Electronic Transactions on Numerical Analysis.
Volume 58, pp. 450–469, 2023.

ISSN 1068–9613.
DOI: 10.1553/etna_vol58s450

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ITERATIVE LAVRENTIEV REGULARIZATION METHOD UNDER A HEURISTIC RULE FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS*

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Abstract. In this paper, we consider the iterative Lavrentiev regularization method for obtaining a stable approximate solution for a nonlinear ill-posed operator equation $F(x) = y$, where $F : D(F) \subset X \to X$ is a nonlinear monotone operator on the Hilbert spaces X. In order to obtain a stable approximate solution using iterative regularization methods, it is important to use a suitable stopping rule to terminate the iterations at the appropriate step. Recently, Qinian Jin and Wei Wang (2018) have proposed a heuristic rule to stop the iterations for the iteratively regularized Gauss-Newton method. The advantage of a heuristic rule over the existing a priori and a posteriori rules is that it does not require accurate information on the noise level, which may not be available or reliable in practical applications. In this paper, we propose a heuristic stopping rule for an iterated Lavrentiev regularization method. We derive error estimates under suitable nonlinearity conditions on the operator F.

Key words. Lavrentiev regularization, nonlinear ill-posed problems, heuristic parameter choice rules

AMS subject classifications. 65J20, 47J06

1. Introduction.

In this paper, we are interested in finding a stable approximate solution for an ill-posed operator equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset X \to X$ is a nonlinear monotone operator on a Hilbert space X. We recall that the operator F is said to be monotone if it satisfies

$$\langle F(x) - F(\tilde{x}), x - \tilde{x}\rangle \geq 0 \qquad \text{for all } x, \tilde{x} \in D(F).$$

We shall denote the inner product and the corresponding norm on the Hilbert space X by $\langle ., .\rangle$ and $\|.\|$, respectively. Throughout the paper we assume that (1.1) has a unique solution, namely $x^\dagger$, and that in place of the exact data y we only have perturbed data y^δ available satisfying

$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$

As the operator equation considered in (1.1) is ill-posed, due to the instability of the problem, one has to use regularization methods to find a stable approximate solution. Tikhonov regularization is one of the classical continuous regularization methods used in literature for calculating an approximate solution for nonlinear ill-posed problems (see [4, 5, 6, 15, 16, 28]). For the case of monotone operators F, one can use a simpler method, namely Lavrentiev regularization [31, 32]. In Lavrentiev regularization, the approximate solution is obtained by solving the equation

$$F(x) - y^\delta + \alpha(x - x_0) = 0, \tag{1.3}$$

where $\alpha > 0$ is a suitably chosen regularization parameter and x_0 is an initial guess for the exact solution $x^\dagger$. If the operator F is Frèchet differentiable in an appropriate neighbourhood of $x^\dagger$, then the equation (1.3) can be expressed as

$$x = x_0 + (F'(x) + \alpha I)^{-1}[y^\delta - F(x) + F'(x)(x - x_0)],$$

*Received June 7, 2022. Accepted March 18, 2023. Published online on July 3, 2023. Recommended by Sergei Pereverzyev.

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where $F'(x)$ denotes the Frèchet derivative of F at x. In the literature on ill-posed operator equations, Lavrentiev regularization have been studied under various discrepancy principles as regularization parameter choice rules (see [11, 22, 23, 25, 29]). Another class of regularization methods, namely iterative regularization methods, have gained a lot of popularity due to their straightforward implementation. In [3], Bakushinsky and Smirnova used an iterative form of Lavrentiev regularization in which the iterations are defined as

$$x_{k+1}^\delta = x_k^\delta - (A_k^\delta + \alpha_k I)^{-1}[F(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)], \tag{1.4}$$

where $A_k^\delta := F'(x_k^\delta)$ and (α_k) is a sequence of regularization parameters satisfying

$$\alpha_k > 0, \qquad 1 \le \frac{\alpha_k}{\alpha_{k+1}} \le \mu, \qquad \text{and} \qquad \lim_{k\to\infty} \alpha_k = 0, \tag{1.5}$$

for some $\mu > 1$. A generalized discrepancy principle (see [1, 2]) has been used by them to get a stopping index k_δ satisfying

$$\|F(x_{k_\delta}^\delta) - y^\delta\|^2 \le \tau\delta \le \|F(x_k^\delta) - y^\delta\|^2 \qquad \text{for } \ 0 \le k < k_\delta, \tag{1.6}$$

for some suitable $\tau > 1$. They have shown convergence of the approximate solution $x_{k_\delta}^\delta$ to the exact solution $x^\dagger$ as $\delta \to 0$. In [22], Mahale and Nair have considered the method defined in (1.4) under the following stopping rule: Select k_δ as the first integer satisfying

$$\|\alpha_{k_\delta}(F'(x_{k_\delta}^\delta) + \alpha_{k_\delta} I)^{-1/2}(F(x_{k_\delta}^\delta) - y^\delta)\| \le \tau\delta \tag{1.7}$$

for some $\tau \ge 1$. They have shown convergence of the method and also obtained order-optimal error estimates under suitable nonlinearity conditions for the operator F. We observe that the stopping rule defined in (1.6), (1.7) and many frequently used stopping rules in the literature on ill-posed operator equations require accurate information about the noise $\delta = \|y - y^\delta\|$ to get a stable approximate solution. In practical applications, due to experimental errors or due to some physical constraints, such noise level information may not be accurate or reliable. Such incorrect or incomplete information may result in a bad approximation of the solution. To overcome this issue, heuristic rules have been used in the literature by various authors (cf. [5, 9, 10, 14, 33, 34]). Recently Jin and Wang in [18] have considered a heuristic selection rule for the iteratively regularized Gauss-Newton method. They have presented a convergence analysis of the method and have also derived error bounds under various forms of source conditions. They have formulated the heuristic rule by first defining the integer $k_\infty := k_\infty(y^\delta)$ that satisfies $k_\infty := \max\{k : x_l^\delta \in D(F) \text{ for all } 0 \le l \le k\}$, where $D(F)$ denotes the domain of F, and then the rule is defined as follows:

Let

$$\theta(k, y^\delta) = \frac{\|F(x_k^\delta) - y^\delta\|^2}{\alpha_k},$$

and let $k_* = k_*(y^\delta)$ be the integer satisfying

$$k_* \in \arg\min\{\theta(k, y^\delta) : k = 0, 1, \dots, k_\infty\}.$$

Then use $x_{k_*}^\delta$ as approximation of the exact solution $x^\dagger$. In this paper, we formulate a heuristic selection rule for the iterated Lavrentiev regularization method defined in (1.4). We will obtain error bounds for the proposed method under Hölder-type source conditions using suitable nonlinearity conditions for the operator F.

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2. Formulation of the heuristic rule. Before formulating our heuristic rule for monotone ill-posed operator equations, we introduce an integer $k_\infty := k_\infty(y^\delta)$ satisfying

$$k_\infty := \max\{k : x_l^\delta \in D(F) \text{ for all } 0 \le l \le k\}.$$

From the above definition of k_∞, it is clear that it is the largest positive integer for which the iterates x_k^δ belong to the domain of F for each k satisfying $0 \le k \le k_\infty$. Note that the integer k_∞ can take the value ∞. This choice of $k_\infty := k_\infty(y^\delta)$ ensures that the iterates (x_k^δ) defined by (1.4) belong to $D(F)$. Now we define our heuristic rule as follows:

Let

$$\theta(k, y^\delta) = \frac{\|F(x_k^\delta) - y^\delta\|^2}{\alpha_k^2}.$$

We define $k_* = k_*(y^\delta)$ to be the integer satisfying

$$k_* \in \arg\min\{\theta(k, y^\delta) : k = 0, 1, \ldots, k_\infty\} \tag{2.1}$$

and use $x_{k_*}^\delta$ as approximation of the exact solution $x^\dagger$.

In order to show the well-definedness of the heuristic rule and to obtain some crucial results for the convergence analysis, we will assume the following condition for the noise $\|y - y^\delta\|$.

The motivation for this condition comes from the work of Hanke and Raus [9] for linear ill-posed operator equations in Hilbert spaces of the form (1.1), where heuristic rules were developed for the case when F is a bounded linear operator between the Hilbert spaces X and Y. In their work, they compared the values of the exact error $\|x_\alpha^\delta - x^\dagger\|^2$, where x_α^δ denotes the regularized solution obtained using the Tikhonov regularization method, and a multiple of the residual $\frac{\|F(x_\alpha^\delta) - y^\delta\|^2}{\alpha}$ with respect to the regularization parameter α that has been suitably chosen so that it minimizes $\|x_\alpha^\delta - x^\dagger\|^2$. Although in reality the exact solution $x^\dagger$ is not known and hence the exact error is not computable, by using a posteriori error bounds for Tikhonov regularization it can be derived that $\|x_\alpha^\delta - x^\dagger\|^2 \approx \frac{1}{\sqrt{\alpha}}$. Thus, the term on the right-hand side can serve as a surrogate for the actual error. In [9], the regularization parameter has been chosen as $\alpha_* = \mathrm{argmin}_{\alpha > 0} \frac{\|F(x_\alpha^\delta) - y^\delta\|^2}{\sqrt{\alpha}}$, and x_{α_*} has been used as an approximate solution. Moreover, using a source condition of the form $x^\dagger - x_0 \in R((F^*F)^\nu)$ for some $\nu > 0$, it has been shown that the rule admits error estimates of the form

$$\|x_{\alpha_*}^\delta - x^\dagger\| \le C\left(1 + \frac{\delta}{\delta^*}\right) \max\{\delta^*, \delta\}^{\frac{2\nu}{(2\nu+1)}}, \tag{2.2}$$

where C is a suitable constant, $\delta^* := \|F(x_{\alpha_*}^\delta) - y^\delta\|$, and $\delta = \|y^\delta - y\|$. We note that the error estimate in (2.2) involves two quantities, namely δ^* and δ. The value of δ^* gives the following test for this parameter choice rule: if $\delta^* \le \delta$ (or what is believed to be the noise level δ), then one should be careful about the chosen parameter $\alpha(y^\delta, \delta)$ since the factor $\frac{\delta}{\delta^*}$ is large, causing (2.2) to blow up. On the contrary, if $\delta^* \ge \delta$, then this situation is not critical as the magnitude of δ^* determines the noise level. Therefore, the value of δ^* should always be monitored, and the computed approximation should be discarded if δ^* is significantly smaller than the expected noise level. Apparently, with additional conditions, the factor $\frac{\delta}{\delta^*}$ can be removed, and we get optimal-order resembling convergence rates of the form

$$\|x_{\alpha_*}^\delta - x^\dagger\| \le C \max\{\delta^*, \delta\}^{\frac{2\nu}{(2\nu+1)}}.$$

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In this paper, we will use the following assumption for the noise $\|y-y^\delta\|$:

ASSUMPTION 1. There exists a constant κ, where $0<\kappa<1$, satisfying

$$\|y^\delta-\tilde{y}\| \geq \kappa\|y^\delta-y\|, \tag{2.3}$$

for any $\tilde{y}\in\{F(x): x\in S(y^\delta)\}$, where $S(y^\delta)$ is the set $\{x_k^\delta : 0\leq k\leq k_\infty\}$ and where the sequence x_k^δ is defined by the method (1.4) using the data y^δ.

