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A NOTE ON "ERROR BOUNDS OF GAUSSIAN QUADRATURE FORMULAE WITH LEGENDRE WEIGHT FUNCTION FOR ANALYTIC INTEGRANDS" BY M. M. SPALEVIĆ ET AL.*

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Dedicated to Prof. Lothar Reichel on the occasion of entering his 8-th decade and to Prof. Miodrag M. Spalević on the occasion of entering his 7-th decade.

Abstract. In paper D. LJ. ĐUKIĆ, R. M. MUTAVDŽIĆ ĐUKIĆ, A. V. PEJČEV, AND M. M. SPALEVIĆ, *Error estimates of Gaussian-type quadrature formulae for analytic functions on ellipses – a survey of recent results*, Electron. Trans. Numer. Anal., 53 (2020), pp. 352–382, Lemma 4.1 can be applied to show the asymptotic behaviour of the modulus of the complex kernel in the remainder term of all the quadrature formulas in the recent papers that are concerned with error estimates of Gaussian-type quadrature formulae for analytic functions on ellipses. However, in the paper D. R. JANDRLIĆ, DJ. M. KRTINIĆ, LJ. V. MIHIĆ, A. V. PEJČEV, M. M. SPALEVIĆ, *Error bounds of Gaussian quadrature formulae with Legendre weight function for analytic integrands*, Electron. Trans. Anal. 55 (2022), pp. 424–437, which this note is concerned with, there is a kernel whose numerator contains an infinite series, and in this case the mentioned lemma cannot be applied. This note shows that the modulus of the latter kernel attains its maximum as conjectured in the latter paper.

Key words. error bound, quadrature formula, Legendre weight function

AMS subject classifications. 65D32, 65D30, 41A55

1. Introduction. Numerical integration is the process of approximating a definite integral of a given function. The n-point quadrature formula (q.f.) with respect to some positive weight function ω on a finite interval, which we normalize to be $[-1, 1]$, has the form

$$\int_{-1}^{1} \omega(t) f(t)\, dt = \sum_{i=1}^{n} \omega_i f(\xi_i) + R_n(f) \tag{1.1}$$

for some set of nodes ξ_i and weights ω_i. Here, we are concerned with n-point Gauss-Legendre q.f. associated with the Legendre weight function. In the work [12], which inspired the work [7], the authors were concerned with Jacobi polynomials $P_n^{(\alpha,\beta)}$, which are orthogonal over $[-1, 1]$ with respect to the Jacobi weight function

$$\omega(t) = (1 - t)^{\alpha}(1 + t)^{\beta}, \quad \alpha, \beta > -1.$$

In the work [7], the authors considered the particular case when $\alpha = \beta = 0$, which leads to the Gauss-Legendre q.f. (1.1) with the weight function $\omega(t) = 1$, and estimated the remainder term $R_n(f)$ of this quadrature rule. When f is an analytic function, the remainder term can be represented as a contour integral with a complex kernel. The authors studied the kernel on elliptic contours with foci at the points ∓ 1 and with the sum of semi-axes $\rho > 1$. They determined an explicit expression for the kernel and on the basis of the assumption that its modulus attains its maximum in the way presented in a form of a conjecture (without proof) [7, pp. 429–430], determined the location on the ellipses where maximum modulus of the kernel is attained for large enough ρ. This allowed the derivation of effective error bounds for this q.f. It is the purpose of this note to prove this conjecture.

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2. Error bounds of q.f. for analytic functions. Let Γ be a simple closed curve in the complex plane encompassing the interval $[-1,1]$ and let $\mathcal{D}$ be its interior. Suppose that f is a function that is analytic in $\mathcal{D}$ and continuous on its closure $\overline{\mathcal{D}} = \mathcal{D} \cup \Gamma$. It follows from the Lagrange interpolation formula (see, e. g., [10, Chapter 2]) that

$$r_n(f;t) = f(t) - \sum_{i=1}^{n} l_i(t) f(\xi_i) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)\Omega_n(t)}{(z-t)\Omega_n(z)}\, dz, \tag{2.1}$$

where

$$\Omega_n(z) = c \prod_{i=1}^{n} (z - \xi_i) \quad (c \neq 0),$$

where ξ_i are the zeros of the corresponding orthogonal polynomial $\Omega_n(t)$ (which is monic when $c = 1$) and the l_i are the fundamental polynomials of Lagrange interpolation. By multiplying (2.1) by the weight function $\omega(t)$ and integrating in t over $(-1,1)$, we get a contour integral representation for the remainder term $R_n(f)$ in the Gauss q.f. (1.1),

