Electronic Transactions on Numerical Analysis.
Volume 60, pp. 238–255, 2024.

ISSN 1068–9613.
DOI: 10.1553/etna_vol60s238

ETNA
Kent State University and
Johann Radon Institute (RICAM)

CONVERGENCE OF THE EBERLEIN DIAGONALIZATION METHOD UNDER GENERALIZED SERIAL PIVOT STRATEGIES*

ERNA BEGOVIĆ KOVAČ† AND ANA PERKOVIĆ†

Abstract. The Eberlein method is a Jacobi-type process for solving the eigenvalue problem of an arbitrary matrix. In each iteration two transformations are applied to the underlying matrix, a plane rotation and a non-unitary core transformation. The paper studies the method under the broad class of generalized serial pivot strategies. We prove global convergence of the Eberlein method under the generalized serial pivot strategies with permutations and present several numerical examples.

Key words. Jacobi-type methods, matrix diagonalization, pivot strategies, global convergence

AMS subject classifications. 65F15

1. Introduction.

The Jacobi diagonalization method is an iterative method for solving the eigenproblem for dense symmetric matrices. Compared to other state-of-the-art diagonalization methods, the main advantage of the Jacobi method is its high relative accuracy; see [4, 5, 22, 26]. The method has been modified to deal with different matrix structures in [10, 16, 19, 20, 23] and to tackle various problems of numerical linear algebra in [3, 6, 7, 24]. Its convergence has been extensively studied; see, e.g., [13, 21]. One of the generalizations of the Jacobi method is known as the Eberlein method.

The Eberlein method, originally proposed in [8] in 1962, is a Jacobi-type process for solving the eigenvalue problem of an arbitrary matrix. It is one of the first efficient norm-reducing methods of this type. The iterative process for a general matrix $A \in \mathbb{C}^{n\times n}$ takes the form

$$A^{(k+1)} = T_k^{-1} A^{(k)} T_k, \quad k \geq 0, \tag{1.1}$$

where $A^{(0)} = A$ and $T_k = R_k S_k$ are non-singular core transformations. By core transformations we understand matrices differing from the identity in a 2×2 submatrix; see [1]. In particular, the matrices R_k are plane rotations, and the S_k are non-unitary core transformations. The transformations R_k are chosen to annihilate the pivot element of the matrix $\frac{1}{2}(A^{(k)} + (A^{(k)})^*)$, while the transformations S_k reduce the Frobenius norm of $A^{(k)}$. In Eberlein's experiments, the matrices $A^{(k)}$, $k \geq 0$, given by the process (1.1) converge to a normal matrix. Eberlein proved this convergence but only under a specific pivot strategy.

In [27], Veselić studied a slightly altered Eberlein algorithm where in the kth step only one transformation is applied, either R_k or S_k, but not both at the same time. He proved the convergence of this modified method under the classical Jacobi pivot strategy. Specifically, he showed that, for an arbitrary $n \times n$ starting matrix $A^{(0)}$, the sequence $A^{(k)}$, $k \geq 0$, converges to a block diagonal normal matrix. At the same time, the sequence $\frac{1}{2}(A^{(k)} + (A^{(k)})^*)$ converges to a diagonal matrix $\operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, where $\{\mu_1, \mu_2, \ldots, \mu_n\}$ are the real parts of the eigenvalues of A. Later in [12], Hari proved global convergence of the original method under the column/row cyclic pivot strategy for real matrices. In [25] Hari and Pupovci proved convergence of the Eberlein method for complex matrices with the pivot strategies that are weakly equivalent to the row cyclic strategy. Moreover, they considered a parallel method

*Received May 30, 2023. Accepted February 28, 2024. Published online on May 6, 2024. Recommended by Thomas Mach. This work has been supported in part by Croatian Science Foundation under the project UIP-2019-04-5200.

†University of Zagreb Faculty of Chemical Engineering and Technology, Marulićev trg 19, 10000 Zagreb, Croatia ({ebegovic,aperkov}@fkit.hr).

ETNA
Kent State University and
Johann Radon Institute (RICAM)

and proved its convergence under pivot strategies that are weakly equivalent to the modulus strategy.

In this paper we expand the global convergence result for the Eberlein method to a significantly broader class of cyclic pivot strategies, i.e., the generalized serial strategies with permutations studied in [2, 14, 15]. We consider the method in the form given in [25]. The novel result on global convergence of the Eberlein method under the generalized serial pivot strategies with permutations is given in Theorem 4.4. Precisely, we show that for an arbitrary $n \times n$ starting matrix $A^{(0)}$, the sequence $A^{(k)}$, $k \geq 0$, converges to a block diagonal normal matrix and that the sequence $\frac{1}{2}(A^{(k)} + (A^{(k)})^*)$ converges to a diagonal matrix $\operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, where $\{\mu_1, \mu_2, \ldots, \mu_n\}$ are the real parts of the eigenvalues of A. Moreover, we present several numerical examples and discuss the cases of unique and multiple eigenvalues.

The paper is organized as follows. In Section 2 we describe the Eberlein method, both in its complex and real forms, while in Section 3 we characterize the set of pivot strategies we deal with. The main part of the paper is contained in Section 4, where we prove convergence of the method under the strategies presented in Section 3. Finally, in Section 5 we discuss results of our numerical tests.

2. The Eberlein method. As it was mentioned in the introduction, there are several variations of the Eberlein method. The method can be applied to complex matrices using the transformations $T_k \in \mathbb{C}^{n\times n}$, $k \geq 0$, or one can obtain the real version with $T_k \in \mathbb{R}^{n\times n}$, $k \geq 0$. In this paper we mostly focus on the complex method described in Section 2.1, but we outline the real case as well in Section 2.2.

Throughout, we denote the imaginary unit with $\imath = \sqrt{-1}$. Besides, given a complex number x, $\operatorname{Re}(x)$ stands for the real part of x, and $\operatorname{Im}(x)$ stands for its imaginary part.

2.1. The complex case. The Eberlein method is an iterative Jacobi-type method used to find the eigenvalues and eigenvectors of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$. One iteration step of the method is given by the relation (1.1). In the kth iteration, the transformation T_k is a core transformation that differs from the identity only in one of its 2×2 principal submatrices $\widehat{T}_k$ determined by the pivot pair $(p(k), q(k))$ and given by

$$\widehat{T}_k = \begin{bmatrix} t^{(k)}_{p(k)p(k)} & t^{(k)}_{p(k)q(k)} \\ t^{(k)}_{q(k)p(k)} & t^{(k)}_{q(k)q(k)} \end{bmatrix}.$$

The matrix T_k is set to be the product of two nonsingular matrices, a plane rotation R_k and a non-unitary core transformation S_k, that is, $T_k = R_k S_k$. Denote the kth pivot pair by $(p, q) = (p(k), q(k))$. The pivot pair is the same for both R_k and S_k and consequently for T_k. In addition to (p, q), the matrices R_k and S_k depend on the transformation angles α_k, φ_k and β_k, ψ_k, respectively. The pivot submatrix $\widehat{T}_k$ is equal to $\widehat{T}_k = \widehat{R}_k \widehat{S}_k \in \mathbb{C}^{2\times 2}$, where

$$\widehat{R}_k = \begin{bmatrix} \cos\varphi_k & -e^{\imath\alpha_k}\sin\varphi_k \\ e^{-\imath\alpha_k}\sin\varphi_k & \cos\varphi_k \end{bmatrix}, \quad \widehat{S}_k = \begin{bmatrix} \cosh\psi_k & -\imath e^{\imath\beta_k}\sinh\psi_k \\ \imath e^{-\imath\beta_k}\sinh\psi_k & \cosh\psi_k \end{bmatrix}. \tag{2.1}$$

The process (1.1) can be written with an intermediate step as

$$\widetilde{A}^{(k)} = R_k^* A^{(k)} R_k, \quad A^{(k+1)} = S_k^{-1} \widetilde{A}^{(k)} S_k, \quad k \geq 0.$$

Let

$$B^{(k)} = \frac{1}{2}(A^{(k)} + (A^{(k)})^*), \tag{2.2}$$
$$\widetilde{B}^{(k)} = R_k^* B^{(k)} R_k.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

By definition, the matrix $B^{(k)}$ is the Hermitian part of $A^{(k)}$, and likewise, $\widetilde{B}^{(k)}$ is the Hermitian part of $\widetilde{A}^{(k)}$. Now, let C be the operator defined by

$$C(A) = AA^* - A^*A. \tag{2.3}$$

We denote $C(A^{(k)}) = (c_{ij}^{(k)})$, $C(\widetilde{A}^{(k)}) = (\tilde{c}_{ij}^{(k)})$, and $\widetilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})$. Obviously, $C(A) = 0$ if and only if A is a normal matrix. The definition of $C(A)$ is linked to one of the measures of nonnormallity of matrices given by Elsner and Paardekooper in [9].