The above assumption has been used in [7, 18, 27, 35] for the error analysis of the iterative regularized Gauss-Newton method in both Banach spaces as well as in Hilbert space settings. In these references, the above condition has been interpreted as follows: For the ill-posed operator equation, due to the smoothing effect of the operator F, $F(x)$ has a certain regularity. On the other hand, the noisy data y^δ, in general, are corrupted by random noise and hence contain many high frequency components so that it may exhibit salient irregularity. The condition (2.3) roughly means that subtracting any regular function of the form $F(x)$ with $x\in S\left(y^\delta\right)$ from the noisy data y^δ can not significantly remove the randomness of the noise. Authors do agree with the fact that, in general, the monotonicity and continuity of the residual function $\left\|F\left(x_k^\delta\right)-y^\delta\right\|$ may not be guaranteed (see [8]). But, during numerical computations, it is possible to verify Assumption 1 by keeping track of the quantity $\left\|F\left(x_k^\delta\right)-y^\delta\right\|$ with respect to an increase in k. So, it can be concluded that if $\left\|F\left(x_k^\delta\right)-y^\delta\right\|$ does not fall below a very small number, then one can be confident that Assumption 1 holds. Also, Assumption 1 is a slight modification of the following assumption that is used in [14] for variational regularization: there exists a constant $\tilde{\kappa}$, with $0<\tilde{\kappa}<1$, satisfying

$$\|y^\delta-y-v\| \geq \tilde{\kappa}\|y^\delta-y\|, \tag{2.4}$$

for any $v\in\{F(x)-y : x\in D(F)\}$. In [12], for the case when Y is a Hilbert space and F is a bounded linear operator, the following condition

$$\|Q(y^\delta-y)\| \geq \sigma\|y^\delta-y\|,$$

with $\sigma>0$, has been used to quantify the randomness of the noise, where Q denotes the orthogonal projection onto the orthogonal complement of the range of F. A weaker form of this condition has been employed in [9]: there exists $0<\sigma<1$ such that

$$\langle y^\delta-y, v\rangle \leq (1-\sigma)\|y^\delta-y\|\|v\| \tag{2.5}$$

for all $v\in\{F(x)-y : x\in D(F)\}$. We note that (2.5) implies (2.4). In fact, by the Cauchy-Schwarz inequality, we observe that

$$\begin{aligned}
\|y^\delta-y-v\|^2 &= \|v\|^2+\|y^\delta-y\|^2-2\langle y^\delta-y, v\rangle\\
&\geq \|v\|^2+\|y^\delta-y\|^2-2(1-\sigma)\|y^\delta-y\|\|v\|\\
&\geq \|v\|^2+\|y^\delta-y\|^2-(1-\sigma)\left(\|y^\delta-y\|^2+\|v\|^2\right)\\
&\geq \sigma\|y^\delta-y\|^2,
\end{aligned}$$

with $\tilde{\kappa}=\sigma^{1/2}$.

Here, we also cite the thesis of Real (2021) [27], in which convergence results for variational regularization and Landweber iteration under heuristic rules have been derived. Throughout the thesis, condition (2.3) has been assumed in the convergence analysis. It has also been mentioned that there is no way to verify such conditions in reality unless the exact data y are accessible. Despite this, heuristic parameter choice rules are useful as long as there

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is a good understanding of the noise in a given problem. To be added is that random noise is a desirable instance when using heuristic parameter choice rules. The least desirable instance happens with a mildly ill-posed forward operator and smooth data noise. This is because the noise condition tells us that the noise should be sufficiently "nonsmooth" (see [20, 21]). We may say that Assumption 1 means that there does not exist overfitting for the noisy data. During numerical computations, the quantity $\left\|F\left(x_k^\delta\right)-y^\delta\right\|$ should be monitored with respect to an increase in k to ensure that Assumption 1 holds [18, 27, 35]. However, the theoretical verification of this assumption still remains an open question.

The following result deals with the well-definedness of the heuristic rule considered in (2.1).

LEMMA 2.1. *Suppose that Assumption 1 holds with $y^\delta \neq y$. Then there exist a finite integer k_* satisfying* (2.1).

Proof. If k_∞ is finite, then the result holds. Let us assume that $k_\infty = \infty$. Using Assumption 1 and the condition on α_k given in (1.5), we get

$$\theta(k, y^\delta) = \frac{\|F(x_k^\delta) - y^\delta\|^2}{\alpha_k^2} \geq \frac{\kappa^2\|y - y^\delta\|^2}{\alpha_k^2} \to \infty$$

as $k \to \infty$. This ensures the existence of a finite integer k_*, where the minimum of $\theta(k, y^\delta)$ occurs. □

3. Assumptions and well-definedness of the method. In this section, we will show the well-definedness of the method. For this, we assume the following nonlinearity condition for the operator F:

ASSUMPTION 2.