$$R_n(f) = I(f;\omega) - \sum_{i=1}^{n} \omega_i f(\xi_i) = \frac{1}{2\pi i} \oint_\Gamma K_n(z;\omega) f(z)\, dz, \tag{2.2}$$

where

$$I(f;\omega) = \int_{-1}^{1} \omega(t)\, f(t)\, dt, \quad \omega_i = \int_{-1}^{1} \omega(t)\, l_i(t)\, dt,$$

and the *kernel* $K_n(z) = K_n(z;\omega)$ can be expressed in the form

$$K_n(z;\omega) = \frac{\varrho_n(z;\omega)}{\Omega_n(z)}, \quad \varrho_n(z;\omega) = \int_{-1}^{1} \omega(t)\, \frac{\Omega_n(t)}{z-t}\, dt, \quad z \in \mathbb{C} \setminus [-1,1]. \tag{2.3}$$

The integral representation (2.2) leads directly to the error bound

$$|R_n(f)| \leq \frac{l(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_n(z)| \right) \left(\max_{z\in\Gamma} |f(z)| \right), \tag{2.4}$$

where $l(\Gamma)$ denotes the length of the contour Γ (see [2]).

A common choice for the contour Γ is one of the confocal ellipses with foci at the points ∓ 1, also known as Bernstein ellipses, and the sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} :\; z = \frac{1}{2}\left(u + u^{-1}\right), u = \rho e^{i\theta},\; 0 \leq \theta < 2\pi \right\}. \tag{2.5}$$

The Bernstein ellipse has the major and minor semi-axes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$, respectively. A recent survey of error bounds (2.4) as well as of related bound, when $\Gamma = \mathcal{E}_\rho$, for Gaussian type q.f. can be found in [2].

The behavior of Legendre polynomials, $\pi_n(t) = P_n^{(0,0)}(t)$ (a particular case of the well-known Jacobi polynomials, $P_n^{(\alpha,\beta)}(t)$ with $\alpha = \beta = 0$), denoted by $\Omega_n(t)$ in the paper [7], was studied in the context of the rate of convergence of spectral interpolation and spectral collocation methods for solving integral and differential equations. By studying these problems, an explicit representation of $P_n^{(\alpha,\beta)}(t)$ was derived in the variable of parametrization and the extrema of $|P_n^{(\alpha,\beta)}(t)|$ on the Bernstein ellipse are identified for some parameters and refined

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asymptotic estimate were provided in [12]. An explicit formula of Jacobi polynomials obtained in [12, Lemma 3.2 and formula (3.11)],

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k}, \quad z = \frac{1}{2}\left(u + u^{-1}\right), \tag{2.6}$$

where, in the case $\alpha = \beta > -1$,

$$d_{k,n} = \begin{cases} \dfrac{2^{2\alpha}\Gamma(n+\alpha+1)\Gamma\left(\frac{k+n+1}{2}+\alpha\right)\Gamma\left(\frac{n-k+1}{2}+\alpha\right)}{\sqrt{\pi}\Gamma(n+2\alpha+1)\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{n-k}{2}+1\right)\Gamma(\alpha+1/2)}, & n-k \text{ even}, \\ 0, & n-k \text{ odd}. \end{cases} \tag{2.7}$$

was used and this general case was adopted to the particular case of Legendre polynomials.

Using properties of Chebyshev polynomials of the first and second kinds

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad n \geq 0$$

and the representations of them when $z \in \mathcal{E}_\rho$ in (cf. (2.5)), $z = \frac{1}{2}(u+u^{-1}), u = \rho e^{i\theta}$ $(0 \leq \theta < 2\pi)$, (see [6, pp. 1176–1177]),

$$T_n(z) = \frac{1}{2}(u^n + u^{-n}), \qquad U_n(z) = \frac{u^{n+1} - u^{-n-1}}{u - u^{-1}},$$

the authors of [12] obtained trigonometric representations of (2.6):

$$P_n^{(\alpha,\beta)}(\cos\theta) = d_{0,n} + 2\sum_{k=1}^{n} d_{k,n}\cos(k\theta). \tag{2.8}$$

This allowed to establish a connection with Chebyshev polynomials. Some of the properties of Chebyshev polynomials are described by Notaris in [9]. It was shown that the integrals

$$\int_{-1}^{1} \frac{p_n(t)}{z \mp t}\, dt, \quad |z| \neq 1,$$

with the p_n being any one of the Chebysev polynomials of degree n, can be computed explicitly; see [9, Proposition 2.2]. This particular property finds direct application in the computation of an explicit kernel formula in our case (and further in the estimation of the quadrature error). This is discussed in the following sections.