The rotation R_k is chosen such that the element of $B^{(k)}$ in position (p, q) is annihilated. The real number α_k, as well as the sine and cosine of φ_k in (2.1), are calculated from the following expressions:

$$\alpha_k = \arg(b_{pq}^{(k)}), \tag{2.4}$$

$$\tan 2\varphi_k = \frac{2|b_{pq}^{(k)}|}{b_{pp}^{(k)} - b_{qq}^{(k)}}, \quad |\varphi_k| \le \frac{\pi}{4}. \tag{2.5}$$

These formulas are the same as for the complex Jacobi method for Hermitian matrices. Then, in order to get $\sin\varphi_k$ and $\cos\varphi_k$, we use the formulas

$$\begin{aligned} \tan\varphi_k &= \frac{2|b_{pq}^{(k)}|\,\operatorname{sign}(b_{pp}^{(k)} - b_{qq}^{(k)})}{|b_{pp}^{(k)} - b_{qq}^{(k)}| + \sqrt{|b_{pp}^{(k)} - b_{qq}^{(k)}|^2 + 4|b_{pq}^{(k)}|^2}}, \\ \cos\varphi_k &= \frac{1}{\sqrt{1+\tan^2\varphi_k}}, \qquad \sin\varphi_k = \frac{\tan\varphi_k}{\sqrt{1+\tan^2\varphi_k}}. \end{aligned} \tag{2.6}$$

On the other hand, S_k is chosen to reduce the Frobenius norm of $A^{(k)}$. Set

$$\Delta_k = \|A^{(k)}\|_F^2 - \|A^{(k+1)}\|_F^2.$$

In [8], Eberlein proved that

$$\begin{aligned} \Delta_k &= \|\widetilde{A}^{(k)}\|_F^2 - \|A^{(k+1)}\|_F^2 \\ &= g_{pq}^{(k)}(1-\cosh 2\psi_k) - h_{pq}^{(k)}\sinh 2\psi_k + \frac{1}{2}(|\tilde{\xi}_{pq}^{(k)}|^2 + |\tilde{d}_{pq}^{(k)}|^2)(1-\cosh 4\psi_k) \\ &\qquad + \operatorname{Im}(\tilde{\xi}_{pq}^{(k)}\tilde{d}_{pq}^{(k)*})\sinh 4\psi_k, \end{aligned}$$

where

$$\begin{aligned} g_{pq}^{(k)} &= \sum_{\substack{i=1\\ i\neq p,q}}^{n} |\tilde{a}_{ip}^{(k)}|^2 + |\tilde{a}_{pi}^{(k)}|^2 + |\tilde{a}_{iq}^{(k)}|^2 + |\tilde{a}_{qi}^{(k)}|^2, \\ h_{pq}^{(k)} &= -\operatorname{Re}(l_{pq}^{(k)})\sin\beta_k + \operatorname{Im}(l_{pq}^{(k)})\cos\beta_k, \\ l_{pq}^{(k)} &= 2\sum_{\substack{i=1\\ i\neq p,q}}^{n} (\tilde{a}_{pi}^{(k)}\tilde{a}_{qi}^{(k)*} - \tilde{a}_{ip}^{(k)*}\tilde{a}_{iq}^{(k)}), \\ \tilde{d}_{pq}^{(k)} &= \tilde{a}_{pp}^{(k)} - \tilde{a}_{qq}^{(k)}, \\ \tilde{\xi}_{pq}^{(k)} &= (\tilde{a}_{pq}^{(k)} + \tilde{a}_{qp}^{(k)})\cos\beta_k - \imath(\tilde{a}_{pq}^{(k)} - \tilde{a}_{qp}^{(k)})\sin\beta_k. \end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

It is shown in [8] that the choice of β_k and ψ_k such that

$$\tan\beta_k = -\frac{\mathrm{Re}(\tilde{c}_{pq}^{(k)})}{\mathrm{Im}(\tilde{c}_{pq}^{(k)})}, \tag{2.7}$$

$$\tanh\psi_k = \frac{1}{2}\frac{2\mathrm{Im}(\tilde{\xi}_{pq}^{(k)}\tilde{d}_{pq}^{(k)*}) - h_{pq}^{(k)}}{g_{pq}^{(k)} + 2(|\tilde{\xi}_{pq}^{(k)}|^2 + |\tilde{d}_{pq}^{(k)}|^2)},$$

$$\cosh\psi_k = \frac{1}{\sqrt{1-\tanh^2\psi_k}}, \qquad \sinh\psi_k = \frac{\tanh\psi_k}{\sqrt{1-\tanh^2\psi_k}} \tag{2.8}$$

implies

$$\Delta_k \geq \frac{1}{3}\frac{|\tilde{c}_{pq}^{(k)}|^2}{\|A^{(k)}\|_F^2} \geq \frac{1}{3}\frac{|\tilde{c}_{pq}^{(k)}|^2}{\|A\|_F^2}, \quad k \geq 1. \tag{2.9}$$

The values of β_k and ψ_k determined by (2.7) and (2.8) are approximations of values that maximizes Δ_k. We summarize this procedure in Algorithm 2.1.

ALGORITHM 2.1 (Eberlein method).

Input: $A \in \mathbb{C}^{n\times n}$
Output: *matrix* $A^{(k)}$
$A^{(0)} = A$;
$k = 0$;
repeat
  Choose a pivot pair (p,q) *according to the pivot strategy;*
  Find α_k *using* (2.4) *and* $\sin\varphi_k$, $\cos\varphi_k$ *using* (2.6);
  $\widetilde{A}^{(k)} = R_k^* A^{(k)} R_k$;
  Find β_k *using* (2.7) *and* $\sin\psi_k$, $\cos\psi_k$ *using* (2.8);
  $A^{(k+1)} = S_k^{-1}\widetilde{A}^{(k)} S_k$;
  $k = k+1$;
until *convergence*

One should keep in mind that it is not necessary to build the matrices $\widetilde{A}^{(k)}$ explicitly but only their pth and qth rows and columns are needed.

2.2. The real case. Suppose that A is a real matrix. We then modify the complex algorithm to ensure that the iterates $A^{(k)}$ stay real during the process (1.1). Firstly, we can take $\alpha_k = \pi$ and $\beta_k = \pi/2$. This implies

$$\widehat{R}_k = \begin{bmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{bmatrix}, \quad \widehat{S}_k = \begin{bmatrix} \cosh\psi_k & \sinh\psi_k \\ \sinh\psi_k & \cosh\psi_k \end{bmatrix}.$$

Same as before, we do not need to calculate the angles φ_k and ψ_k directly. It is sufficient to find the matrices $\widehat{R}_k$ and $\widehat{S}_k$.

As in the complex case, φ_k is selected to annihilate the pivot element of $B^{(k)}$ while ψ_k is chosen to reduce $\|A^{(k)}\|_F$. The angle φ_k is calculated from the following relation similar to (2.5):

$$\tan 2\varphi_k = \frac{2b_{pq}^{(k)}}{b_{qq}^{(k)} - b_{pp}^{(k)}}, \quad |\varphi_k| \leq \frac{\pi}{4}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Considering that $\beta_k = \pi/2$ and that all the elements of $A^{(k)}$ are real, the formula for ψ_k is transformed into

$$\tanh \psi_k = \frac{\tilde{c}_{pq}^{(k)}}{g_{pq}^{(k)} + 2((\tilde{e}_{pq}^{(k)})^2 + (\tilde{d}_{pq}^{(k)})^2)},$$

where

$$\tilde{e}_{pq}^{(k)} = \tilde{a}_{pq}^{(k)} - \tilde{a}_{qp}^{(k)},$$

while $g_{pq}^{(k)}$ and $\tilde{d}_{pq}^{(k)}$ are the same as in the complex case.

3. Generalized serial pivot strategies. In each iteration k of Algorithm 2.1, the pivot position is selected according to a pivot strategy. In this section we describe the large class of pivot strategies we are going to deal with, i.e., the generalized serial pivot strategies with permutations defined in [14].

For an $n \times n$ matrix, possible pivot pairs are those of the upper triangular part of the matrix, i.e., $\mathcal{P}_n := \{(i,j) : 1 \le i < j \le n\}$. A pivot strategy is a function $I : \mathbb{N}_0 \to \mathcal{P}_n$, with $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We work with *cyclic pivot strategies*, thus we take I as a periodic function with period $T = N \equiv \frac{n(n-1)}{2}$ and image $\mathcal{P}_n$.