(i) *The nonlinear operator F is Frèchet differentiable in a ball around the exact solution $B_r(x^\dagger) := \{x \in X : \|x - x^\dagger\| < r\} \subset D(F)$ for some number $r > 0$.*

(ii) *The nonlinear operator F is properly scaled, i.e., there exist $\alpha_0 > 0$ such that $\|F'(x^\dagger)\| \leq \alpha_0$.*

(iii) *There exists constants $K_0 > 0$ and $K_1 > 0$ such that*

$$\|(F'(x) - F'(\tilde{x}))w\| \leq K_0\|x - \tilde{x}\|\|F'(\tilde{x})w\| + K_1\|F'(\tilde{x})(x - \tilde{x})\|\|w\| \tag{3.1}$$

for all $x, \tilde{x} \in B_r(x^\dagger)$ and $w \in X$.

The nonlinearity condition given in Assumption 2(iii) has been used by various authors in the literature (see [13, 17, 18, 19]). Using the fundamental theorem of integral calculus, it can be shown that the following two inequalities are consequences of the nonlinearity condition given in (3.1):

$$\|F(x) - F(\tilde{x}) - F'(\tilde{x})(x - \tilde{x})\| \leq \frac{1}{2}(K_0 + K_1)\|x - \tilde{x}\|\|F'(\tilde{x})(x - \tilde{x})\| \tag{3.2}$$

and

$$\|F(x) - F(\tilde{x}) - F'(\tilde{x})(x - \tilde{x})\| \leq \frac{3}{2}(K_0 + K_1)\|x - \tilde{x}\|\|F'(x)(x - \tilde{x})\| \tag{3.3}$$

for all $x, \tilde{x} \in B_r(x^\dagger)$. Now we consider a source condition that is needed for obtaining error estimates for the method.

ASSUMPTION 3. There exists $v \in X$ and $p > 0$ such that

$$x_0 - x^\dagger = (F'(x^\dagger))^p v.$$

In the above source condition, the fractional power $(F'(x^\dagger))^p$ is defined by (see [30])

$$(F'(x^\dagger))^p v := \frac{\sin p\pi}{\pi} \int_0^\infty s^{p-1}(F'(x^\dagger) + sI)^{-1} F'(x^\dagger) v \, \mathrm{d}s, \qquad v \in X.$$

In case when the operator $F'(x^\dagger)$ is a self-adjoint, the power $(F'(x^\dagger))^p$ can be defined using spectral theory (see [24]). For getting convergence rate for the method, the next proposition, which is a generalization of the well-known interpolation inequality for self adjoint operators, will be used, and its proof can be found in [26]:

PROPOSITION 3.1. *Let $B : X \to X$ be linear monotone operator. Then, for $s \geq 0$,*

$$\|B^s x\| \leq \tilde{c} \|B^{s+1} x\|^{s/s+1} \|x\|^{1/s+1},$$

where $\tilde{c} = s^{-s/(s+1)} + s^{1/s+1} \leq 2$.

Proof. See [26] and also [31, Proposition 2.1]. □

For the simplicity of expression, we use the following notation throughout the paper:

$$r_\alpha(\lambda) = \frac{\alpha}{(\alpha + \lambda)}, \quad e_k^\delta = x_k^\delta - x^\dagger, \quad e_0 = x_0 - x^\dagger, \quad A = F'(x^\dagger), \quad \text{and} \quad A_k^\delta = F'(x_k^\delta).$$

For the convergence analysis, we choose $\hat{k}$ to be the first integer satisfying

$$\alpha_{\hat{k}} \leq \frac{c_0 \delta}{\|e_0\|} < \alpha_k, \tag{3.4}$$

for $0 \leq k < \hat{k}$, where $c_0 > 1$ is an appropriate constant and $\alpha_0 > \frac{c_0 \delta}{\|e_0\|}$. The conditions on α_k given in (1.5) ensure the existence of $\hat{k}$ satisfying (3.4).

In the following lemma, we show the well-definedness of the iterations defined in (1.4) when they are terminated according to the stopping rule (3.4).

LEMMA 3.2. *Let Assumption 2 be satisfied, and let $\{\alpha_k\}$ be chosen according to (3.4). If*

$$\max\{\frac{1}{c_0} + 3\gamma(K_0 + K_1)\|e_0\|, 5(K_0 + K_1)\|e_0\|\mu\} < 1,$$

then

$$\|e_k^\delta\| \leq 2\|e_0\| \qquad \text{and} \qquad \|A e_k^\delta\| \leq \gamma \alpha_k \|e_0\|, \tag{3.5}$$

for $0 \leq k \leq \hat{k}$, where $\gamma = \frac{(1+\frac{1}{c_0})\mu}{1 - 5(K_0+K_1)\|e_0\|\mu}$. Moreover, $x_k^\delta \in B_r(x^\dagger)$ for $r > 2\|e_0\|$ and for $0 \leq k \leq \hat{k}$.

Proof. We prove (3.5) using induction. From the scaling condition, it is trivial for $k = 0$. Now, we assume the result holds for some $0 < k < \hat{k}$, and we show that it holds for $k + 1$. From the iteration defined in (1.4), we have

$$x_{k+1}^\delta = x_k^\delta - (\alpha_k I + A_k^\delta)^{-1}(\alpha_k(x_k^\delta - x_0) + F(x_k^\delta) - y^\delta), \tag{3.6}$$