3. The maximum of the modulus of the kernel. The kernel studied in [7] is given by $K_n(z) = \frac{\varrho_n(z)}{\pi_n(z)}$, $z \notin [-1,1]$, where $\pi_n(z)$ is defined by (2.6) with

$$d_{k,n} = \begin{cases} \dfrac{(n+k-1)!!\,(n-k-1)!!}{(n+k)!!\,(n-k)!!}, & n-k \text{ even}, \\ 0, & n-k \text{ odd}, \end{cases}. \tag{3.1}$$

and (see [7, pp. 426–427])

$$k!! = \begin{cases} \prod_{i=0}^{k/2-1} (k-2i), & k \text{ even}, \\ \prod_{i=0}^{(k-1)/2} (k-2i), & k \text{ odd}, \end{cases}.$$

while, due to [7, Remark 2.2], for n odd

$$\varrho_n(z) = \sum_{j=1}^{\infty} \frac{c_j}{u^{2j}},$$

with

$$c_j = -16j \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-1}{}'' \frac{d_{k,n}}{k^2 - (2j)^2},$$

and analogously for n even

$$\varrho_n(z) = \sum_{j=1}^{\infty} \frac{c_j}{u^{2j+1}} = \frac{1}{u} \sum_{j=1}^{\infty} \frac{c_j}{u^{2j}},$$

with

$$c_j = -8j \sum_{\substack{k=0 \\ k \text{ even}}}^{n}{}'' \frac{d_{k,n}}{k^2 - (2j+1)^2},$$

where the notation $\sum''$ means that the first term $d_{0,n}$ in the sum must be halved (if it appears); see [7, pp. 426]. The conjecture in [7] is concerned with expressions of the form

$$I(\rho,\theta) = \frac{u^2 + A + o\left(\frac{1}{\rho}\right)}{u^2 + B + o\left(\frac{1}{\rho}\right)} = \frac{\frac{1}{u^2} + \frac{A}{u^4} + o\left(\frac{1}{\rho^5}\right)}{\frac{1}{u^2} + \frac{B}{u^4} + o\left(\frac{1}{\rho^5}\right)} \qquad (\rho \to \infty),$$

where A and B are real numbers different each other, $u = \rho e^{i\theta}$, and $I(\rho,\theta) \equiv I(\rho,\theta+\pi)$. The conjecture claims that, under some aditional conditions, if $A > B$, then there exists $\rho^* > 1$ such that $\max_{\theta\in[0,\pi)} |I(\rho,\theta)| = |I(\rho,0)|$ for each $\rho > \rho^*$ and if $A < B$, then there exists $\rho^* > 1$ such that $\max_{\theta\in[0,\pi)} |I(\rho,\theta)| = |I(\rho,\pi/2)|$ for each $\rho > \rho^*$. These statements were confirmed empirically in the case of the modulus of our kernel in [7], but no proof was provided.

In our case the numerator is an infinite series in $\frac{1}{u^2}$,

$$\sum_{i=1}^{+\infty} \frac{A_i}{u^{2i}}, \quad A_1 = 1,$$

while the denominator is a polynomial expression in $\frac{1}{u^2}$,

$$\sum_{i=1}^{n+1} \frac{B_i}{u^{2i}}, \quad B_1 = 1,$$

and we will see that the “aditional conditions” mentioned imply uniform boundness of the coefficients of the series in the numerator. Since

$$\left|\sum_{j=1}^{\infty} \frac{c_j}{u^{2j}}\right| = \sqrt{\sum_{j=1}^{\infty} c_j^2 \rho^{-4j} + 2\sum_{i=2}^{\infty}\sum_{j=1}^{i-1} c_i c_j \rho^{-2i-2j} \cos 2(i-j)\theta},$$

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after (for the sake of compactness of the notation) denoting

$$\pi_n(z) = \sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k} = u^{n+2} \sum_{j=1}^{n+1} \frac{b_j}{u^{2j}},$$

and further

$$\sum_{j=1}^{\infty} \frac{c_j}{u^{2j}} = c_1 \sum_{j=1}^{\infty} \frac{\overline{c}_j}{u^{2j}}, \qquad \overline{c}_j = \frac{c_j}{c_1}, \qquad (\overline{c}_1 = 1),$$

$$\sum_{j=1}^{n+1} \frac{b_j}{u^{2j}} = b_1 \sum_{j=1}^{n+1} \frac{\overline{b}_j}{u^{2j}}, \qquad \overline{b}_j = \frac{b_j}{b_1}, \qquad (\overline{b}_1 = 1),$$

it appears that we have to prove the following.