Pivot strategies are often better understood using pivot orderings. A cyclic strategy I defines a sequence $\mathcal{O}_I$ that is an ordering of $\mathcal{P}_n$ of the form

$$\mathcal{O}_I = I(0), I(1), \ldots, I(N-1) \in \mathcal{O}(\mathcal{P}_n),$$

where $\mathcal{O}(\mathcal{P}_n)$ stands for the set of all finite sequences of elements of $\mathcal{P}_n$, provided that each pair belonging to $\mathcal{P}_n$ appears at least once in every sequence. An *admissible transposition* in a pivot sequence $\mathcal{O}$ is any transposition of two adjacent pivot pairs

$$(i_r, j_r), (i_{r+1}, j_{r+1}) \to (i_{r+1}, j_{r+1}), (i_r, j_r),$$

assuming that the sets $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint. The following definition concerns several equivalence relations for pivot orderings; see, e.g., [14].

DEFINITION 3.1. *Two pivot sequences $\mathcal{O}$ and $\mathcal{O}'$, with $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_r, j_r)$, are said to be*

1. equivalent ($\mathcal{O} \sim \mathcal{O}'$) *if one can be obtained from the other by a finite set of admissible transpositions;*
2. shift-equivalent ($\mathcal{O} \overset{s}{\sim} \mathcal{O}'$) *if $\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2]$ and $\mathcal{O}' = [\mathcal{O}_2, \mathcal{O}_1]$, where $[\ ,\ ]$ denotes the concatenation. The length of $\mathcal{O}_1$ is called the shift length;*
3. weak equivalent ($\mathcal{O} \overset{w}{\sim} \mathcal{O}'$) *if there exist $\mathcal{O}_i \in \mathcal{O}(\mathcal{S})$, $0 \le i \le t$, such that every two adjacent terms in the sequence $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_t = \mathcal{O}'$ are equivalent or shift-equivalent;*
4. permutation equivalent ($\mathcal{O} \overset{p}{\sim} \mathcal{O}'$ *or* $\mathcal{O}' = \mathcal{O}(\mathrm{q})$) *if there is a permutation* q *of the set $\mathcal{O}$ such that $\mathcal{O}' = (\mathrm{q}(i_0), \mathrm{q}(j_0)), (\mathrm{q}(i_1), \mathrm{q}(j_1)), \ldots, (\mathrm{q}(i_r), \mathrm{q}(j_r))$;*
5. reverse ($\mathcal{O}' = \mathcal{O}^{\leftarrow}$) *if $\mathcal{O}' = (i_r, j_r), \ldots, (i_1, j_1), (i_0, j_0)$.*

Two pivot strategies $I_{\mathcal{O}}$ and $I_{\mathcal{O}'}$ are equivalent (shift-equivalent, weak equivalent, permutation equivalent, reverse) if the same is true for their corresponding pivot orderings $\mathcal{O}$ and $\mathcal{O}'$.

It is easy to see that if $\mathcal{O}$ and $\mathcal{O}'$ are weak equivalent $\mathcal{O} \overset{w}{\sim} \mathcal{O}'$, then there exists a finite sequence $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_t = \mathcal{O}'$ such that

$$\mathcal{O} \sim \mathcal{O}_1 \overset{s}{\sim} \mathcal{O}_2 \sim \mathcal{O}_3 \overset{s}{\sim} \mathcal{O}_4 \cdots \mathcal{O}' \quad \text{or} \quad \mathcal{O} \overset{s}{\sim} \mathcal{O}_1 \sim \mathcal{O}_2 \overset{s}{\sim} \mathcal{O}_3 \sim \mathcal{O}_4 \cdots \mathcal{O}'. \tag{3.1}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

The chains in (3.1) connecting $\mathcal{O}$ and $\mathcal{O}'$ are said to be in *canonical form*.

The most intuitive cyclic strategies are the *row-cyclic*, $I_{row} = I_{\mathcal{O}_{row}}$, and the *column-cyclic strategy*, $I_{col} = I_{\mathcal{O}_{col}}$, collectively named *serial pivot strategies*. Cyclic strategies that are equivalent to serial pivot strategies are called *wavefront strategies*.

In [8], Eberlein adopted the strategy where the pivot pair (p, q) is chosen such that the quantity $4|c_{pq}^{(k)}|^2 + (c_{pp}^{(k)} - c_{qq}^{(k)})^2$ is larger or equal to the average of all possible results for $4|c_{ij}^{(k)}|^2 + (c_{ii}^{(k)} - c_{jj}^{(k)})^2$, $1 \leq i < j \leq n$. Veselić [27] used the classical Jacobi pivot strategy, which takes the pivot pair that is largest in absolute value. Employing both such strategies results in slowing the algorithm down for large matrices. Later in [12], Hari proved convergence for the real method under the wavefront strategies. In [25] Pupovci and Hari provided a convergence proof for the complex method using the parallel modulus strategy and the strategies that are weakly equivalent to it.

Now let us describe the generalized serial pivot strategies with permutations; for more details, see [14, 15]. Given $l_1 < l_2$, denote by $\Pi^{(l_1,l_2)}$ the set of all permutations of the set $\{l_1, l_1+1, l_1+2, \ldots, l_2\}$. Moreover, let

$$\mathcal{C}_c^{(n)} = \Big\{\mathcal{O} \in \mathcal{O}(\mathcal{P}_n) \mid \mathcal{O} =(1,2),(\tau_3(1),3),(\tau_3(2),3),\ldots,(\tau_n(1),n),\ldots \\ \ldots,(\tau_n(n-1),n),\ \tau_j \in \Pi^{(1,j-1)},\ 3 \leq j \leq n\Big\}.$$

The orderings from $\mathcal{C}_c^{(n)}$ go through the matrix column by column, starting from the second one, just like in the standard column strategy I_{col}. However, in each column, pivot elements are chosen in some arbitrary order. If $\mathcal{O} \in \mathcal{C}_c^{(n)}$, then $\mathcal{O}$ is called a column-wise ordering with permutations. Similarly, the set of row-wise orderings with permutations is defined as

$$\mathcal{C}_r^{(n)} = \Big\{\mathcal{O} \in \mathcal{O}(\mathcal{P}_n) \mid \mathcal{O} =(n-1,n),(n-2,\tau_{n-2}(n-1)),(n-2,\tau_{n-2}(n)),\ldots \\ \ldots,(1,\tau_1(2)),\ldots,(1,\tau_1(n)),\ \tau_i \in \Pi^{(i+1,n)},\ 1 \leq i \leq n-2\Big\}.$$

By employing these two sets of orderings and their reverses, we define the set of *serial orderings with permutations* as

$$\mathcal{C}_{sp}^{(n)} = \mathcal{C}_c^{(n)} \cup \overleftarrow{\mathcal{C}}_c^{(n)} \cup \mathcal{C}_r^{(n)} \cup \overleftarrow{\mathcal{C}}_r^{(n)}.$$

We now expand the set $\mathcal{C}_{sp}^{(n)}$ by using the equivalence relations in Definition 3.1. Let

$$\mathcal{C}_{sg}^{(n)} = \Big\{\mathcal{O} \in \mathcal{O}(\mathcal{P}_n) \mid \mathcal{O} \overset{w}{\sim} \mathcal{O}' \overset{p}{\sim} \mathcal{O}'' \text{ or } \mathcal{O} \overset{p}{\sim} \mathcal{O}' \overset{w}{\sim} \mathcal{O}'', \mathcal{O}'' \in \mathcal{O}_{sp}^{(n)}\Big\},$$

with $\mathcal{O}' \in \mathcal{O}(\mathcal{P}_n)$. Strategies defined by orderings from $\mathcal{C}_{sg}^{(n)}$ are called *generalized serial pivot strategies with permutations*.

4. Convergence of the Eberlein method. In this section we prove that the iterative process (1.1) converges under any pivot ordering $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. In what follows, we use the notation introduced in Section 2.

Let us start by listing some auxiliary results from the literature and their direct implications:

- As shown by Eberlein in [8], for $\|A^{(k)}\|_F^2$ we have

$$\Delta_k = \|A^{(k)}\|_F^2 - \|A^{(k+1)}\|_F^2 = \|\widetilde{A}^{(k)}\|_F^2 - \|A^{(k+1)}\|_F^2 \geq 0. \tag{4.1}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

- Since the sequence $(\|A^{(k)}\|_F^2,\ k \geq 0)$ is nonincreasing and bounded, it is convergent. Therefore, the inequalities (4.1) and (2.9) imply

$$\lim_{k\to\infty} \tilde{c}_{pq}^{(k)} = 0. \tag{4.2}$$

- As shown by Hari in [12], for $\widetilde{A}^{(k)} = R_k^* A^{(k)} R_k$, $k \geq 0$, and

$$E^{(k)} = A^{(k+1)} - \widetilde{A}^{(k)}, \tag{4.3}$$

we have

$$\|E^{(k)}\|_F^2 \leq \frac{3}{2} n^2 |\tilde{c}_{pq}^{(k)}|. \tag{4.4}$$

- Again in [12] it has been proven that for $\widetilde{B}^{(k)} = R_k^* B^{(k)} R_k$, $k \geq 0$, and

$$F^{(k)} = B^{(k+1)} - \widetilde{B}^{(k)}, \tag{4.5}$$

we have

$$\|F^{(k)}\|_F^2 \leq \frac{3}{2} n^2 |\tilde{c}_{pq}^{(k)}|. \tag{4.6}$$

- For any $k \geq 0$, we have

$$\begin{aligned} C(\widetilde{A}^{(k)}) &= C(R_k^* A^{(k)} R_k) \\ &= R_k^* A^{(k)} (A^{(k)})^* R_k - R_k^* (A^{(k)})^* A^{(k)} R_k \\ &= R_k^* (A^{(k)} (A^{(k)})^* - (A^{(k)})^* A^{(k)}) R_k \\ &= R_k^* C(A^{(k)}) R_k. \end{aligned} \tag{4.7}$$

We define the *off-norm* of an $n \times n$ matrix X as the Frobenius norm of its off-diagonal part, that is,

$$\operatorname{off}^2(X) = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |x_{ij}|^2.$$

The matrix X is diagonal if and only if $\operatorname{off}(X) = 0$. Of course, the off-norm is not a matrix norm as $\operatorname{off}(X) = 0$ does not imply $X = 0$.

