associated norm.

3. Numerical analysis of a fully discrete approximation. In this section, we study, from the numerical point of view, the variational problem defined by the variational equations (2.5) and (2.6) and the relation (2.7).

3.1. Fully discrete approximation. In this section, we introduce a fully discrete approximation of the above weak formulation. We do this in two steps. In order to provide the spatial approximation, let us define a uniform partition of the spatial domain denoted by $a_0 = 0 < \ldots < a_M = \ell$, and let us construct the finite element space:

$$V^h = \left\{ w^h \in C([0,\ell]) \cap V\,;\; w^h_{|_{[a_i,a_{i+1}]}} \in P_1([a_i,a_{i+1}]), \text{ for } i = 0,\ldots,M-1 \right\}.$$

In this definition, we have denoted by $P_1([a_i,a_{i+1}])$ the space of affine functions in the subinterval $[a_i,a_{i+1}]$, and, as usual, we let $h=a_{i+1}-a_i=\ell/M$ be the mesh size. Without doubt, we could extend the analysis to the case of higher-order finite elements, but we omit it for the sake of readability.

Now, we can define the discrete initial conditions as

$$u^{0h}=P^h u^0,\quad v^{0h}=P^h v^0,\quad \theta^{0h}=P^h\theta^0,$$

where P^h represents the finite element interpolation operator over V^h (see [2] for details).

For the discretization of the time derivatives, we denote by $t_0=0<\ldots<t_N=T$ a uniform partition of the time interval with time step $k=t_1-t_0=T/N$ and nodes $t_n=nk$, for $n=0,\ldots,N$. Of course, the analysis presented in this section could be extended to the case of non-uniform partitions. We employ the usual notation: let $w_n=w(t_n)$ be the value of a continuous function $w(t)$ at time $t=t_n$, and, for a sequence $\{w_n\}_{n=0}^N$, let $\delta w_n=(w_n-w_{n-1})/k$ be its divided differences.

By using the well-known implicit Euler scheme, we have the following fully discrete problem:

Find the discrete velocity $\{v_n^{hk}\}_{n=0}^N\subset V^h$ and the discrete temperature $\{\theta_n^{hk}\}_{n=0}^N\subset V^h$ such that $v_0^{hk}=v^{0h}$, $\theta_0^{hk}=\theta^{0h}$, and, for $n=1,\ldots,N$ and for all $w^h,r^h\in V^h$,

$$\rho(\delta v_n^{hk},w^h)_Y+\rho\varepsilon^2((\delta v_n^{hk})_x,w_x^h)_Y-\mu^*((u_n^{hk})_x,w_x^h)_Y+\beta((\theta_n^{hk})_x,w^h)_Y=0, \tag{3.1}$$

$$c(\delta\theta_n^{hk},r^h)_Y+m((\theta_n^{hk})_x,r_x^h)_Y+\beta((v_n^{hk})_x,r^h)_Y=0, \tag{3.2}$$

where the discrete displacement field u_n^{hk} is obtained from the equation:

$$u_n^{hk}=k\sum_{j=1}^n v_j^{hk}+u^{0h}. \tag{3.3}$$

Thanks to the definition of the coefficients of the problem and applying the classical Lax-Milgram lemma, it is easy to show that the above fully discrete problem has a unique solution.

3.2. Discrete stability and a priori error analysis. Now the aim of this section is to prove the main theoretical results: the discrete stability of the solution to the problem (3.1)–(3.3) and an a priori error estimates. First, discrete stability is summarized in the following lemma:

LEMMA 3.1. *There exists a positive constant C which is independent of the discretization parameters h and k, but depending on the constitutive data and the final time, such that*

$$\|v_n^{hk}\|_V+\|u_n^{hk}\|_V+\|\theta_n^{hk}\|_Y\le C,\qquad \textit{for}\ \ n=1,\ldots,N.$$

Proof. Taking $w^h=v_n^{hk}$ as a test function in equation (3.1), we find that

$$\rho(\delta v_n^{hk},v_n^{hk})_Y+\rho\varepsilon^2((\delta v_n^{hk})_x,(v_n^{hk})_x)_Y-\mu^*((u_n^{hk})_x,(v_n^{hk})_x)_Y+\beta((\theta_n^{hk})_x,v_n^{hk})_Y=0,$$

and keeping in mind that

$$\begin{aligned}
(\delta v_n^{hk},v_n^{hk})_Y&\ge\frac{1}{2k}\Big[\|v_n^{hk}\|_Y^2-\|v_{n-1}^{hk}\|_Y^2\Big],\\
((\delta v_n^{hk})_x,(v_n^{hk})_x)_Y&\ge\frac{1}{2k}\Big[\|(v_n^{hk})_x\|_Y^2-\|(v_{n-1}^{hk})_x\|_Y^2\Big],\\
\mu^*((u_n^{hk})_x,(v_n^{hk})_x)_Y&\le C(\|(u_n^{hk})_x\|_Y^2+\|(v_n^{hk})_x\|_Y^2),\\
\beta((\theta_n^{hk})_x,v_n^{hk})_Y&=-\beta(\theta_n^{hk},(v_n^{hk})_x)_Y\le C(\|\theta_n^{hk}\|_Y^2+\|(v_n^{hk})_x\|_Y^2),
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality several times as well as Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon^2} b^2, \qquad a, b, \epsilon \in \mathbb{R},\, \epsilon > 0, \tag{3.4}$$

and where C is a generic positive constant independent of h and k that may change its value at each occurrence, it follows that

$$\begin{aligned}\frac{\rho}{2k}\Big[\|v_n^{hk}\|_Y^2 - \|v_{n-1}^{hk}\|_Y^2\Big] + \frac{\rho\varepsilon^2}{2k}\Big[\|(v_n^{hk})_x\|_Y^2 - \|(v_{n-1}^{hk})_x\|_Y^2\Big]& \\ \leq C\Big(\|(u_n^{hk})_x\|_Y^2 + \|(v_n^{hk})_x\|_Y^2 + \|\theta_n^{hk}\|_Y^2\Big)&.\end{aligned}$$