THEOREM 3.1. *For a fixed natural number n and the coefficients $\overline{c}_1, \overline{c}_2, \ldots$, and $\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_{n+1}$ defined depending on the parity of n as described above, if $\overline{c}_2 > \overline{b}_2$, then there exists ρ^* such that for each $\rho > \rho*$ and each $\theta \in [0, \pi)$, it holds*

$$\frac{\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2\sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} \rho^{-2i_1-2j_1} \cos 2(i_1 - j_1)\theta}{\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2\sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} \rho^{-2i_2-2j_2} \cos 2(i_2 - j_2)\theta}$$
$$\leq \frac{\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2\sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} \rho^{-2i_1-2j_1}}{\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2\sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} \rho^{-2i_2-2j_2}},$$

and if $\overline{c}_2 < \overline{b}_2$, then there exists ρ^* such that for each $\rho > \rho*$ and each $\theta \in [0, \pi)$, it holds

$$\frac{\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2\sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} \rho^{-2i_1-2j_1} \cos 2(i_1 - j_1)\theta}{\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2\sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} \rho^{-2i_2-2j_2} \cos 2(i_2 - j_2)\theta}$$
$$\leq \frac{\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2\sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} (-1)^{i_1-j_1} \rho^{-2i_1-2j_1}}{\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2\sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} (-1)^{i_2-j_2} \rho^{-2i_2-2j_2}}.$$

Proof. When $\overline{c}_2 > \overline{b}_2$, we have to show that the expression

$$\left(\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2 \sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} \rho^{-2i_1-2j_1} \cos 2(i_1 - j_1)\theta \right) \times$$
$$\left(\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2 \sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} \rho^{-2i_2-2j_2} \right)$$
$$- \left(\sum_{j_2=1}^{n+1} \overline{b}_{j_2}^2 \rho^{-4j_2} + 2 \sum_{i_2=2}^{n+1} \sum_{j_2=1}^{i_2-1} \overline{b}_{i_2} \overline{b}_{j_2} \rho^{-2i_2-2j_2} \cos 2(i_2 - j_2)\theta \right) \times$$
$$\left(\sum_{j_1=1}^{\infty} \overline{c}_{j_1}^2 \rho^{-4j_1} + 2 \sum_{i_1=2}^{\infty} \sum_{j_1=1}^{i_1-1} \overline{c}_{i_1} \overline{c}_{j_1} \rho^{-2i_1-2j_1} \right)$$
$$= 2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta - 1)\rho^{-10} + \sum_{s=6}^{\infty} p_s \rho^{-2s}$$
$$= 2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta - 1) \left(\rho^{-10} + \sum_{s=6}^{+\infty} \frac{p_s}{2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta - 1)} \rho^{-2s} \right),$$

is negative for each $\theta \in (0,\pi)$ when ρ is large enough, where

$$\begin{aligned} p_s =& 4 \sum_{\substack{1\le j_1\le i_1-1\\ 1\le j_2\le i_2-1\\ i_1+j_1+i_2+j_2=s}} \overline{c}_{i_1}\overline{c}_{j_1}\overline{b}_{i_2}\overline{b}_{j_2}\left(\cos 2(i_1-j_1)\theta - \cos 2(i_2-j_2)\theta\right)\\ &+2\sum_{\substack{1\le j_1\\ 1\le j_2\le i_2-1\\ 2j_1+i_2+j_2=s}} \overline{c}_{j_1}^2\overline{b}_{i_2}\overline{b}_{j_2}\left(1-\cos 2(i_2-j_2)\theta\right)\\ &+2\sum_{\substack{1\le j_2\\ 1\le j_1\le i_1-1\\ 2j_2+i_1+j_1=s}} \overline{b}_{j_2}^2\overline{c}_{i_2}\overline{c}_{i_2}\left(\cos 2(i_1-j_1)\theta - 1\right), \end{aligned}$$

since, when we multiply the corresponding parentheses monomial by monomial, ρ^{-2s} appears in the following cases:

1. $2\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}\cos 2(i_1-j_1)\theta \cdot 2\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}$ and $-2\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}\cdot \cos 2(i_2-j_2)\theta \cdot 2\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}$, where $1\le j_1\le i_1-1; 1\le j_2\le i_2-1; i_1+j_1+i_2+j_2=s$;
2. $\overline{c}_{j_1}^2\rho^{-4j_1}\cdot 2\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}$ and $-\overline{c}_{j_1}^2\rho^{-4j_1}\cdot 2\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}\cos 2(i_2-j_2)\theta$, where $1\le j_1; 1\le j_2\le i_2-1; 2j_1+i_2+j_2=s$;
3. $\overline{b}_{j_2}^2\rho^{-4j_2}\cdot 2\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}\cos 2(i_1-j_1)\theta$ and $-\overline{b}_{j_2}^2\rho^{-4j_2}\cdot 2\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}$, where $1\le j_2; 1\le j_1\le i_1-1; 2j_2+i_1+j_1=s$.

We have to prove that there exists ρ^*, independent of θ, such that for each $\theta\in(0,\pi)$ it holds that

$$\rho^{-10}+\sum_{s=6}^{+\infty}\frac{p_s}{2(\overline{c}_2-\overline{b}_2)(\cos 2\theta-1)}\rho^{-2s}>0$$

for each $\rho\in(\rho^*,+\infty)$. To show this, we will first prove the following lemma.

LEMMA 3.2. *There exists a constant $C>1$ such that for each $s\in\mathbb{N}$, $s>5$, it holds that*

$$\left|\frac{p_s}{2(\overline{c}_2-\overline{b}_2)(\cos 2\theta-1)}\right|<C^s.$$

Proof. The following bounds hold:

(3.2)
$$\begin{aligned} &\left|\frac{p_s}{2(\overline{c}_2-\overline{b}_2)(\cos 2\theta-1)}\right| \le \frac{1}{2(\overline{c}_2-\overline{b}_2)}\times\\ &\left(4\sum_{\substack{1\le j_1\le i_1-1\\ 1\le j_2\le i_2-1\\ i_1+j_1+i_2+j_2=s}} |\overline{c}_{i_1}\overline{c}_{j_1}\overline{b}_{i_2}\overline{b}_{j_2}|\frac{|\cos 2(i_1-j_1)\theta-\cos 2(i_2-j_2)\theta|}{1-\cos 2\theta}\right.\\ &\left.+2\sum_{\substack{1\le j_1\\ 1\le j_2\le i_2-1\\ 2j_1+i_2+j_2=s}}\overline{c}_{j_1}^2|\overline{b}_{i_2}\overline{b}_{j_2}|\frac{1-\cos 2(i_2-j_2)\theta}{1-\cos 2\theta}+2\sum_{\substack{1\le j_2\\ 1\le j_1\le i_1-1\\ 2j_2+i_1+j_1=s}}\overline{b}_{j_2}^2|\overline{c}_{i_2}\overline{c}_{i_2}|\frac{1-\cos 2(i_1-j_1)\theta}{1-\cos 2\theta}\right). \end{aligned}$$

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The number of terms om each of the three sums is obviously bounded by the number of positive integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = s$, which is, as we know from the high school, equals to $\binom{s-1}{3}$, which is a polynomial function in s.

The number of b-coefficients is finite, so the set of their absolute values is uniformly bounded. Whenever $2j \geq n$, we have

$$\begin{aligned}|c_j| &= 16j \sideset{}{''}\sum_{\substack{k=1\\k \text{ odd}}}^{n-1} \frac{d_{k,n}}{(2j)^2 - k^2} \leq 16 \sum_{\substack{k=1\\k \text{ odd}}}^{n-1} d_{k,n} \frac{j}{(2j)^2 - (2j-1)^2} \\ &= 16 \sum_{\substack{k=1\\k \text{ odd}}}^{n-1} d_{k,n} \frac{j}{4j-1} \leq \frac{16}{3} \sum_{\substack{k=1\\k \text{ odd}}}^{n-1} d_{k,n}\end{aligned}$$

if n is odd (since $\frac{j}{4j-1} \leq \frac{1}{3}$ for each $j \in \mathbb{N}$), and

$$\begin{aligned}|c_j| &= 8j \sideset{}{''}\sum_{\substack{k=0\\k \text{ even}}}^{n} \frac{d_{k,n}}{(2j+1)^2 - k^2} \leq 8 \sum_{\substack{k=0\\k \text{ even}}}^{n} d_{k,n} \frac{j}{(2j+1)^2 - (2j)^2} \\ &= 8 \sum_{\substack{k=0\\k \text{ even}}}^{n} d_{k,n} \frac{j}{4j+1} \leq 2 \sum_{\substack{k=0\\k \text{ even}}}^{n} d_{k,n}\end{aligned}$$

if n even (since $\frac{j}{4j+1} \leq \frac{1}{4}$ for each $j \in \mathbb{N}$). The set of the c_j for which $2j < n$ is finite, which means that the set of the absolute values of all c-coefficients is uniformly bounded, i.e., there exists a constant $M > 0$ such that

$$c_j < M, \quad \text{and} \quad b_i < M, \quad \text{for each } i, j \in \mathbb{N}. \tag{3.3}$$