We will also use a result from [15] for the complex Jacobi operators. Jacobi annihilators and operators were introduced in [17] and later generalized in [13]. Here we give a simplified definition of the complex Jacobi annihilators and operators, the one designed to meet our needs.

For an $n \times n$ matrix B we define its vectorization as the vector $b = \operatorname{vecoff}(B) \in \mathbb{C}^{2N}$, $N = n(n-1)/2$, containing all off-diagonal elements of B. In our case we assume that B is Hermitian. Let R be an $n \times n$ rotation matrix that differs from the identity matrix in its 2×2 submatrix $\widehat{R}$ defined by the pivot position (p, q), as in (2.1), such that the rotation angle φ satisfies $|\varphi| \leq \frac{\pi}{4}$. Moreover, let $\mathcal{N}_{pq} : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^{n\times n}$ be an operator that sets to zero the elements at positions (p, q) and (q, p) in a given matrix. A *complex Jacobi annihilator* $\mathcal{R}_{p,q}(R) \in \mathbb{C}^{2N\times 2N}$ is defined by the rule

$$\mathcal{R}_{p,q}(R)\operatorname{vecoff}(B) = \operatorname{vecoff}(\mathcal{N}_{pq}(R^* B R)). \tag{4.8}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For a pivot ordering $\mathcal{O} = (p_0, q_0), (p_1, q_1), \ldots, (p_{N-1}, q_{N-1}) \in \mathcal{O}(\mathcal{P}_n)$, a *complex Jacobi operator* determined by the ordering $\mathcal{O}$ is defined as a product of N Jacobi annihilators

$$\mathcal{J}_{\mathcal{O}} = \mathcal{R}_{p_{N-1},q_{N-1}}(R_{N-1}) \cdots \mathcal{R}_{p_1,q_1}(R_1)\mathcal{R}_{p_0,q_0}(R_0).$$

The definitions given above are special cases of those in [15]. It is useful to recall that the spectral norm of a Jacobi annihilator is equal to one, except for the case of the 2×2 annihilator which is a zero-matrix. This follows from the structure of the annihilators; see, e.g., [15].

PROPOSITION 4.1. *Let $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. Suppose that $\mathcal{O} \overset{p}{\sim} \mathcal{O}' \overset{w}{\sim} \mathcal{O}''$ or $\mathcal{O} \overset{w}{\sim} \mathcal{O}' \overset{p}{\sim} \mathcal{O}''$, $\mathcal{O}'' \in \mathcal{C}_{sp}^{(n)}$ and that the weak equivalence relation is in canonical form containing exactly d shift equivalences. Then, for any $d+1$ Jacobi operators $\mathcal{J}_{\mathcal{O},1}, \mathcal{J}_{\mathcal{O},2}, \ldots, \mathcal{J}_{\mathcal{O},d+1}$, there is a constant γ_n depending only on n such that*

$$\|\mathcal{J}_{\mathcal{O},1}\mathcal{J}_{\mathcal{O},2} \cdots \mathcal{J}_{\mathcal{O},d+1}\|_2 \leq \gamma_n, \quad 0 \leq \gamma_n < 1.$$

Proof. This is a special case of [15, Theorem 3.6]. ∎

Further on, we prove the following two auxiliary propositions.

PROPOSITION 4.2. *Let $(x_k, k \geq 0)$ be a sequence of nonnegative real numbers such that*

$$x_{k+1} = \gamma x_k + c_k, \quad 0 \leq \gamma < 1. \tag{4.9}$$

If $\lim_{k\to\infty} c_k = 0$, then

$$\lim_{k\to\infty} x_k = 0.$$

Proof. First, we show that the sequence (4.9) is bounded from above. Take

$$C = \max\{x_0, \sup_k c_k\}.$$

We prove the boundedness by mathematical induction. For $k = 0$,

$$x_0 \leq C \leq \frac{C}{1-\gamma} =: M, \quad \text{for } 0 \leq \gamma < 1.$$

Assume that $x_k \leq M$ for some given k. Then, for $k+1$, we have

$$x_{k+1} = \gamma x_k + c_k \leq \gamma M + C = \gamma M + (1-\gamma)M = M.$$

As a consequence, $x_k \leq M$ for any $k \geq 0$ and $\limsup_{k\to\infty} x_k = L \in \mathbb{R}$. Then,

$$L = \limsup_{k\to\infty} x_{k+1} \leq \gamma \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} c_k = \gamma L.$$

Due to $0 \leq \gamma < 1$, the upper inequality can only hold with $L = 0$. Since $(x_k)_k$ is a sequence of nonnegative real numbers, $\liminf_{k\to\infty} x_k \geq 0$, which implies that

$$\limsup_{k\to\infty} x_k = \liminf_{k\to\infty} x_k = 0$$

and then that $\lim_{k\to\infty} x_k = 0$. ∎

PROPOSITION 4.3. *Let $H \neq 0$ be a Hermitian matrix. Let $(H^{(k)}, k \geq 0)$ be a sequence generated by applying the following iterative process to H:*

$$H^{(k+1)} = R_k^* H^{(k)} R_k + M^{(k)}, \quad H^{(0)} = H, \quad k \geq 0, \tag{4.10}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

where R_k are complex plane rotations acting in the $(p(k), q(k))$ plane, $p(k) < q(k)$, with the rotation angles $|\varphi_k| \leq \frac{\pi}{4}$, $k \geq 0$. Suppose that the pivot strategy is defined by an ordering $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$ and that

$$\lim_{k\to\infty} \operatorname{off}(M^{(k)}) = 0. \tag{4.11}$$

Then,

$$\lim_{k\to\infty} \left| h_{p(k)q(k)}^{(k+1)} \right| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \left| h_{q(k)p(k)}^{(k+1)} \right| = 0 \tag{4.12}$$

implies

$$\lim_{k\to\infty} \operatorname{off}(H^{(k)}) = 0.$$

Proof. The proof follows the same lines of arguments as [15, Theorem 3.8]. To simplify the notation, let $(p, q) = (p(k), q(k))$ denote the pivot pair at step k. The transformation $R_k^* H^{(k)} R_k$ does not annihilate the elements at positions (p, q) and (q, p) of $H^{(k)}$, but we can write it as

$$R_k^* H^{(k)} R_k = \mathcal{N}_{pq}(R_k^* H^{(k)} R_k) + (R_k^* H^{(k)} R_k)_{pq}(e_p e_q^*) + (R_k^* H^{(k)} R_k)_{qp}(e_q e_p^*), \tag{4.13}$$

where e_r is the rth column vector of the identity matrix I_n and $\mathcal{N}_{pq}$ is as in (4.8). By using the vecoff operator for equation (4.10) and the definition of a Jacobi annihilator (4.8), we obtain from relation (4.13)

$$\chi^{(k+1)} = \mathcal{R}_{p_k,q_k}(R_k)\chi^{(k)} + m^{(k)}, \quad k \geq 0, \tag{4.14}$$

where $\chi^{(k)} = \operatorname{vecoff}(H^{(k)})$, and

$$\begin{aligned} m^{(k)} &= \operatorname{vecoff}(M^{(k)}) + (R_k^* H^{(k)} R_k)_{pq} e_{\tau(p,q)} + (R_k^* H^{(k)} R_k)_{qp} e_{\tau(q,p)} \\ &= \operatorname{vecoff}(M^{(k)}) + (h_{pq}^{(k+1)} - M_{pq}^{(k)}) e_{\tau(p,q)} + (h_{qp}^{(k+1)} - M_{qp}^{(k)}) e_{\tau(q,p)}. \end{aligned} \tag{4.15}$$