Now, we obtain the estimates for the discrete temperature. Taking $r^h = \theta_n^{hk}$ as a test function in the discrete equation (3.2), we obtain

$$c(\delta\theta_n^{hk}, \theta_n^{hk})_Y + m((\theta_n^{hk})_x, (\theta_n^{hk})_x)_Y + \beta((v_n^{hk})_x, \theta_n^{hk})_Y = 0.$$

If we take into account that

$$\begin{aligned}(\delta\theta_n^{hk}, \theta_n^{hk}))_Y &\geq \frac{1}{2k}\Big[\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big],\\ \beta((v_n^{hk})_x, \theta_n^{hk})_Y &\leq C(\|(v_n^{hk})_x\|_Y^2 + \|\theta_n^{hk}\|_Y^2),\end{aligned}$$

we find that

$$\frac{c}{2k}\Big[\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big] \leq C(\|(v_n^{hk})_x\|_Y^2 + \|\theta_n^{hk}\|_Y^2).$$

Combining the estimates for the discrete velocity field and the discrete temperature, it follows that

$$\begin{aligned}\frac{\rho}{2k}\Big[\|v_n^{hk}\|_Y^2 - \|v_{n-1}^{hk}\|_Y^2\Big] + \frac{\rho\varepsilon^2}{2k}\Big[\|(v_n^{hk})_x\|_Y^2 - \|(v_{n-1}^{hk})_x\|_Y^2\Big] + \frac{c}{2k}\Big[\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big]& \\ \leq C\Big(\|(u_n^{hk})_x\|_Y^2 + \|(v_n^{hk})_x\|_Y^2 + \|\theta_n^{hk}\|_Y^2\Big)&.\end{aligned}$$

After a multiplication by k and a summation up to n, we have

$$\begin{aligned}\|v_n^{hk}\|_Y^2 + \|(v_n^{hk})_x\|_Y^2 + \|\theta_n^{hk}\|_Y^2 \leq Ck\sum_{j=1}^{n}\Big(\|(u_j^{hk})_x\|_Y^2 + \|(v_j^{hk})_x\|_Y^2 + \|\theta_j^{hk}\|_Y^2\Big)& \\ + C(\|v^{0h}\|_V^2 + \|\theta^{0h}\|_Y^2)&.\end{aligned}$$

Finally, keeping in mind that

$$\|(u_n^{hk})_x\|_Y^2 \leq Ck\sum_{j=1}^{n}\|(v_j^{hk})_x\|_Y^2 + C\|(u^{0h})_x\|_Y^2,$$

using a discrete version of Gronwall's inequality (see [1] for example), we obtain the desired discrete stability. □

Secondly, we provide the main a priori error estimates result. This is stated in the following theorem:

THEOREM 3.2. *If we denote by (u, v, θ) the solution to the variational problem (2.5)–(2.7) and by $\{u_n^{hk}, v_n^{hk}, \theta_n^{hk}\}_{n=0}^N$ the solution to the discrete variational problem (3.1)–(3.3), then we have the following a priori error estimates, for all $\{w_n^h\}_{n=0}^N$, $\{r_n^h\}_{n=0}^N \subset V^h$:*

$$
\begin{aligned}
\max_{0\le n\le N}&\Big\{\|v_n - v_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2\Big\}\\
&\le Ck\sum_{j=1}^N \Big(\|\dot v_j - \delta v_j\|_V^2\\
&\qquad\qquad + \|\dot\theta_j - \delta\theta_j\|_Y^2 + \|v_j - w_j^h\|_V^2 + \|\theta_j - r_j^h\|_V^2 + I_j\Big)\\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1}\Big(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|_V^2 + \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|_Y^2\Big)\\
&\quad + C\max_{0\le n\le N}\Big(\|v_n - w_n^h\|_V^2 + \|\theta_n - r_n^h\|_Y^2\Big)\\
&\quad + C\Big(\|v^0 - v^{0h}\|_V^2 + \|u^0 - u^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|_Y^2\Big),
\end{aligned}
$$

where I_j represents the integration error defined as

$$
I_j = \left\| \int_0^{t_j} v(s)\,ds - k\sum_{l=1}^j v_j \right\|_V^2 \tag{3.5}
$$

and C is again a generic positive constant independent of h and k but depending on the constitutive data and the final time.