which also implies

$$\begin{aligned}
x_{k+1}^\delta - x^\dagger &= (x_k^\delta - x^\dagger) - (\alpha_k I + A_k^\delta)^{-1}(\alpha_k(x_k^\delta - x_0) + F(x_k^\delta) - y^\delta) \\
&= (\alpha_k I + A_k^\delta)^{-1}\{(\alpha_k I + A_k^\delta)(x_k^\delta - x^\dagger) - (F(x_k^\delta) - y^\delta - \alpha_k(x_k^\delta - x_0)\} \\
&= (\alpha_k I + A_k^\delta)^{-1}\{\alpha_k(x_0 - x^\dagger) - (F(x_k^\delta) - y^\delta - A_k^\delta(x_k^\delta - x^\dagger))\} \\
&= r_{\alpha_k}(A_k^\delta)(x_0 - x^\dagger) - (\alpha_k I + A_k^\delta)^{-1}\{F(x_k^\delta) - y^\delta - A_k^\delta(x_k^\delta - x^\dagger)\}.
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
x_{k+1}^\delta - x^\dagger &= r_{\alpha_k}(A_k^\delta)(x_0 - x^\dagger) \\
&\quad - (\alpha_k I + A_k^\delta)^{-1}\{F(x_k^\delta) - y - A_k^\delta(x_k^\delta - x^\dagger)\} + (\alpha_k I + A_k^\delta)^{-1}(y - y^\delta).
\end{aligned}$$

From the above equation, we get

$$\begin{aligned}
\|e_{k+1}^\delta\| \le \|r_{\alpha_k}(A_k^\delta)e_0\| &+ \|(\alpha_k I + A_k^\delta)^{-1}(y - y^\delta)\| \\
&+ \|(\alpha_k I + A_k^\delta)^{-1}\{F(x_k^\delta) - y - A_k^\delta(x_k^\delta - x^\dagger)\}\|.
\end{aligned}$$

Using Assumption 2 and (3.4), we have

$$\begin{aligned}
\|e_{k+1}^\delta\| &\le \|e_0\| + \frac{\delta}{\alpha_k} + \frac{3}{2}\frac{K_0 + K_1}{\alpha_k}\|e_k^\delta\|\|Ae_k^\delta\| \\
&\le \left(1 + \frac{1}{c_0}\right)\|e_0\| + \frac{3}{2}\frac{K_0 + K_1}{\alpha_k}\|e_k^\delta\|\|Ae_k^\delta\|.
\end{aligned} \tag{3.7}$$

Using (3.5) and the induction hypothesis, we obtain

$$\begin{aligned}
\|e_{k+1}^\delta\| &\le \left(1 + \frac{1}{c_0}\right)\|e_0\| + 3\gamma(K_0 + K_1)\|e_0\|^2 \\
&\le \left(1 + \frac{1}{c_0} + 3\gamma(K_0 + K_1)\|e_0\|\right)\|e_0\|.
\end{aligned}$$

By the hypothesis of the lemma, we get

$$\|e_{k+1}^\delta\| \le 2\|e_0\|. \tag{3.8}$$

Thus, we have $\|e_k^\delta\| \le 2\|e_0\|$ for all $0 \le k \le \hat{k}$, and if we choose $r > 2\|e_0\|$, then it follows that $x_k^\delta \in B_r(x^\dagger)$. Again from (3.6), it can be seen that

$$A_k^\delta e_{k+1}^\delta = A_k^\delta r_{\alpha_k}(A_k^\delta)e_0 - A_k^\delta(\alpha_k I + A_k^\delta)^{-1}\{F(x_k^\delta) - y^\delta - A_k^\delta(x_k^\delta - x^\dagger)\}.$$

From the above equation, we get

$$\|A_k^\delta e_{k+1}^\delta\| \le \alpha_k\|e_0\| + \|F(x_k^\delta) - y^\delta - A_k^\delta(x_k^\delta - x^\dagger)\|.$$

Using (1.2) and (3.3), we can write

$$\|A_k^\delta e_{k+1}^\delta\| \le \alpha_k\|e_0\| + \frac{3}{2}(K_0 + K_1)\|e_k^\delta\|\|Ae_k^\delta\| + \delta. \tag{3.9}$$

Using Assumption 2, we have

$$\|(A - A_k^\delta)e_{k+1}^\delta\| \le K_0\|e_k^\delta\|\|Ae_{k+1}^\delta\| + K_1\|e_{k+1}^\delta\|\|Ae_k^\delta\|. \tag{3.10}$$

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Note that

$$\|Ae_{k+1}^{\delta}\| \leq \|A_k^{\delta}e_{k+1}^{\delta}\| + \|(A - A_k^{\delta})e_{k+1}^{\delta}\|. \tag{3.11}$$

Now using (3.9) and (3.10) in (3.11), we get

$$\begin{aligned}\|Ae_{k+1}^{\delta}\| &\leq \alpha_k\|e_0\| + \frac{3}{2}(K_0 + K_1)\|e_k^{\delta}\|\|Ae_k^{\delta}\| \\ &\qquad + \delta + K_0\|e_k^{\delta}\|\|Ae_{k+1}^{\delta}\| + K_1\|e_{k+1}^{\delta}\|\|Ae_k^{\delta}\| \\ &\leq \alpha_k\|e_0\| + \left\{\frac{3}{2}(K_0 + K_1)\|e_k^{\delta}\| + K_1\|e_{k+1}^{\delta}\|\right\}\|Ae_k^{\delta}\| \\ &\qquad + \delta + 2K_0\mu\|e_k^{\delta}\|\|Ae_{k+1}^{\delta}\|.\end{aligned} \tag{3.12}$$

This together with (3.5) implies

$$(1 - 2\mu K_0\|e_0\|)\|Ae_{k+1}^{\delta}\| \leq \left(1 + \frac{1}{c_0}\right)\alpha_k\|e_0\| + (3K_0 + 5K_1)\|e_0\|\|Ae_k^{\delta}\|.$$