Here, $\tau(p, q)$ stands for the position of the matrix element x_{pq} in the vectorization $\operatorname{vecoff}(X)$, and $e_{\tau(p,q)}$ is the column vector of the identity matrix I_{2N} with one in position $\tau(p, q)$. The relation (4.15) and the assumptions (4.11), (4.12) imply that

$$\lim_{k\to\infty} m^{(k)} = 0. \tag{4.16}$$

We denote the matrix obtained from H after t cycles of the process (4.10) by $H^{(tN)}$. The vector $\chi^{(tN)} = \operatorname{vecoff}(H^{(tN)})$ can be written as

$$\chi^{(tN)} = \mathcal{J}_{\mathcal{O}}^{[tN]} \chi^{((t-1)N)} + m^{[tN]}, \quad t \geq 1.$$

The Jacobi operator $\mathcal{J}_{\mathcal{O}}^{[tN]}$ that appears in the previous equation is determined by the ordering $\mathcal{O} = (p_0, q_0), (p_1, q_1), \ldots, (p_{N-1}, q_{N-1}) \in \mathcal{O}(\mathcal{P}_n)$ and by the Jacobi annihilators

$$\mathcal{J}_{\mathcal{O}}^{[tN]} = \mathcal{R}_{p_{N-1},q_{N-1}}(R_{tN-1}) \cdots \mathcal{R}_{p_1,q_1}(R_{(t-1)N+1}) \mathcal{R}_{p_0,q_0}(R_{(t-1)N}),$$

while

$$\begin{aligned} m^{[tN]} = {} & \mathcal{R}_{p_{N-1},q_{N-1}}(R_{tN-1}) \cdots \mathcal{R}_{p_1,q_1}(R_{(t-1)N+1}) \mathcal{R}_{p_0,q_0}(R_{(t-1)N}) m^{((t-1)N)} \\ & + \cdots + \mathcal{R}_{p_{N-1},q_{N-1}}(R_{tN-1}) m^{(tN-2)} + m^{(tN-1)}. \end{aligned} \tag{4.17}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Due to the fact that the spectral norm of any Jacobi annihilator is equal to one (or zero if it is a 2×2 annihilator), the relation (4.17) implies that

$$\|m^{[tN]}\|_2 \leq \|m^{((t-1)N)}\|_2 + \cdots + \|m^{(tN-2)}\|_2 + \|m^{(tN-1)}\|_2, \quad t \geq 1.$$

Thus, from the limit (4.16) it holds

$$\lim_{t\to\infty} m^{[tN]} = 0. \tag{4.18}$$

For $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$, i.e., the pivot strategy is generalized serial, suppose that

$$\chi^{((t+d)N)} = \mathcal{J}_{\mathcal{O}}^{[(t+d)N]} \cdots \mathcal{J}_{\mathcal{O}}^{[(t+1)N]} \mathcal{J}_{\mathcal{O}}^{[tN]} \chi^{((t-1)N)} + m_{[d+1]}^{[tN]}, \quad t \geq 1, \tag{4.19}$$

where

$$m_{[d+1]}^{[tN]} = \mathcal{J}_{\mathcal{O}}^{[(t+d)N]} \cdots \mathcal{J}_{\mathcal{O}}^{[(t+1)N]} m^{[tN]} + \cdots + \mathcal{J}_{\mathcal{O}}^{[(t+d)N]} m^{[(t+d-1)N]} + m^{[(t+d)N]}.$$

Again because of the property of the spectral norm of the Jacobi operator, we have

$$\|m_{[d+1]}^{[tN]}\|_2 \leq \|m^{[tN]}\|_2 + \|m^{[(t+1)N]}\|_2 + \cdots + \|m^{[(t+d)N]}\|_2,$$

and by leveraging on the limit (4.18), we obtain

$$\lim_{t\to\infty} m_{[d+1]}^{[tN]} = 0.$$

Applying Proposition 4.1 to the Jacobi operators in (4.19) it holds that

$$\|\mathcal{J}_{\mathcal{O}}^{[(t+d)N]} \cdots \mathcal{J}_{\mathcal{O}}^{[(t+1)N]} \mathcal{J}_{\mathcal{O}}^{[tN]}\|_2 \leq \gamma_n, \quad 0 \leq \gamma_n < 1. \tag{4.20}$$

Therefore, thanks to the bound in (4.20), the spectral norm of (4.19) satisfies

$$\begin{aligned}\|\chi^{[(t+d)N)}\|_2 &\leq \|\mathcal{J}_{\mathcal{O}}^{[(t+d)N]} \cdots \mathcal{J}_{\mathcal{O}}^{[(t+1)N]} \mathcal{J}_{\mathcal{O}}^{[tN]}\|_2 \|\chi^{((t-1)N)}\|_2 + \|m_{[d+1]}^{[tN]}\|_2 \\ &\leq \gamma_n \|\chi^{[(t-1)N]}\|_2 + \|m_{[d+1]}^{[tN]}\|_2.\end{aligned}$$

Considering that $0 \leq \gamma_n < 1$ and $\|m_{[d+1]}^{[tN]}\|_2 \to 0$, as $t \to \infty$, by Proposition 4.2 it holds that $\lim_{t\to\infty} \chi^{(tN)} = 0$. Therefore, the iterations obtained after each cycle converge to zero.

Additionally, for the iterations $0 < k < N$ within one cycle, from relation (4.14) we have

$$\begin{aligned}&\chi^{((t-1)N+k)} \\ &\quad = \mathcal{R}_{p_{k-1},q_{k-1}}(R_{(t-1)N+k-1}) \cdots \mathcal{R}_{p_1,q_1}(R_{(t-1)N+1}) \mathcal{R}_{p_0,q_0}(R_{(t-1)N}) \chi^{((t-1)N)} \\ &\qquad + \mathcal{R}_{p_{k-1},q_{k-1}}(R_{(t-1)N+k-1}) \cdots \mathcal{R}_{p_1,q_1}(R_{(t-1)N+1}) \mathcal{R}_{p_0,q_0}(R_{(t-1)N}) m^{((t-1)N)} \\ &\qquad + \cdots + \mathcal{R}_{p_{k-1},q_{k-1}}(R_{(t-1)N+k-1}) m^{((t-1)N+k-2)} + m^{((t-1)N+k-1)}\end{aligned}$$

and

$$\begin{aligned}&\|\chi^{((t-1)N+k)}\|_2 \\ &\quad \leq \|\chi^{((t-1)N)}\|_2 + \|m^{((t-1)N)}\|_2 + \cdots + \|m^{((t-1)N+k-2)}\|_2 + \|m^{((t-1)N+k-1)}\|_2 \\ &\quad \leq \|\chi^{((t-1)N)}\|_2 + k \max_{0\leq r\leq k-1} \|m^{((t-1)N+r)}\|_2.\end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Thus, $\lim_{t\to\infty} \|\chi^{((t-1)N+k)}\|_2 = 0$, and then

$$\lim_{k\to\infty} \|\chi^{(k)}\|_2 = 0.$$

Finally, since $\text{off}(H^{(k)}) = \|\chi^{(k)}\|_2$, $k \geq 0$, we have $\lim_{k\to\infty} \text{off}(H^{(k)}) = 0$, which completes the proof. □

Now we can prove the convergence theorem for the Eberlein method under the serial orderings with permutations $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. In what follows, $B^{(k)}$ is defined as in equation (2.2) and $C(B^{(k)})$ is as in (2.3).

THEOREM 4.4. *Let $A \in \mathbb{C}^{n\times n}$, and let $(A^{(k)}, k \geq 0)$ be a sequence generated by the Eberlein method under a generalized serial pivot strategy defined by an ordering $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. Then*

(i) The sequence of the off-norms $(\text{off}(B^{(k)}), k \geq 0)$ tends to zero, that is,

$$\lim_{k\to\infty} \text{off}(B^{(k)}) = 0.$$

(ii) The sequence $(A^{(k)}, k \geq 0)$ tends to a normal matrix, that is,

$$\lim_{k\to\infty} C(A^{(k)}) = 0.$$

(iii) The sequence of matrices $(B^{(k)}, k \geq 0)$ tends to a fixed diagonal matrix, that is,

$$\lim_{k\to\infty} B^{(k)} = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n),$$

where μ_i, $1 \leq i \leq n$, are the real parts of the eigenvalues of A.

(iv) If $\mu_i \neq \mu_j$, then $\lim_{k\to\infty} a_{ij}^{(k)} = 0$ and $\lim_{k\to\infty} a_{ji}^{(k)} = 0$.

Proof.

Item (i). For $F^{(k)}$ defined as in (4.5) we have

$$B^{(k+1)} = R_k^* B^{(k)} R_k + F^{(k)}, \quad k \geq 0. \tag{4.21}$$

At the pivot position (p, q) in step k we have

$$b_{pq}^{(k+1)} = \tilde{b}_{pq}^{(k)} + f_{pq}^{(k)},$$

where $F^{(k)} = (f_{ij}^{(k)})$.