Proof. First, we obtain the a priori error estimates for the velocity field. Subtracting the variational equation (2.5) at time $t = t_n$ and for a test function $w = w^h \in V^h \subset V$ and the discrete variational equation (3.1), we have, for all $w^h \in V^h$,

$$
\begin{aligned}
\rho(\dot v_n - \delta v_n^{hk}, w^h)_Y + \rho\varepsilon^2((\dot v_n - \delta v_n^{hk})_x, w_x^h)_Y - \mu^*((u_n - u_n^{hk})_x, w_x^h)_Y&\\
+ \beta((\theta_n - \theta_n^{hk})_x, w^h)_Y = 0&,
\end{aligned}
$$

and so we find that, for all $w^h \in V^h$,

$$
\begin{aligned}
\rho(\dot v_n - \delta v_n^{hk}, v_n - v_n^{hk})_Y &+ \rho\varepsilon^2((\dot v_n - \delta v_n^{hk})_x, (v_n - v_n^{hk})_x)_Y\\
&- \mu^*((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x)_Y + \beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk})_Y\\
&= \rho(\dot v_n - \delta v_n^{hk}, v_n - w^h)_Y + \rho\varepsilon^2((\dot v_n - \delta v_n^{hk})_x, (v_n - w^h)_x)_Y\\
&- \mu^*((u_n - u_n^{hk})_x, (v_n - w^h)_x)_Y + \beta((\theta_n - \theta_n^{hk})_x, v_n - w^h)_Y.
\end{aligned}
$$

Now, taking into account that

$$
\begin{aligned}
(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}))_Y &\ge \frac{1}{2k}\Big[\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big],\\
((\delta v_n - \delta v_n^{hk})_x, (v_n - v_n^{hk})_x)_Y &\ge \frac{1}{2k}\Big[\|(v_n - v_n^{hk})_x\|_Y^2 - \|(v_{n-1} - v_{n-1}^{hk})_x\|_Y^2\Big],\\
\mu^*((u_n - u_n^{hk})_x, (v_n - v_n^{hk}))_x)_Y &\le C(\|(u_n - u_n^{hk})_x\|_Y^2 + \|(v_n - v_n^{hk})_x\|_Y^2),\\
\beta((\theta_n - \theta_n^{hk})_x, w)_Y = -\beta(\theta_n - \theta_n^{hk}, w_x)_Y &\le C(\|\theta_n - \theta_n^{hk}\|_Y^2 + \|w_x\|_Y^2),
\end{aligned}
$$

where C is a generic positive constant independent of h and k that may change its value at each occurrence, using several times the Cauchy-Schwarz and Cauchy's inequality (3.4), it follows that, for all $w^h \in V^h$,

$$\frac{1}{2k}\Big[\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big] + \frac{1}{2k}\Big[\|(v_n - v_n^{hk})_x\|_Y^2 - \|(v_{n-1} - v_{n-1}^{hk})_x\|_Y^2\Big]$$
$$\leq C\Big(\|\dot{v}_n - \delta v_n\|_V^2 + \|v_n - w^h\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|v_n - v_n^{hk}\|_Y^2 + \|(v_n - v_n^{hk})_x\|_Y^2$$
$$+ \|(u_n - u_n^{hk})_x\|_Y^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)_Y + ((\delta v_n - \delta v_n^{hk})_x, (v_n - w^h)_x)_Y\Big).$$

Secondly, we derive the error estimates for the temperature. Therefore, if we subtract the variational equation (2.6) at time $t = t_n$ and for a test function $r = r^h \in V^h \subset V$ and the discrete variational equation (3.2), we obtain, for all $r^h \in V^h$,

$$c(\dot{\theta}_n - \delta\theta_n^{hk}, r^h)_Y + m((\theta_n - \theta_n^{hk})_x, r_x^h)_Y + \beta((v_n - v_n^{hk})_x, r^h)_Y = 0,$$

and so it follows that, for all $r^h \in V^h$,

$$c(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y + m((\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x)_Y$$
$$+ \beta((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk})_Y$$
$$= c(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y + m((\theta_n - \theta_n^{hk})_x, (\theta_n - r^h)_x)_Y$$
$$+ \beta((v_n - v_n^{hk})_x, \theta_n - r^h)_Y.$$

Now, keeping in mind that

$$(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y \geq \frac{1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big],$$
$$m((\theta_n - \theta_n^{hk})_x, (\theta_n - r^h)_x)_Y \leq \epsilon\|(\theta_n - \theta_n^{hk})_x\|_Y^2 + C\|\theta_n - r^h\|_V^2,$$
$$\beta((v_n - v_n^{hk})_x, w)_Y \leq C(\|(v_n - v_n^{hk})_x\|_Y^2 + \|w\|_Y^2),$$

where $\epsilon > 0$ is assumed small enough, using again the Cauchy-Schwarz and Cauchy's inequality (3.4), we have, for all $r^h \in V^h$,

$$\frac{1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big]$$
$$\leq C\Big(\|\dot{\theta}_n - \delta\theta_n\|_Y^2 + \|\theta_n - r^h\|_V^2$$
$$+ \|\theta_n - \theta_n^{hk}\|_Y^2 + \|(v_n - v_n^{hk})_x\|_Y^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y\Big).$$

Combining the estimates for the velocity field and the temperature, we find that, for all w^h, $r^h \in V^h$,

$$\frac{1}{2k}\Big[\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big] + \frac{1}{2k}\Big[\|(v_n - v_n^{hk})_x\|_Y^2 - \|(v_{n-1} - v_{n-1}^{hk})_x\|_Y^2\Big]$$
$$+ \frac{1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big]$$
$$\leq C\Big(\|\dot{v}_n - \delta v_n\|_V^2 + \|v_n - w^h\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|v_n - v_n^{hk}\|_Y^2 + \|(v_n - v_n^{hk})_x\|_Y^2$$
$$+ (\delta v_n - \delta v_n^{hk}, v_n - w^h)_Y + ((\delta v_n - \delta v_n^{hk})_x, (v_n - w^h)_x)_Y + \|\dot{\theta}_n - \delta\theta_n\|_Y^2$$
$$+ \|(u_n - u_n^{hk})_x\|_Y^2 + \|\theta_n - r^h\|_V^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y\Big).$$