We know that the expressions

$$\frac{1 - \cos 2i\theta}{1 - \cos 2\theta},$$

are polynomials in $\cos 2\theta$, since for each $i \in \mathbb{N}$ it is known that $\cos 2i\theta$ is the polynomial T_i (the Chebyshev polynomial of the first kind) in $\cos 2\theta$ of the degree i. Moreover, whenever $\cos 2\theta = 1$, it holds that $\cos 2i\theta = 1$, which means that $1 - \cos 2i\theta = 1 - T_i(\cos 2\theta)$ is a polynomial in $\cos 2\theta$ which takes value zero when $\cos 2\theta = 1$. We conlude using Bezuot's theorem that the polynomial $1 - T_i(x)$ is divisible by the polynomial $1 - x$, and the quotient is a polynomial in x. It now follows that

$$\frac{\cos 2j\theta - \cos 2i\theta}{1 - \cos 2\theta} = \frac{(1 - \cos 2i\theta) - (1 - \cos 2j\theta)}{1 - \cos 2\theta} = \frac{1 - \cos 2i\theta}{1 - \cos 2\theta} - \frac{1 - \cos 2j\theta}{1 - \cos 2\theta}$$

also is a polynomial in $\cos 2\theta$. The degrees of the polynomials in the expression under the consideration are $i - 1$, i.e., $\max\{i-1, j-1\}$ (or zero when $i = j$), which is less than s. For each polynomial $Q_d(\cos 2\theta) = \sum_{t=0}^{d} a_t \left(\cos 2\theta\right)^t$, we have

$$|Q_d(\cos 2\theta)| \leq \sum_{t=0}^{d} |a_t| |\cos 2\theta|^t \leq \sum_{t=0}^{d} |a_t| = \|Q_d\|.$$

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Since the polynomial $Q_{i-1}(\cos 2\theta)$ (for each integer i, $i \geq 3$) satisfies the same recurence as the Chebyshev polynomials,

$$Q_i(\cos 2\theta) = 2\cos 2\theta\, Q_{i-1}(\cos 2\theta) - Q_{i-2}(\cos 2\theta),$$

the sum of the absolute values of the coefficients, $\|Q_i\|$, satisfies $\|Q_i\| \leq 2\|Q_{i-1}\| + \|Q_{i-2}\|$, for which we know that it has an exponential bound in i, i.e., there exists a constant $C_1 > 1$ such that $\|Q_i\| < C_1^i$ for each $i \in \mathbb{N}$, which is further bounded by C_1^s for each $s \in \mathbb{N}$, $s > 5$, since $s > i$. Also,

$$\frac{\cos 2j\theta - \cos 2i\theta}{1 - \cos 2\theta} = Q_j(\cos 2\theta) - Q_i(\cos 2\theta),$$

and $\|Q_j(\cos 2\theta) - Q_i(\cos 2\theta\| \leq \|Q_j\| + \|Q_i\| \leq C_1^i + C_1^j < 2C_1^s$, for each $s > 5$, since it always holds $s > i, j$.

Now, from (3.3) and (3.2) it follows that

$$\left|\frac{p_s}{2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta{-}1)}\right| \leq \frac{1}{2(\overline{c}_2 - \overline{b}_2)}\binom{s-1}{3}\left(\frac{M}{c_1}\right)^2\left(\frac{M}{b_1}\right)^2(4\cdot 2C_1^s + 2C_1^s + 2C_1^s) < C^s$$

for some constant $C > 1$ and each $s \in \mathbb{N}$, $s > 5$, which shows our lemma. ∎