The relations (4.6) and (4.2) imply $\lim_{k\to\infty} F^{(k)} = 0$ and $\lim_{k\to\infty} f_{pq}^{(k)} = 0$. Furthermore, the rotation R_k is chosen to annihilate $\tilde{b}_{pq}^{(k)}$. It annihilates $\tilde{b}_{qp}^{(k)}$ as well because $B^{(k)}$ is Hermitian. Therefore, $\lim_{k\to\infty} b_{pq}^{(k+1)} = 0$ and $\lim_{k\to\infty} b_{qp}^{(k+1)} = 0$. The matrix $B^{(0)} = B$ is Hermitian by definition, and the iterative process (4.21) satisfies the assumptions of Proposition 4.3. Hence,

$$\lim_{k\to\infty} \text{off}(B^{(k)}) = 0.$$

Item (ii). For $E^{(k)}$ defined as in (4.3) we have

$$C(A^{(k+1)}) = C(\tilde{A}^{(k)} + E^{(k)}).$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Then,

$$\begin{aligned} C(A^{(k+1)}) &= (\tilde{A}^{(k)} + E^{(k)})(\tilde{A}^{(k)} + E^{(k)})^* - (\tilde{A}^{(k)} + E^{(k)})^*(\tilde{A}^{(k)} + E^{(k)}) \\ &= \tilde{A}^{(k)}(\tilde{A}^{(k)})^* + E^{(k)}(\tilde{A}^{(k)})^* + (\tilde{A}^{(k)} + E^{(k)})(E^{(k)})^* \\ &\qquad - (\tilde{A}^{(k)})^*\tilde{A}^{(k)} - (E^{(k)})^*\tilde{A}^{(k)} - (\tilde{A}^{(k)} + E^{(k)})^*E^{(k)} \\ &= C(\tilde{A}^{(k)}) + A^{(k+1)}(E^{(k)})^* - (A^{(k+1)})^*E^{(k)} \\ &\qquad + E^{(k)}(\tilde{A}^{(k)})^* - (E^{(k)})^*\tilde{A}^{(k)} \\ &= C(\tilde{A}^{(k)}) + W^{(k)}, \end{aligned} \tag{4.22}$$

where

$$W^{(k)} = A^{(k+1)}(E^{(k)})^* - (A^{(k+1)})^*E^{(k)} + E^{(k)}(\tilde{A}^{(k)})^* - (E^{(k)})^*\tilde{A}^{(k)}.$$

Moreover, applying relation (4.7), we can write (4.22) as

$$C(A^{(k+1)}) = R_k^* C(A^{(k)}) R_k + W^{(k)}.$$

Using the properties of the norm and inequality (4.1) we obtain

$$\begin{aligned} \|W^{(k)}\|_F &\le \|A^{(k+1)}(E^{(k)})^*\|_F + \|(A^{(k+1)})^*E^{(k)}\|_F \\ &\qquad + \|E^{(k)}(\tilde{A}^{(k)})^*\|_F + \|(E^{(k)})^*\tilde{A}^{(k)}\|_F \\ &= 2\|E^{(k)}\|_F(\|A^{(k+1)}\|_F + \|\tilde{A}^{(k)}\|_F) \le 4\|E^{(k)}\|_F\|\tilde{A}^{(k)}\|_F \end{aligned}$$

and

$$\|W^{(k)}\|_F^2 \le 16\|E^{(k)}\|_F^2\|\tilde{A}^{(k)}\|_F^2.$$

It follows from relations (4.3) and (4.4) that

$$\|W^{(k)}\|_F^2 \le 16\|E^{(k)}\|_F^2\|A\|_F^2 \le 24n^2|\tilde{c}_{pq}^{(k)}|\|A\|_F^2.$$

Thus, relation (4.2) implies

$$\lim_{k\to\infty} \|W^{(k)}\|_F = 0. \tag{4.23}$$

We consider the off-diagonal and the diagonal part of $C(A^{(k)})$ separately. Similarly as for matrices $B^{(k)}$, at the pivot position (p, q) in step k we have

$$c_{pq}^{(k+1)} = \tilde{c}_{pq}^{(k)} + w_{pq}^{(k)},$$

where $W^{(k)} = (w_{ij}^{(k)})$. The relations (4.2) and (4.23) imply $\lim_{k\to\infty} c_{pq}^{(k+1)} = 0$. It is easy to verify that the matrices $C(A^{(k)})$, $k \ge 0$, are Hermitian. Then $\lim_{k\to\infty} c_{qp}^{(k+1)} = 0$, and we can use Proposition 4.3 again to obtain

$$\lim_{k\to\infty} \operatorname{off}(C(A^{(k)})) = 0. \tag{4.24}$$

It remains to show that

$$\lim_{k\to\infty} c_{ii}^{(k)} = 0.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Set $A^{(k)} = B^{(k)} + Z^{(k)}$, where $B^{(k)}$ is Hermitian and given in (2.2) and $Z^{(k)}$ is skew-Hermitian. Then,

$$C(A^{(k)}) = 2(Z^{(k)}B^{(k)} - B^{(k)}Z^{(k)}). \tag{4.25}$$

The diagonal element of $C(A^{(k)})$ is given by

$$c_{ii}^{(k)} = 2\sum_{j=1}^{n}(z_{ij}^{(k)}b_{ji}^{(k)} - b_{ij}^{(k)}z_{ji}^{(k)}).$$

In proving assertion *(i)*, it is shown that $\lim_{k\to\infty} \mathrm{off}(B^{(k)}) = 0$, that is,

$$\lim_{k\to\infty} b_{ij}^{(k)} = 0, \quad \text{for } i \neq j.$$

Thus,

$$\lim_{k\to\infty} c_{ii}^{(k)} = 2(z_{ii}^{(k)}b_{ii}^{(k)} - b_{ii}^{(k)}z_{ii}^{(k)}) = 0. \tag{4.26}$$

The relations (4.24) and (4.26) imply the assertion *(ii)* of the theorem.

Item (iii). In proving assertion *(i)* we showed that the matrices $B^{(k)}$ tend to a diagonal matrix. The fact that the diagonal elements of the matrix $\lim_{k\to\infty} B^{(k)}$ correspond to the real parts of the eigenvalues of A is then proved as in [25], using the assertion *(ii)* of this theorem.

Item (iv). Using the relation (4.25) and the assertions *(i)* and *(ii)* of the theorem it follows that

$$\begin{aligned} 0 = \lim_{k\to\infty} c_{ij}^{(k)} &= 2\lim_{k\to\infty}\sum_{r=1}^{n}(z_{ir}^{(k)}b_{rj}^{(k)} - b_{ir}^{(k)}z_{rj}^{(k)}) \\ &= 2\lim_{k\to\infty}(z_{ij}^{(k)}b_{jj}^{(k)} - b_{ii}^{(k)}z_{ij}^{(k)}) = 2(\mu_j - \mu_i)\lim_{k\to\infty} z_{ij}^{(k)}, \quad 1 \le i, j \le n. \end{aligned}$$

If $\mu_i \neq \mu_j$, then $\lim_{k\to\infty} z_{ij}^{(k)} = 0$. Finally, since $\lim_{k\to\infty} b_{ij}^{(k)} = 0$, for $i \neq j$, we have

$$a_{ij}^{(k)} = b_{ij}^{(k)} + z_{ij}^{(k)} \to 0 \quad \text{and} \quad a_{ji}^{(k)} = (b_{ij}^{(k)})^* - (z_{ij}^{(k)})^* \to 0, \quad \text{as } k \to \infty. \qquad \square$$