Thus, we get

$$\begin{aligned}\|Ae_{k+1}^{\delta}\| &\leq \frac{1}{1 - 2\mu K_0\|e_0\|}\left\{1 + \frac{1}{c_0} + (3K_0 + 5K_1)\gamma\|e_0\|\right\}\alpha_k\|e_0\| \\ &\leq \left\{\frac{1 + \frac{1}{c_0}}{1 - 5(K_0 + K_1)\|e_0\|\mu}\right\}\alpha_k\|e_0\| \leq \gamma\alpha_{k+1}\|e_0\|.\end{aligned}$$

Hence,

$$\|Ae_k^{\delta}\| \leq \gamma\alpha_k\|e_0\|,$$

for all $0 \leq k \leq \hat{k}$. ∎

LEMMA 3.3. *Let the conditions of Lemma 3.2 be satisfied, and let the integer $\hat{k}$ be chosen as in (3.4). Then $\|F(x_{\hat{k}}^{\delta}) - y^{\delta}\| \leq \mathcal{C}\delta$, where $\mathcal{C} = 1 + 2\gamma c_0$ with γ as in Lemma 3.2.*

Proof. From Lemma 3.2 and Assumption 2, we have

$$\begin{aligned}\|F(x_{\hat{k}}^{\delta}) - y^{\delta}\| &\leq \|y^{\delta} - y\| + \|Ae_{\hat{k}}^{\delta}\| + \|F(x_{\hat{k}}^{\delta}) - y - Ae_{\hat{k}}^{\delta}\| \\ &\leq \delta + \gamma\alpha_{\hat{k}}\|e_0\| + \frac{K_0 + K_1}{2}\|e_{\hat{k}}^{\delta}\|\|Ae_{\hat{k}}^{\delta}\| \\ &\leq \delta + \gamma\alpha_{\hat{k}}\|e_0\| + \gamma(K_0 + K_1)\alpha_{\hat{k}}\|e_0\|^2 \\ &\leq \delta + \gamma(1 + (K_0 + K_1)\|e_0\|)\|e_0\|\alpha_{\hat{k}}.\end{aligned}$$

From the condition in Lemma 3.2, we have $(K_0 + K_1)\|e_0\| < 1$. Thus,

$$\|F(x_{\hat{k}}^{\delta}) - y^{\delta}\| \leq \delta + 2\gamma\|e_0\|\alpha_{\hat{k}}. \tag{3.13}$$

Using (3.4) in (3.13), we get

$$\|F(x_{\hat{k}}^{\delta}) - y^{\delta}\| \leq \mathcal{C}\delta,$$

where $\mathcal{C} = 1 + 2\gamma c_0$. ∎

In the following result, we prove the well-definedness of the heuristic rule given in (2.1) and also obtain a lower bound for α_{k_*}.

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LEMMA 3.4. *Let the conditions of Lemma 3.2 and Assumption 1 be satisfied. Then the integer k_* in the heuristic rule (2.1) is finite, and for this integer k_*, it holds that $\alpha_{k_*} \geq \frac{c_2\delta}{\|e_0\|}$ with $c_2 := \frac{\kappa c_0}{\mu \mathcal{C}\|e_0\|}$, where $\mathcal{C}$ is as in Lemma 3.3.*

Proof. From Lemma 3.3 and the definition of k_* and $\hat{k}$, it follows that

$$\theta(k_*, y^\delta) \leq \theta(\hat{k}, y^\delta) = \frac{\|F(x_{\hat{k}}^\delta) - y^\delta\|^2}{\alpha_{\hat{k}}^2} \leq \frac{\mathcal{C}^2\delta^2\mu^2}{\alpha_{\hat{k}-1}^2} \leq \frac{\mathcal{C}^2\mu^2\|e_0\|^2}{c_0^2}.$$

Hence by Assumption 1 we have

$$\frac{\kappa^2\delta^2}{\alpha_{k_*}^2} \leq \theta(k_*, y^\delta) \leq \frac{\mathcal{C}^2\mu^2\|e_0\|^2}{c_0^2},$$

which implies $\alpha_{k_*} \geq \dfrac{\kappa c_0 \delta}{\mu\mathcal{C}\|e_0\|}$. Thus, we get the desired estimate with $c_2 = \frac{\kappa c_0}{\mu\mathcal{C}\|e_0\|}$. ∎

For showing the well-definedness of the iterates given in (1.4) for $0 \leq k \leq k_*$, we introduce a new integer k_*^δ using the constant c_2 involved in Lemma 3.4 in the following way: Choose k_*^δ to be the first positive integer satisfying

$$\alpha_{k_*^\delta} \leq \frac{c_2\delta}{\|e_0\|} < \alpha_k, \qquad 0 \leq k < k_*^\delta. \tag{3.14}$$

From Lemma 3.4, it is evident that $k_* \leq k_*^\delta$ and $0 < c_2 < 1$. In the next result, we show the well-definedness of the iterates x_k^δ, for all $0 \leq k \leq k_*^\delta$, by obtaining estimates for $\|e_k^\delta\|$ and $\|Ae_k^\delta\|$. The proof of the result is similar to the proof of Lemma 3.2. For the sake of completeness, we provide a brief sketch of it here.

LEMMA 3.5. *Let Assumption 2 and the conditions of Lemma 3.4 be satisfied, and let $\{\alpha_k\}$ be chosen according to (3.14). If*

$$\max\{c_2 + \frac{3(K_0+K_1)\gamma_1}{c_2}\|e_0\|, \frac{5(K_0+K_1)\mu}{c_2}\|e_0\|\} < 1,$$

then

$$\|e_k^\delta\| \leq \frac{2\|e_0\|}{c_2} \qquad \text{and} \qquad \|Ae_k^\delta\| \leq \frac{\gamma_1}{c_2}\alpha_k\|e_0\|, \tag{3.15}$$

for $0 \leq k \leq k_*^\delta$, where $\gamma_1 = \frac{(1+\frac{1}{c_2})\mu}{1-\frac{5(K_0+K_1)\|e_0\|\mu}{c_2}}$. Moreover, $x_k^\delta \in B_r(x^\dagger)$ for $r > \frac{2\|e_0\|}{c_2}$.