We now can finish the proof of the theorem. It holds that

$$\begin{aligned}
&\rho^{-10} + \sum_{s=6}^{+\infty}\frac{p_s}{2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta{-}1)}\rho^{-2s} \geq \rho^{-10} - \left|\sum_{s=6}^{+\infty}\frac{p_s}{2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta{-}1)}\rho^{-2s}\right| \\
&\geq \rho^{-10} - \sum_{s=6}^{+\infty} C^s\rho^{-2s} \\
&= \rho^{-10}\left(1 - C^5\sum_{j=1}^{+\infty}\left(C\rho^{-2}\right)^j\right) = \rho^{-10}\left(1 - C^5\frac{\frac{C}{\rho^2}}{1 - \frac{C}{\rho^2}}\right) = \rho^{-10}\frac{\rho^2 - C^6 - C}{\rho^2 - C},
\end{aligned}$$

which obviously is positive for each $\rho > \sqrt{C^6 + C} = \rho^*$ and does not depend on θ.

The case $\overline{c}_2 < \overline{b}_2$ is analogous, i.e., we have to show that the expression

$$\begin{aligned}
&\left(\sum_{j_1=1}^{\infty}\overline{c}_{j_1}^2\rho^{-4j_1} + 2\sum_{i_1=2}^{\infty}\sum_{j_1=1}^{i_1-1}\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}\cos 2(i_1 - j_1)\theta\right)\times \\
&\qquad\left(\sum_{j_2=1}^{n+1}\overline{b}_{j_2}^2\rho^{-4j_2} + 2\sum_{i_2=2}^{n+1}\sum_{j_2=1}^{i_2-1}\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}(-1)^{i_2-j_2}\right) \\
&-\left(\sum_{j_2=1}^{n+1}\overline{b}_{j_2}^2\rho^{-4j_2} + 2\sum_{i_2=2}^{n+1}\sum_{j_2=1}^{i_2-1}\overline{b}_{i_2}\overline{b}_{j_2}\rho^{-2i_2-2j_2}\cos 2(i_2 - j_2)\theta\right)\times \\
&\qquad\left(\sum_{j_1=1}^{\infty}\overline{c}_{j_1}^2\rho^{-4j_1} + 2\sum_{i_1=2}^{\infty}\sum_{j_1=1}^{i_1-1}\overline{c}_{i_1}\overline{c}_{j_1}\rho^{-2i_1-2j_1}(-1)^{i_1-j_1}\right) \\
&= 2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta + 1)\rho^{-10} + \sum_{s=6}^{+\infty}p_s\rho^{-2s} \\
&= 2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta + 1)\left(\rho^{-10} + \sum_{s=6}^{+\infty}\frac{p_s}{2(\overline{c}_2 - \overline{b}_2)(\cos 2\theta + 1)}\rho^{-2s}\right),
\end{aligned}$$

will be negative for each $\theta \in [0,\pi)\{\pi/2\}$ when ρ is large enough, where

$$\begin{aligned}
p_s =& 4 \sum_{\substack{1\le j_1\le i_1-1\\ 1\le j_2\le i_2-1\\ i_1+j_1+i_2+j_2=s}} \overline{c}_{i_1}\overline{c}_{j_1}\overline{b}_{i_2}\overline{b}_{j_2}\left((-1)^{i_2-j_2}\cos 2(i_1-j_1)\theta-(-1)^{i_1-j_1}\cos 2(i_2-j_2)\theta\right)\\
&+2\sum_{\substack{1\le j_1\\ 1\le j_2\le i_2-1\\ 2j_1+i_2+j_2=s}} \overline{c}_{j_1}^2\overline{b}_{i_2}\overline{b}_{j_2}\left((-1)^{i_2-j_2}-\cos 2(i_2-j_2)\theta\right)\\
&+2\sum_{\substack{1\le j_2\\ 1\le j_1\le i_1-1\\ 2j_2+i_1+j_1=s}} \overline{b}_{j_2}^2\overline{c}_{i_2}\overline{c}_{i_2}\left(\cos 2(i_1-j_1)\theta-(-1)^{i_1-j_1}\right).
\end{aligned}$$

In this case we have to prove that there exists ρ^*, independent of θ, such that for each $\theta \in [0,\pi)/\{\pi/2\}$ it holds

$$\rho^{-10}+\sum_{s=6}^{+\infty}\frac{p_s}{2(\overline{c}_2-\overline{b}_2)(\cos 2\theta+1)}\rho^{-2s}>0 \ \text{ for each } \ \rho\in(\rho^*,+\infty).$$