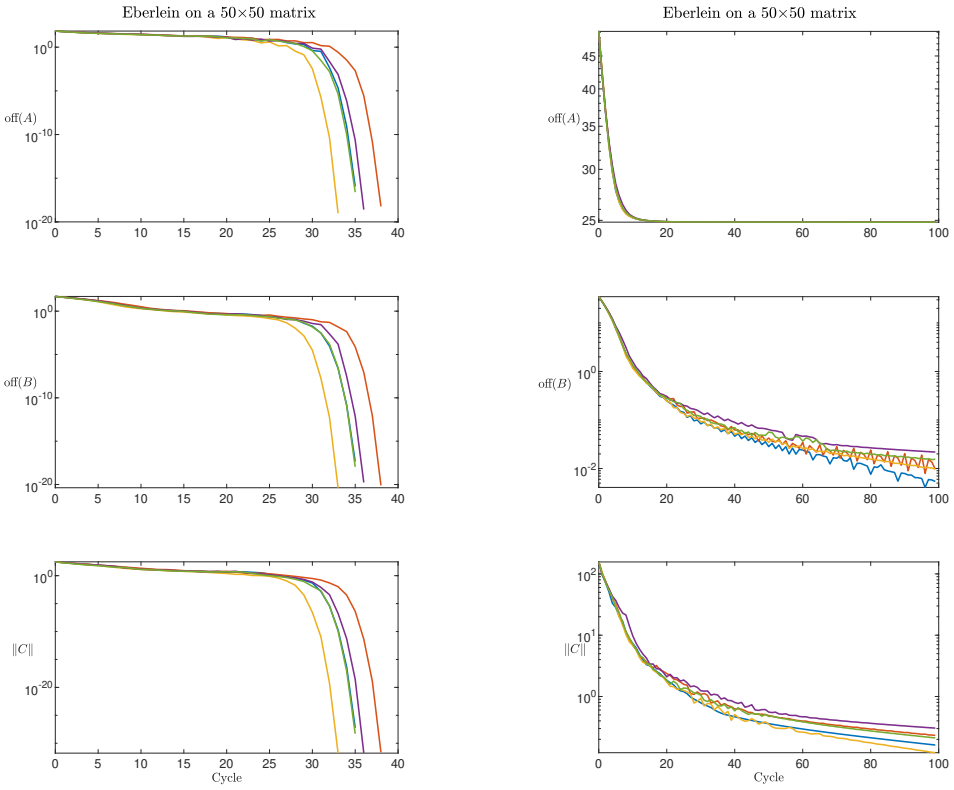
Therefore, starting with an $n \times n$ matrix A, the Eberlein method under a pivot strategy defined by any generalized serial pivot ordering converges to some matrix Λ. If all of the real parts of the eigenvalues of A are distinct, then Λ is a diagonal matrix. If the real parts μ_i and μ_j of the eigenvalues of $A^{(0)}$ are the same, then we can not claim that the corresponding off-diagonal elements $a_{ij}^{(k)}$ and $a_{ji}^{(k)}$ tend to zero. This can result in the presence of blocks on the diagonal of Λ. Assuming that the diagonal elements of Λ are arranged such that their real parts appear in decreasing order, based on Theorem 4.4, we get to the following conclusion: the matrix Λ is a block diagonal matrix with block-sizes corresponding to the number of times the same real part appears in the spectrum of A.

The eigenvalues with distinct real parts can be read from the diagonal of Λ. Pairs of complex conjugate eigenvalues with nonrepeating real parts, if they create a block, will correspond to 2×2 matrices with the real parts on the diagonal. Such eigenvalues are easy to be read from 2×2 blocks. For repeating real parts, the blocks can have larger size. In order to find all eigenvalues of A, it is then enough to find the eigenvalues of such blocks. To this end, e.g., the nonsymmetric Jacobi algorithm for the computation of the Schur form discussed in [24] can be applied.

In our numerical tests, we observed that blocks appear in the cases when there are complex eigenvalues having the same real part but different imaginary parts. Repeating real or complex eigenvalues did not create blocks in practice.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

5. Numerical results. In this section, we numerically test Algorithm 2.1 under the generalized pivot strategies with permutations. All experiments are done in Matlab R2021a. To depict the performance of the Eberlein algorithm, we observe three quantities: off$(A^{(k)})$, off$(B^{(k)})$, and $\|C(A^{(k)})\|_F$. The results are presented in logarithmic scale. The algorithm is terminated when the change in the off-norm of $B^{(k)}$ becomes small enough, here, 10^{-8}. According to Theorem 4.4, both off$(B^{(k)})$ and $\|C(A^{(k)})\|_F$ should converge to zero.



(a) Complex algorithm, random $A \in \mathbb{C}^{50\times 50}$.

(b) Real algorithm, random $A \in \mathbb{R}^{50\times 50}$.

FIG. 5.1. *Change in* off$(A^{(k)})$, off$(B^{(k)})$, *and* $\|C(A^{(k)})\|_F$ *for different pivot strategies.*

Figure 5.1 displays the results of the Eberlein algorithm applied to a non-structured random complex matrix, as well as the results of the real Eberlein algorithm applied to a non-structured random real matrix. We test the algorithm under different pivot strategies. Each line represents the results of a different pivot strategy $I_{\mathcal{O}}$, $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. The strategies are randomly chosen at the beginning of the algorithm. No pivot strategy is superior to others. A strategy that leads to the fastest convergence for one matrix will be slow for a different matrix. We observe that off$(B^{(k)})$ and $\|C(A^{(k)})\|_F$ converge to zero in both the complex and the real algorithm, although the convergence is slower for the real algorithm. In the complex case, off(A) converges to zero as well. In other words, the matrix is diagonalized. On the other hand, this is not the case for the real algorithm. The reason is that for the real algorithm,

ETNA
Kent State University and
Johann Radon Institute (RICAM)

blocks are formed in correspondence with the eigenvalues with equal real part.

The algorithm is significantly faster if it is applied to a normal matrix; see, e.g., [11, 18]. We construct a unitarily diagonalizable 400×400 matrix $A = A^{(0)}$ by multiplying some chosen complex diagonal matrix from the left- and right-hand sides by a random unitary matrix. In Figure 5.2, we display the results of the Eberlein method under a randomly chosen pivot strategy $I_{\mathcal{O}}$, $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$, applied to a diagonalizable complex matrix. Here we do not show $\|C(A^{(k)})\|_F$ because $A^{(0)}$ is normal, that is, $C(A^{(0)}) = 0$, and it stays normal during the process.

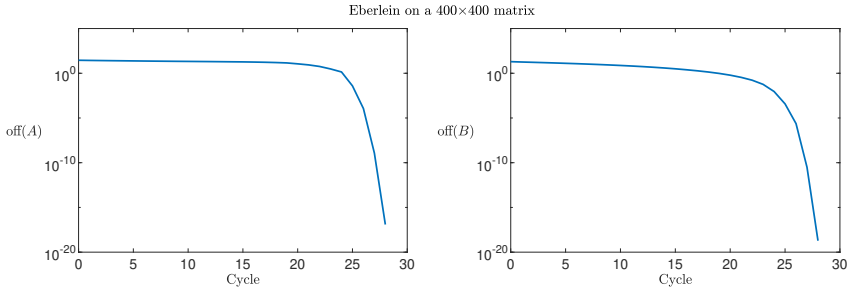


FIG. 5.2. *Progress of the off-norms of $A^{(k)}$ and $B^{(k)}$ for a unitarily diagonalizable complex matrix.*

In order to show the block diagonal structure of $A^{(k)}$ discussed at the end of the previous section, we applied the Eberlein method to matrices in $\mathbb{C}^{10\times 10}$ and $\mathbb{C}^{50\times 50}$. To generate the starting matrix A, first we set the upper-triangular matrix T to have the specified diagonal elements. Then we multiply T by a random unitary matrix Q, that is, $A = Q^*TQ$. In our implementation of the algorithm, we introduce an additional condition so that the real part of the diagonal elements appear in decreasing order. If necessary, this is achieved by translating the angle α_k by $\pi/2$ in the kth step of the process. The evolution of the matrix structure of the iterates is shown in Figure 5.3. Specifically, the figure shows the logarithm of the absolute values of the elements of $A^{(k)}$. Lighter squares represent the elements with larger absolute value. According to Theorem 4.4, the algorithm should converge to a block diagonal matrix in both cases described below.

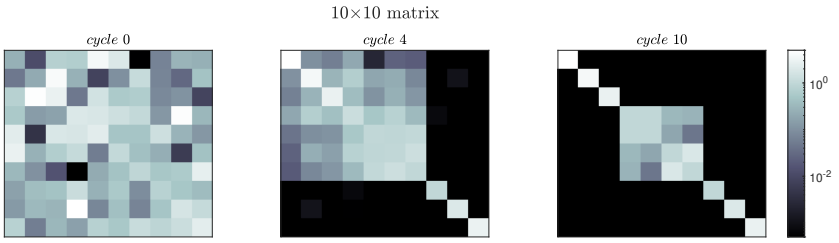
In Figure 5.3a, the matrix has distinct eigenvalues with spectrum

$$\{5, 4, 3, 1 \pm 2\imath, 1 \pm \imath, -1, -2, -3\}.$$

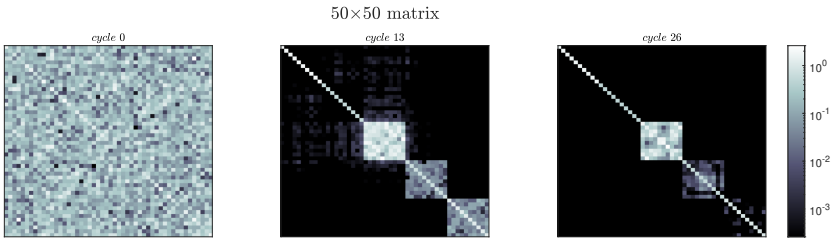
Thus, we deal with two complex conjugate pairs of eigenvalues with the same real part. On the other hand, in Figure 5.3b, the spectrum of a 50×50 matrix consists of two random complex numbers of multiplicity ten and three pairs of complex conjugate complex numbers, each of multiplicity five.