Proof. We prove the result by an induction argument. For $k = 0$, the result is obvious. Now let us assume that it holds for some $0 \leq k < k_*^\delta$, and we will prove it for $k+1$. From (3.7) and (3.14), we have

$$\begin{aligned}\|e_{k+1}^\delta\| &\leq \|e_0\| + \frac{\delta}{\alpha_k} + \frac{3}{2}\frac{K_0+K_1}{\alpha_k}\|e_k^\delta\|\|Ae_k^\delta\| \\ &\leq \left(1 + \frac{1}{c_2}\right)\|e_0\| + \frac{3}{2}\frac{K_0+K_1}{\alpha_k}\|e_k^\delta\|\|Ae_k^\delta\|.\end{aligned}$$

Using the induction hypothesis, we get

$$\|e_{k+1}^\delta\| \leq \left(1 + c_2 + \frac{3(K_0+K_1)\gamma_1}{c_2}\|e_0\|\right)\frac{\|e_0\|}{c_2}.$$

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Thus, using the condition of the lemma, we obtain

$$\|e_{k+1}^\delta\| \le \frac{2}{c_2}\|e_0\|.$$

Hence, we have $\|e_k^\delta\| \le \frac{2}{c_2}\|e_0\|$ for all $0 \le k \le k_*^\delta$, and if we choose $r > \frac{2}{c_2}\|e_0\|$, then $x_k^\delta \in B_r(x^\dagger)$. From (3.12), we get

$$\begin{aligned}
\|Ae_{k+1}^\delta\| &\le \alpha_k\|e_0\| + \left\{\frac{3}{2}(K_0+K_1)\|e_k^\delta\| + K_1\|e_{k+1}^\delta\|\right\}\|Ae_k^\delta\| + \delta + K_0\|e_k^\delta\|\|Ae_{k+1}^\delta\| \\
&\le \alpha_k\|e_0\| + \frac{3K_0+5K_1}{c_2}\|e_0\|\|Ae_k^\delta\| + \delta + \frac{2K_0\|e_0\|}{c_2}\|Ae_{k+1}^\delta\| \\
&\le \left(1+\frac{1}{c_2}\right)\alpha_k\|e_0\| + \frac{3K_0+5K_1}{c_2}\|e_0\|\|Ae_k^\delta\| + \frac{2K_0\|e_0\|}{c_2}\|Ae_{k+1}^\delta\|.
\end{aligned}$$

Thus, we find

$$\begin{aligned}
\|Ae_{k+1}^\delta\| &\le \frac{1}{1-\frac{2\mu K_0\|e_0\|}{c_2}}\left\{1+\frac{1}{c_2}+\frac{3K_0+5K_1}{2}\gamma_1\|e_0\|\right\}\alpha_k\|e_0\| \\
&\le \left\{\frac{1+\frac{1}{c_2}}{1-\frac{5(K_0+K_1)\|e_0\|\mu}{c_2}}\right\}\alpha_k\|e_0\| \le \gamma_1\alpha_{k+1}\|e_0\| \le \frac{\gamma_1}{c_2}\alpha_{k+1}\|e_0\|.
\end{aligned}$$

As a consequence,

$$\|Ae_k^\delta\| \le \frac{\gamma_1}{c_2}\alpha_k\|e_0\|, \qquad \text{for all } 0 \le k \le k_*^\delta,$$

where $\gamma_1 = \frac{(1+\frac{1}{c_2})\mu}{1-\frac{5(K_0+K_1)\|e_0\|\mu}{c_2}}$. ∎

4. Some crucial estimates. In this section, we obtain some estimates that will be useful for proving the main result of the paper.

LEMMA 4.1. *Let the conditions of Lemma 3.4 be satisfied. Then,*

$$\|e_{k+1}^\delta - r_{\alpha_k}(A)e_0\| \le \mathcal{C}_2\|e_0\|\left\{\|e_k^\delta\| + \frac{\|Ae_k^\delta\|}{\alpha_k}\right\} + \frac{\delta}{\alpha_k}, \tag{4.1}$$

$$\|Ae_{k+1}^\delta - Ar_{\alpha_k}(A)e_0\| \le \mathcal{C}_3\|e_0\|\left\{\|Ar_{\alpha_k}(A)e_0\| + \|Ae_k^\delta\|\right\} + 2\delta \tag{4.2}$$

for all $0 \le k < k_*^\delta$, *where* $\mathcal{C}_2 = \max\{K_0, \mathcal{C}_1\}$ *with*

$$\mathcal{C}_1 = K_1 + \frac{3}{c_2}(K_0+K_1) \quad \textit{and} \quad \mathcal{C}_3 = \max\{\frac{4}{c_2}K_0, 2K_1(1+\frac{1}{c_2}) + \frac{2K_0}{c_2}\}.$$

Proof. From the iterative method, we write

$$\begin{aligned}
e_{k+1}^\delta = r_{\alpha_k}(A)e_0 &+ (r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0 \\
&- (\alpha_k I + A_k^\delta)^{-1}(F(x_k^\delta) - y^\delta + A_k^\delta(x_k^\delta - x^\dagger)).
\end{aligned} \tag{4.3}$$