Multiplying these estimates by k and after a summation up to n, we have, for all $\{w_j^h\}_{j=1}^n$, $\{r_j^h\}_{j=1}^n \subset V^h$,

$$\begin{aligned}
&\|v_n - v_n^{hk}\|_Y^2 + \|(v_n - v_n^{hk})_x\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 \\
&\quad \leq Ck \sum_{j=1}^n \Big(\|\dot{v}_j - \delta v_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|\theta_j - \theta_j^{hk}\|_Y^2 + \|v_j - v_j^{hk}\|_Y^2 \\
&\qquad + \|(v_j - v_j^{hk})_x\|_Y^2 + \|\dot{\theta}_j - \delta\theta_j\|_Y^2 + \|\theta_j - r_j^h\|_V^2 \\
&\qquad + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_Y + ((\delta v_j - \delta v_j^{hk})_x, (v_j - w_j^h)_x)_Y \\
&\qquad + \|(u_j - u_j^{hk})_x\|_Y^2 + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h)_Y \Big) \\
&\quad + C(\|v^0 - v^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|_Y^2).
\end{aligned}$$

Finally, taking into account that

$$\begin{aligned}
&k\sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_Y \\
&\quad = (v_n - v_n^{hk}, v_n - w_n^h)_Y + (v^{0h} - v^0, v_1 - w_1^h)_Y \\
&\qquad + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h))_Y, \\
&k\sum_{j=1}^n ((\delta v_j - \delta v_j^{hk})_x, (v_j - w_j^h)_x)_Y \\
&\quad = ((v_n - v_n^{hk})_x, (v_n - w_n^h)_x)_Y + ((v^{0h} - v^0)_x, (v_1 - w_1^h)_x)_Y \\
&\qquad + \sum_{j=1}^{n-1} ((v_j - v_j^{hk})_x, (v_j - w_j^h - (v_{j+1} - w_{j+1}^h))_x)_Y, \\
&k\sum_{j=1}^n (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h)_Y \\
&\quad = (\theta_n - \theta_n^{hk}, \theta_n - r_n^h)_Y + (\theta^{0h} - \theta^0, \theta_1 - r_1^h)_Y \\
&\qquad + \sum_{j=1}^{n-1} (\theta_j - \theta_j^{hk}, \theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h))_Y, \\
&\|u_n - u_n^{hk}\|_V^2 \leq C\Big(k\sum_{j=1}^N \|v_j - v_j^{hk}\|_V^2 + \|u^0 - u^{0h}\|_V^2 + I_n\Big),
\end{aligned}$$

where I_n is the integration error defined by (3.5), and applying a discrete version of Gronwall's inequality (see again [1]), we arrive at the a priori error estimates. □

As an example of application of the above a priori error estimates result, we derive a convergence order under suitable regularity conditions. So, let us assume that the continuous solution to problem (2.5)–(2.7) has the regularity

$$u \in H^3(0,T;Y) \cap C^2([0,T];H^2(0,\ell)), \quad \theta \in H^2(0,T;Y) \cap C^1([0,T];H^2(0,\ell)).$$

From the previous a priori error estimates, using well-known properties of the approximation by finite elements (see [2]) and some estimates provided in [1] in the study of damage problems, we may conclude that there exists a positive constant C independent of h and k such that

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|_V + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|_Y\Big\} \le C(h+k).$$

4. Numerical results. Finally, we describe the numerical scheme implemented in MATLAB for solving the fully discrete problem (3.1)–(3.3), and we provide some numerical examples to demonstrate the accuracy of the approximations and the behavior of the solution.

4.1. Numerical scheme. As a first step, given the solution v_{n-1}^{hk} and θ_{n-1}^{hk} at time t_{n-1}, the variables v_n^{hk} and θ_n^{hk} are obtained by solving the following discrete linear problem, for all $w^h, r^h \in V^h$,

$$\begin{aligned}
&\rho(\frac{1}{k}v_n^{hk}, w^h)_Y + \rho\varepsilon^2((\frac{1}{k}v_n^{hk})_x, w_x^h)_Y - \mu^*(k(v_n^{hk})_x, w_x^h)_Y + \beta((\theta_n^{hk})_x, w^h)_Y\\
&\qquad = \rho(\frac{1}{k}v_{n-1}^{hk}, w^h)_Y + \rho\varepsilon^2((\frac{1}{k}v_{n-1}^{hk})_x, w_x^h)_Y + \mu^*((u_{n-1}^{hk})_x, w_x^h)_Y,\\
&c(\frac{1}{k}\theta_n^{hk}, r^h)_Y + m((\theta_n^{hk})_x, r_x^h)_Y + \beta((v_n^{hk})_x, r^h)_Y = c(\frac{1}{k}\theta_{n-1}^{hk}, r^h)_Y.
\end{aligned}$$

The numerical scheme was implemented on a 3.2 GHz PC using MATLAB, and a typical run took about 0.22 sec of CPU time using the parameters $h = k = 0.001$.