For both matrices, after a few cycles, we can faintly see the diagonal blocks. After a few more cycles, the block diagonal structure is clear. For the first matrix, the obtained 4×4 block has eigenvalues that are (approximately) $1 \pm i$ and $1 \pm 2i$. The rest of the diagonal carries the real eigenvalues of the original matrix. On the other hand, for the second matrix we observe three blocks that correspond to three pairs of complex conjugate eigenvalues. The rest of the diagonal corresponds to two repeating eigenvalues, and they do not form blocks despite the tenfold multiplicity of each eigenvalue. Compared to the part that formed the blocks, for the repeating eigenvalues there are no other eigenvalues with the same real part but different imaginary part.

ETNA
Kent State University and
Johann Radon Institute (RICAM)



(a) Two complex conjugate pairs of eigenvalues with the same real part that formed a 4×4 diagonal block.



(b) Three complex conjugate eigenvalues formed 10×10 diagonal blocks, while the rest of the diagonal carries two repeating eigenvalues.

FIG. 5.3. *Block diagonal structure.*

In Figure 5.4 we test the accuracy of the Eberlein method. The top graph demonstrates that the Eberlein method for a random 50×50 matrix converged to the same solution as the Matlab `eig` function. The bottom graphs displays the relative errors in the real and imaginary parts of the obtained eigenvalues, regarding the solutions obtained by the Matlab `eig` function.

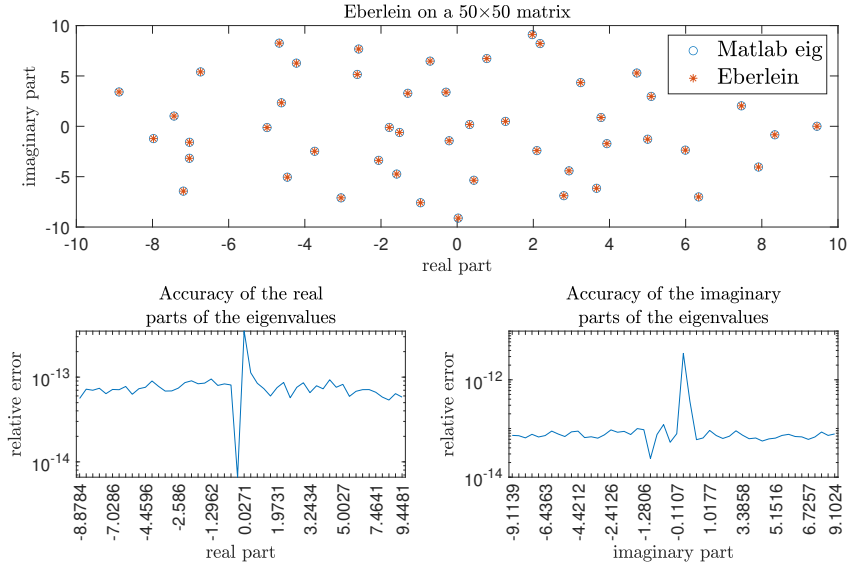


FIG. 5.4. *Accuracy of the Eberlein method in comparison to the Matlab* `eig` *function.*

ETNA
Kent State University and
Johann Radon Institute (RICAM)

To summarize, in this final section we showed the numerical behavior of the Eberlein algorithm. The obtained numerical results depict the theoretical results given in Theorem 4.4. Precisely, for $B^{(k)} = \frac{1}{2}(A^{(k)} + (A^{(k)})^*)$, the sequence $(\text{off}(B^{(k)}), k \geq 0)$ converges to zero, that is, the Hermitian part of $A^{(k)}$ converges to a diagonal matrix. Moreover, for $C(A^{(k)}) = A^{(k)}(A^{(k)})^* - (A^{(k)})^* A^{(k)}$, the sequence $(C(A^{(k)}), k \geq 0)$ converges to zero, that is, $A^{(k)}$ converges to a normal matrix. Moreover, we showed that if the real parts of the eigenvalues of A are distinct, then $A^{(k)}$ converges to a diagonal matrix. Otherwise, blocks corresponding to the repeating eigenvalues could appear. Regarding the accuracy of the method, we compared it to the Matlab `eig` function, and the results are satisfactory.

Acknowledgments. The authors would like to thank the anonymous referees for their comments that improved the paper.

REFERENCES

[1] J. L. AURENTZ, T. MACH, L. ROBOL, R. VANDEBRIL, AND D. S. WATKINS, *Core-Chasing Algorithms for the Eigenvalue Problem*, SIAM, Philadelphia, 2018.
[2] E. BEGOVIĆ, *Konvergencija Blok Jacobijevih Metoda*, PhD. Thesis, Faculty of Science, University of Zagreb, Zagreb, 2014.
[3] A. BUNSE-GERSTNER AND H. FASSBENDER, *A Jacobi-like method for solving algebraic Riccati equations on parallel computers*, IEEE Trans. Automat. Control, 42 (1997), pp. 1071–1084.
[4] J. DEMMEL AND K. VESELIĆ, *Jacobi's method is more accurate than QR*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 1204–1245.
[5] F. M. DOPICO, P. KOEV, AND J. M. MOLERA, *Implicit standard Jacobi gives high relative accuracy*, Numer. Math., 113 (2009), pp. 519–553.
[6] Z. DRMAČ AND K. VESELIĆ, *New fast and accurate Jacobi SVD algorithm. I*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1322–1342.
[7] ———, *New fast and accurate Jacobi SVD algorithm. II*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1343–1362.
[8] P. J. EBERLEIN, *A Jacobi-like method for the automatic computation of eigenvalues and eigenvectors of an arbitrary matrix*, J. Soc. Indust. Appl. Math., 10 (1962), pp. 74–88.
[9] L. ELSNER AND M. H. C. PAARDEKOOPER, *On measures of nonnormality of matrices*, Linear Algebra Appl., 92 (1987), pp. 107–123.
[10] H. FASSBENDER, D. S. MACKEY, AND N. MACKEY, *Hamilton and Jacobi come full circle: Jacobi algorithms for structured Hamiltonian eigenproblems*, Linear Algebra Appl., 332/334 (2001), pp. 37–80.
[11] H. H. GOLDSTINE AND L. P. HORWITZ, *A procedure for the diagonalization of normal matrices*, J. Assoc. Comput. Mach., 6 (1959), pp. 176–195.
[12] V. HARI, *On the global convergence of the Eberlein method for real matrices*, Numer. Math., 39 (1982), pp. 361–369.
[13] V. HARI, *Convergence to diagonal form of block Jacobi-type methods*, Numer. Math., 129 (2015), pp. 449–481.
[14] V. HARI AND E. BEGOVIĆ KOVAČ, *Convergence of the cyclic and quasi-cyclic block Jacobi methods*, Electron. Trans. Numer. Anal., 46 (2017), pp. 107–147. https://etna.ricam.oeaw.ac.at/vol.46.2017/pp107-147.dir/pp107-147.pdf
[15] ———, *On the convergence of complex Jacobi methods*, Linear Multilinear Algebra, 69 (2021), pp. 489–514.
[16] V. HARI, S. SINGER, AND S. SINGER, *Full block J-Jacobi method for Hermitian matrices*, Linear Algebra Appl., 444 (2014), pp. 1–27.
[17] P. HENRICI AND K. ZIMMERMANN, *An estimate for the norms of certain cyclic Jacobi operators*, Linear Algebra Appl., 1 (1968), pp. 489–501.
[18] G. LOIZOU, *On the quadratic convergence of the Jacobi method for normal matrices*, Comput. J., 15 (1972), pp. 274–276.
[19] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Numerical methods for palindromic eigenvalue problems: computing the anti-triangular Schur form*, Numer. Linear Algebra Appl., 16 (2009), pp. 63–86.
[20] D. S. MACKEY, N. MACKEY, AND F. TISSEUR, *Structured tools for structured matrices*, Electron. J. Linear Algebra, 10 (2003), pp. 106–145.
[21] W. F. MASCARENHAS, *On the convergence of the Jacobi method for arbitrary orderings*, SIAM J. Matrix Anal. Appl., 16 (1995), pp. 1197–1209.
[22] J. MATEJAŠ, *Accuracy of the Jacobi method on scaled diagonally dominant symmetric matrices*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 133–153.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

[23] C. MEHL, *Jacobi-like algorithms for the indefinite generalized Hermitian eigenvalue problem*, SIAM J. Matrix Anal. Appl., 25 (2004), pp. 964–985.
[24] ———, *On asymptotic convergence of nonsymmetric Jacobi algorithms*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 291–311.
[25] D. PUPOVCI AND V. HARI, *On the convergence of parallelized Eberlein methods*, Rad. Mat., 8 (1992/98), pp. 249–267.
[26] I. SLAPNIČAR, *Highly accurate symmetric eigenvalue decomposition and hyperbolic SVD*, Linear Algebra Appl., 358 (2003), pp. 387–424.
[27] K. VESELIĆ, *A convergent Jacobi method for solving the eigenproblem of arbitrary real matrices*, Numer. Math., 25 (1975/76), pp. 179–184.