We have the following bounds:

$$\begin{aligned}
&\left|\frac{p_s}{2(\overline{c}_2-\overline{b}_2)(\cos 2\theta+1)}\right| \le \frac{1}{2(\overline{c}_2-\overline{b}_2)}\times\\
&\left(4\sum_{\substack{1\le j_1\le i_1-1\\ 1\le j_2\le i_2-1\\ i_1+j_1+i_2+j_2=s}} |\overline{c}_{i_1}\overline{c}_{j_1}\overline{b}_{i_2}\overline{b}_{j_2}|\frac{|(-1)^{i_2-j_2}\cos 2(i_1-j_1)\theta-(-1)^{i_1-j_1}\cos 2(i_2-j_2)\theta|}{1+\cos 2\theta}\right.\\
&\quad+2\sum_{\substack{1\le j_1\\ 1\le j_2\le i_2-1\\ 2j_1+i_2+j_2=s}} \overline{c}_{j_1}^2|\overline{b}_{i_2}\overline{b}_{j_2}|\frac{1-(-1)^{i_2-j_2}\cos 2(i_2-j_2)\theta}{1+\cos 2\theta}\\
&\quad\left.+2\sum_{\substack{1\le j_2\\ 1\le j_1\le i_1-1\\ 2j_2+i_1+j_1=s}} \overline{b}_{j_2}^2|\overline{c}_{i_2}\overline{c}_{i_2}|\frac{1-(-1)^{i_1-j_1}\cos 2(i_1-j_1)\theta}{1+\cos 2\theta}\right).
\end{aligned} \tag{3.4}$$

When we put $\theta = \pi/2+\theta_1$, the same trigonometric expressions in θ_1 appear as in the previous case in θ, while all the other bounds are identical. This completes the proof. ∎

We observe that in all the kernels K_n from [7, Conjecture 2.1], i.e., for each $n \in \mathbb{N}$, we have the case $\overline{c}_2 < \overline{b}_2$ in Theorem 3.1.

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REFERENCES

[1] D. Lj. Đukić, A. V. Pejčev, and M. M. Spalević, *The error bounds of Gauss-Kronrod quadrature formulae for weight functions of Bernstein-Szegő type*, Numer. Algorithms, 77 (2018), pp. 1003–1028.

[2] D. Lj. Đukić, R. M. Mutavdžić Đukić, A. V. Pejčev, and M. M. Spalević , *Error estimates of Gaussian-type quadrature formulae for analytic functions on ellipses-a survey of recent results*, Electron. Trans. Numer. Anal., 53 (2020), pp. 352–382. https://etna.ricam.oeaw.ac.at/vol.53.2020/pp352-382.dir/pp352-382.pdf

[3] C. F. Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Commentationes Societatis Regiae Scientiarum Göttingensis Recentiores, 3 (1814). Also in Werke III, pp. 163–196.

[4] W. Gautschi, *Orthogonal polynomials: computation and approximation*, Oxford University Press, 2004.

[5] W. Gautschi, *OPQ: a Matlab suite of programs for generating orthogonal polynomials and related quadrature rules*. Available at https://www.cs.purdue.edu/archives/2002/wxg/codes/OPQ.html.

[6] W. Gautschi and R. S. Varga, *Error bounds for Gaussian quadrature of analytic functions,* SIAM J. Numer. Anal., 20 (1983), pp. 1170–1186.

[7] D. R. Jandrlić, Dj. M. Krtinić, Lj. V. Mihić, A. V. Pejčev, M. M. Spalević, *Error bounds of Gaussian quadrature formulae with Legendre weight function for analytic integrands*, Electron. Trans. Anal., 55 (2022), pp. 424–437. https://etna.ricam.oeaw.ac.at/vol.55.2022/pp424-437.dir/pp424-437.pdf

[8] S. E. Notaris, *Error bounds for Gauss-Kronrod quadrature of analytic functions*, Numer. Math., 64 (1993), pp. 371–380.

[9] S. E. Notaris, *Integral formulas for Chebyshev polynomials and the error term of interpolatory quadrature formulae for analytic functions*, Math. Comp., 75 (2006), pp. 1217–1231.

[10] I. P. Mysovskih, *Lectures of Numerical Methods*, Wolters-Noordhoff Publishing, Groningen, 1969.

[11] M. M. Spalević, M. S. Pranić, and A. V. Pejčev, *Maximum of the modulus of kernels of Gaussian quadrature formulae for one class of Bernstein-Szegő weight functions*, Appl. Math. Comput., 218 (2012), pp. 5746–5756.

[12] H. Wang and L. Zhang, *Jacobi polynomials on the Bernstein ellipse*, J. Sci. Comput., 75 (2018), pp. 457–477.