4.2. Numerical convergence. In order to verify the accuracy of the finite element approximations, the first simulation corresponds to an academical example.

We have slightly modified the equations (2.1) and (2.2) into the form

$$\begin{aligned}
\rho\ddot{u} - \rho\varepsilon^2\ddot{u}_{xx} &= -\mu^* u_{xx} - \beta\theta_x + F_1 &&\text{in } (0,\ell)\times(0,T),\\
c\dot{\theta} &= m\theta_{xx} - \beta\dot{u}_x + F_2 &&\text{in } (0,\ell)\times(0,T),
\end{aligned}$$

where the supply terms F_1 and F_2 are given by

$$F_1(x,t) = e^t(6x + x(x-1) - 17), \qquad F_2(x,t) = e^t(6x + x(x-1) - 7).$$

In this example, we have used the following data:

$$\ell = 1,\quad T = 1,\quad \rho = 1,\quad \varepsilon = 3,\quad \mu^* = 2,\quad \beta = 3,\quad c = 1,\quad m = 2,$$

and the initial conditions, for all $x \in (0,1)$,

$$u^0(x) = v^0(x) = \theta^0(x) = x(x-1).$$

Therefore, the exact solution to this one-dimensional problem with homogeneous boundary conditions can be easily calculated, and it is given as, for a.e. $(x,t) \in [0,1]\times[0,1]$,

$$u(x,t) = \theta(x,t) = e^t x(x-1).$$

We consider estimates for the approximation errors of the form:

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|_V + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|_Y\Big\},$$

and we present them in Table 4.1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h + k$ is illustrated in

TABLE 4.1
Example 1: Numerical errors for some values of h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.170084	0.170118	0.170136	0.170147	0.170153	0.170156	0.170156
$1/2^4$	0.084983	0.084966	0.084970	0.084974	0.084977	0.084978	0.084979
$1/2^5$	0.042547	0.042481	0.042475	0.042475	0.042476	0.042477	0.042477
$1/2^6$	0.021395	0.021259	0.021241	0.021237	0.021237	0.021237	0.021237
$1/2^7$	0.010926	0.010666	0.010630	0.010621	0.010619	0.010618	0.010618
$1/2^8$	0.005878	0.005405	0.005333	0.005315	0.005310	0.005309	0.005309
$1/2^9$	0.003608	0.002837	0.002702	0.002666	0.002656	0.002655	0.002655
$1/2^{10}$	0.002690	0.001649	0.001417	0.001351	0.001331	0.001328	0.001327
$1/2^{11}$	0.002342	0.001151	0.000823	0.000708	0.000671	0.000666	0.000664
$1/2^{12}$	0.002214	0.000960	0.000574	0.000411	0.000346	0.000336	0.000332
$1/2^{13}$	0.002172	0.000889	0.000479	0.000287	0.000193	0.000173	0.000166
$1/2^{14}$	0.002160	0.000865	0.000443	0.000239	0.000126	0.000096	0.000083

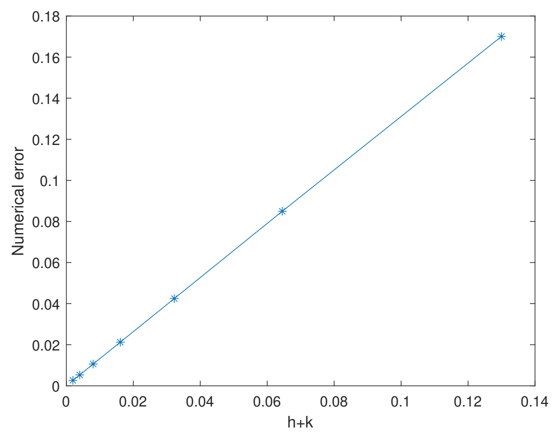

FIG. 4.1. *Example 1: asymptotic constant error.*

Figure 4.1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in the previous section, is achieved.

Now, we ask ourselves what happens if the final time T increases. Since the constant C obtained in Theorem 3.2 depends on the constitutive data but also on T, we modify the previous example by assuming that T varies between the values 1 and 10 and by using the same data but with the discretization parameters $h = k = 10^{-3}$. It is worth noting that, since this dependence is exponential (see [1] and the references cited therein), we have restricted ourselves to a maximum final time $T = 10$ for the sake of clarity. Therefore, in Figure 4.2 we display the evolution of the numerical error with respect to the final time. As it was expected, it seems that an exponential growth is found.

Finally, we consider a new case where the initial conditions do not satisfy the required regularity. So, we solve the same problem as before without supply terms and the final time $T = 1$ but with the following initial conditions, for all $x \in (0, 1)$,

$$u^0(x) = v^0(x) = \theta^0(x) = \begin{cases} x/2 & \text{if } x \in [0, 0.5], \\ -x+1 & \text{if } x \in (0.5, 1]. \end{cases}$$

That is, in this case the initial conditions are not continuous, and they do not have the regularity

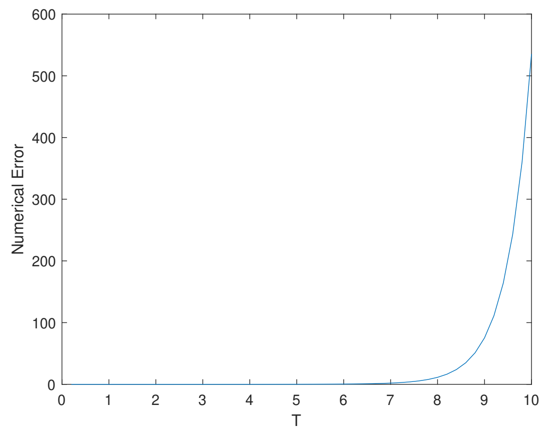

FIG. 4.2. *Example 1: evolution of the numerical error depending on the final time.*

imposed in Theorem 3.2. Since we cannot calculate the exact solution, we replace it by the numerical solution obtained with parameters $h = 2^{-18}$ and $k = 10^{-6}$.

Therefore, in Table 4.1 we provide the numerical errors for several values of the discretization parameters h and k; however, due to the expected loss of regularity, we have used the L^2-norm in all the terms. As can be seen, it seems that convergence is found. Moreover, in Figure 4.2 the evolution of these L^2 numerical errors are displayed depending on the parameter $h + k$. It is worth noting that we have also analyzed the behavior of the H^1-norm, as we did for the regular solution, but it does not decrease with h and k.

TABLE 4.2
Example 1: L^2-numerical errors for some values of h and k with non-regular initial conditions.

$h \downarrow k \rightarrow$	0.001	0.005	0.002	0.0001	0.0005	0.0002	0.00001
$1/2^3$	0.186744	0.186918	0.187023	0.187058	0.187076	0.187086	0.187090
$1/2^4$	0.130162	0.130293	0.130372	0.130399	0.130412	0.130420	0.130422
$1/2^5$	0.091082	0.091177	0.091235	0.091255	0.091264	0.091270	0.091272
$1/2^6$	0.063886	0.063954	0.063997	0.064011	0.064019	0.064023	0.064024
$1/2^7$	0.044858	0.044905	0.044936	0.044947	0.044953	0.044956	0.044957
$1/2^8$	0.031485	0.031514	0.031536	0.031544	0.031548	0.031551	0.031552
$1/2^9$	0.022044	0.022053	0.022066	0.022072	0.022075	0.022077	0.022078
$1/2^{10}$	0.015339	0.015323	0.015327	0.015331	0.015334	0.015335	0.015336
$1/2^{11}$	0.010526	0.010484	0.010476	0.010477	0.010479	0.010480	0.010480
$1/2^{12}$	0.006999	0.006933	0.006911	0.006909	0.006909	0.006910	0.006910
$1/2^{13}$	0.004324	0.004229	0.004194	0.004187	0.004185	0.004185	0.004185
$1/2^{14}$	0.002214	0.002047	0.001990	0.001978	0.001974	0.001972	0.001972

4.3. Energy decay. Following the definition of the continuous case, we can define the discrete energy as

$$E_n^{hk} = \rho||v_n^{hk}||_Y^2 + \rho\epsilon^2||(v_n^{hk})_x||_Y^2 + ||(u_n^{hk})_x||_Y^2 + c||\theta_n^{hk}||_Y^2. \tag{4.1}$$

Now, we simulate two different problems. First, we present the case of an exponentially stable solution. Since the elastic term is not positive, we try to find suitable solutions of the form $u(x,t) = e^{wt}\sin(nx)$ and $\theta(x,t) = e^{wt}\cos(nx)$.

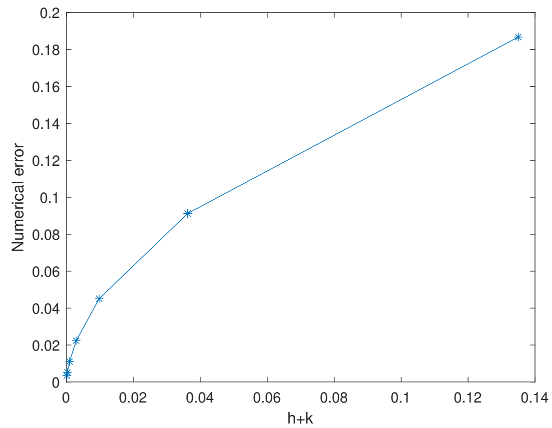

FIG. 4.3. *Example 1: Asymptotic constant error in the L^2-norm with non-regular initial conditions.*

Substituting them into equations (2.1) and (2.2), we obtain

$$\rho w^2 e^{wt}\sin(nx) + \rho\varepsilon^2 w^2 n^2 e^{wt}\sin(nx) = \mu^* e^{wt} n^2 \sin(nx) + \beta e^{wt} n \sin(x),$$
$$c w e^{wt}\cos(nx) = -m e^{wt} n^2 \cos(nx) - \beta w e^{wt} n \cos(nx).$$

That is, we arrive at the system:

$$\rho(1+\varepsilon^2 n^2)w^2 = \mu^* n^2 + \beta n,$$
$$cw = -mn^2 - \beta n w.$$

Since we are looking for a solution that decays exponentially, we need that w is negative, and so we have the following solution:

$$w = -\sqrt{\frac{\mu^* n^2 + \beta n}{1+\epsilon^2 n^2}}, \qquad m = -\frac{w(c+\beta n)}{n^2}. \tag{4.2}$$

From the definition of the material coefficients, we have to assume that $\mu^* n + \beta > 0$ and $c + \beta n > 0$, which is obtained if $\beta > \max\{-c/n, -\mu^* n\}$.

It is worth noting that, in order to impose the above functions as possible solutions, we need to consider Neumann boundary conditions for the temperature variable θ instead of Dirichlet type.

In this first case, we use the following data:

$$T = 12, \quad \ell = \pi, \quad \rho = 1, \quad \varepsilon = 3, \quad \mu^* = 1, \quad \beta = 2, \quad c = 4,$$

where m is obtained from (4.2), and the initial conditions are given by, for all $x \in (0,\pi)$,

$$u^0(x) = \sin(nx), \quad v^0(x) = w\sin(nx), \quad \theta^0(x) = \cos(nx).$$

Taking the discretization parameters $h = \frac{\pi}{1000}$ and $k = 0.00001$, the evolution in time of the discrete energy is illustrated in Figure 4.4 (in both natural and semi-log scales) for three values of the parameter $n = 1, 3, 5$. As can be seen, for every value it converges to zero, and an exponential decay seems to be achieved.

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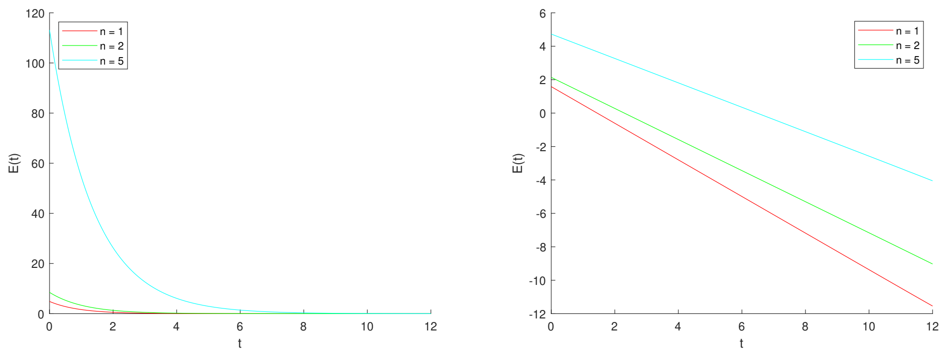

FIG. 4.4. *Evolution in time of the discrete energy (natural and semi-log scales) for the good choice of the parameters.*

Now, we will show what happens in the general case, when the parameters w and m do not satisfy the adequate conditions (4.2). For the sake of simplicity in the presentation, in the rest of this section we restrict ourselves to the simplest case $n = 1$.

If we use the data

$$T = 10, \quad \ell = \pi, \quad \rho = 1, \quad \varepsilon = 3, \quad \beta = 2, \quad c = 4, \quad w = -\sqrt{\frac{\mu^* + \beta}{1 + \epsilon^2}}, \quad m = 2,$$

and the same initial conditions as before, taking the discretization parameters $h = \frac{\pi}{1000}$ and $k = 0.0001$, then the evolution in time of the discrete energy defined in (4.1) is displayed in Figure 4.5 (in both natural and semi-log scales) for different values of μ^*. We note that the parameter m does not satisfy the condition $m = -w(c + \beta)$, and so we cannot expect an exponential decay. In fact, we should have an exponential growth in general. As can be clearly seen, after a finite time (which is smaller when the parameter μ^* increases), the discrete energy explodes, and an exponential growth is found.

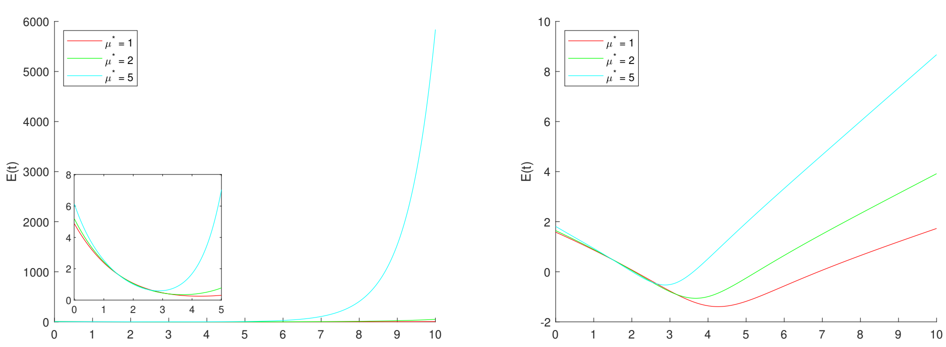

FIG. 4.5. *Evolution in time of the discrete energy (natural and semi-log scales) for several values of the parameter μ^*.*

Finally, using the data

$$T = 2.5, \quad \ell = \pi, \quad \rho = 1, \quad \epsilon = 3, \quad \beta = 2, \quad c = 4, \quad m = 2,$$

and the initial conditions, for all $x \in (0, \pi)$,

$$u^0(x) = v^0(x) = \theta^0(x) = x(x - \pi),$$

and taking the discretization parameters $h = \frac{\pi}{1000}$ and $k = 0.0001$, the evolution in time of the discrete energy defined above is displayed in Figure 4.6 (in both natural and semi-log scales) again for different values of the parameter μ^*. As expected, for all the choices of the parameter μ^*, the solution explodes, and it exhibits an exponential growth which is faster when μ^* increases.

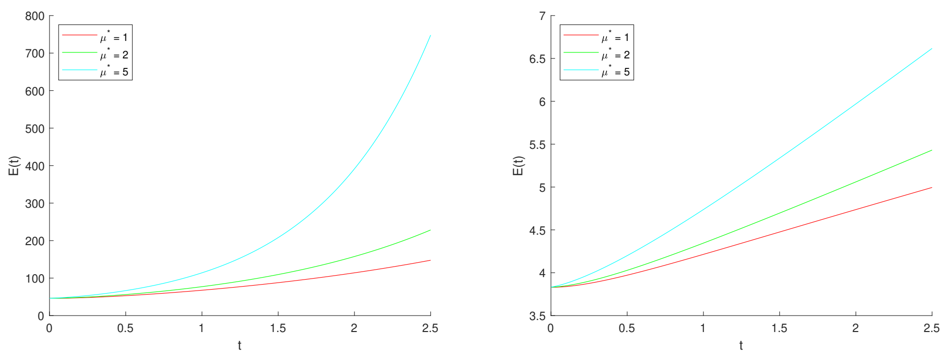

FIG. 4.6. *Evolution in time of the discrete energy (natural and semi-log scales) depending on the parameter μ^*.*

Acknowledgements. The authors want to thank the two anonymous referees for their careful revision and theirs positive comments and suggestions.

Appendix A. Existence and regularity of the solution. In the previous sections of this work, we have studied, from the numerical point of view, the problem determined by the system (2.1)–(2.2) with initial conditions (2.3) and boundary conditions (2.4). Therefore, it will be convenient to guarantee the existence and the regularity of the solutions to this problem.

We are going to study it in the Hilbert space

$$\mathcal{H} = H_0^1(0, \ell) \times H_0^1(0, \ell) \times L^2(0, \ell),$$

and we consider the scalar product associated to the norm

$$\|(u, v, \theta)\|^2 = \int_0^\ell \left(\rho v^2 + \rho\varepsilon^2 v_x^2 + (\lambda - \mu^*)u_x^2 + c\theta^2\right) dx,$$

where λ is a positive number greater than μ^*.

Setting $Lv = v - \rho\varepsilon^2 v_{xx}$, this operator is an isomorphism between $H_0^1(0, \ell) \cap H^2(0, \ell)$ and $L^2(0, \ell)$. Therefore, it admits an inverse L^{-1}. Thus, we can write our problem in the following abstract form:

(A.1) $$\frac{dU}{dt} = \mathcal{A}U, \qquad U(0) = (u^0, v^0, \theta^0),$$

where

$$\mathcal{A}\begin{bmatrix}u\\v\\\theta\end{bmatrix}=\begin{bmatrix}v\\L^{-1}[-\mu^*u_{xx}-\beta\theta_x]\\c^{-1}[m\theta_{xx}-\beta v_x]\end{bmatrix}.$$

We can observe that the domain of this operator is composed by the elements (u, v, θ) of the Hilbert space $\mathcal{H}$ such that $u, \theta \in H^2(0,\ell)\cap H_0^1(0,\ell)$ and $v \in H^1(0,\ell)$. We can also see that

$$\langle \mathcal{A}U, U\rangle = \int_0^\ell (\lambda u_x v_x - m\theta_x^2)\,dx.$$

Therefore, we can guarantee the existence of a positive constant C such that

$$\langle \mathcal{A}U, U\rangle \le C\|U\|^2.$$

Finally, we can also note that there exists a positive constant K such that $(KI-\mathcal{A})$ is surjective. In fact, given $(f_1, f_2, f_3) \in \mathcal{H}$ we obtain the system

$$\begin{aligned}Ku - v &= f_1,\\ KLv + \mu^* u_{xx} + \beta\theta_x &= Lf_2,\\ cK\theta - m\theta_{xx} + \beta v_x &= cf_3.\end{aligned}$$

If we substitute the first equation into the remaining two ones, then we find that

$$\begin{aligned}K^2Lu + \mu^* u_{xx} + \beta\theta_x &= Lf_2 + KLf_1,\\ cK\theta - m\theta_{xx} + \beta K u_x &= cf_3 - \beta f_1.\end{aligned}$$

We can observe that $(Lf_2 + KLf_1, cf_3 - \beta f_1) \in H_0^{-1}(0,\ell)\times H^{-1}(0,\ell)$. At the same time, the bilinear form given by

$$\langle (u,\theta),(\tilde{u},\tilde{\theta})\rangle = \langle K(K^2Lu + \mu^* u_{xx} + \beta\theta_x)\tilde{u}, (cK\theta - m\theta_{xx} + \beta K u_x)\tilde{\theta}\rangle$$

is bounded in $H_0^1(0,\ell)\times H_0^1(0,\ell)$, and, if K is large enough, it is also coercive. Therefore, we can guarantee the existence of a solution in the domain of the operator $\mathcal{A}$.

If we apply the Lumer-Phillips corollary of the Hille-Yosida theorem, we can conclude that the operator $\mathcal{A}$ generates a contractive semigroup. Therefore, we can conclude the following existence and uniqueness result:

THEOREM A.1. *The operator $\mathcal{A}$ defined previously is the infinitesimal generator of a C^0 contractive semigroup on the Hilbert space $\mathcal{H}$. Thus, for any initial data $(u^0, v^0, \theta^0) \in \mathcal{H}$, there exists a solution to the abstract problem (A.1) with the following regularity:*

$$U \in C([0,\infty);\mathrm{Dom}(\mathcal{A}))\cap C^1([0,\infty);\mathcal{H}).$$

It is well-known that the existence of a C^0-semigroup also provides the continuous dependence of the solutions with respect to the initial data and supply terms (when they are imposed). Therefore, we can guarantee that our problem is well posed in the sense of Hadamard.

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