Using Assumption 2 and (1.2), we get

$$\begin{aligned}
\|e_{k+1}^\delta - r_{\alpha_k}(A)e_0\| &\le \|(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \\
&\quad + \|(\alpha_k I + A_k^\delta)^{-1}(F(x_k^\delta) - y^\delta + A_k^\delta(x_k^\delta - x^\dagger))\| \\
&\le \|(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \\
&\quad + \frac{1}{\alpha_k}\left(\frac{3}{2}(K_0+K_1)\|e_k^\delta\|\|Ae_k^\delta\| + \delta\right).
\end{aligned} \tag{4.4}$$

Note that

$$(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0 = \alpha_k(\alpha_k I + A_k^\delta)^{-1}(A - A_k^\delta)(\alpha_k I + A)^{-1}e_0.$$

Thus,

$$\begin{aligned}\|(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| &\le \|\alpha_k(\alpha_k I + A_k^\delta)^{-1}(A - A_k^\delta)(\alpha_k I + A)^{-1}e_0\| \\ &\le \|(A - A_k^\delta)(\alpha_k I + A)^{-1}e_0\|.\end{aligned}$$

Again from Assumption 2, we have

$$\begin{aligned}\|(A - A_k^\delta)(\alpha_k I + A)^{-1}e_0\| &\le K_0\|e_k^\delta\|\|A(\alpha_k I + A)^{-1}e_0\| + K_1\|Ae_k^\delta\|\|(\alpha_k I + A)^{-1}e_0\| \\ &\le K_0\|e_k^\delta\|\|e_0\| + \frac{K_1}{\alpha_k}\|e_0\|\|Ae_k^\delta\|.\end{aligned}$$

Therefore,

$$\|(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \le K_0\|e_k^\delta\|\|e_0\| + \frac{K_1}{\alpha_k}\|e_0\|\|Ae_k^\delta\|.$$

From (4.4), we get

$$\begin{aligned}\|e_{k+1}^\delta - r_{\alpha_k}(A)e_0\| \le K_0\|e_k^\delta\|\|e_0\| &+ \frac{K_1}{\alpha_k}\|e_0\|\|Ae_k^\delta\| \\ &+ \frac{1}{\alpha_k}\left(\frac{3}{c_2}(K_0 + K_1)\|e_0\|\|Ae_k^\delta\| + \delta\right) \\ \le \Bigg\{K_0\|e_k^\delta\| &+ \frac{1}{\alpha_k}\left(K_1 + \frac{3}{c_2}(K_0 + K_1)\right)\|Ae_k^\delta\|\Bigg\}\|e_0\| + \frac{\delta}{\alpha_k}.\end{aligned}$$

Denoting $\mathcal{C}_1 = K_1 + \frac{3}{c_2}(K_0 + K_1)$, we obtain

$$\begin{aligned}\|e_{k+1}^\delta - r_{\alpha_k}(A)e_0\| &\le \|e_0\|\left\{K_0\|e_k^\delta\| + \frac{\mathcal{C}_1}{\alpha_k}\|Ae_k^\delta\|\right\} + \frac{\delta}{\alpha_k} \\ &\le \mathcal{C}_2\|e_0\|\left\{\|e_k^\delta\| + \frac{1}{\alpha_k}\|Ae_k^\delta\|\right\} + \frac{\delta}{\alpha_k},\end{aligned}$$

where $\mathcal{C}_2 = \max\{K_0, \mathcal{C}_1\}$. Applying the operator A on both sides of (4.3), we get

$$\begin{aligned}\|Ae_{k+1}^\delta - Ar_{\alpha_k}(A)e_0\| &\le \|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \\ &\quad + \|A(\alpha_k I + A_k^\delta)^{-1}(F(x_k^\delta) - y^\delta + A_k^\delta(x_k^\delta - x^\dagger))\|.\end{aligned} \tag{4.5}$$

Note that for any $v \in X$,

$$\|A(\alpha_k I + A_k^\delta)^{-1}v\| \le \|A_k^\delta(\alpha_k I + A_k^\delta)^{-1}v\| + \|(A - A_k^\delta)(\alpha_k I + A_k^\delta)^{-1}v\|.$$

Using Assumption (2) and (3.15), we obtain

$$\begin{aligned}&\|A(\alpha_k I + A_k^\delta)^{-1}v\| \\ &\quad\le \|A_k^\delta(\alpha_k I + A_k^\delta)^{-1}v\| + \|(A - A_k^\delta)(\alpha_k I + A_k^\delta)^{-1}v\| \\ &\quad\le \|v\| + K_0\|e_k^\delta\|\|A_k^\delta(\alpha_k I + A_k^\delta)^{-1}v\| + K_1\|A_k^\delta e_k^\delta\|\|(\alpha_k I + A_k^\delta)^{-1}v\| \\ &\quad\le \|v\| + K_0\|e_k^\delta\|\|v\| + \frac{K_1}{\alpha_k}\|A_k^\delta e_k^\delta\|\|v\|,\end{aligned} \tag{4.6}$$

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for any $v \in X$. Also, from the condition of Lemma 3.5, we have $\frac{2}{c_2}(K_0 + K_1)\|e_0\| \leq 1$. Hence, we get

$$\begin{aligned} \|A_k^\delta e_k^\delta\| &\leq \|Ae_k^\delta\| + \|(A - A_k^\delta)e_k^\delta\| \leq (1 + (K_0 + K_1)\|e_k^\delta\|)\|Ae_k^\delta\| \\ &\leq \left(1 + \frac{2}{c_2}(K_0 + K_1)\|e_0\|\right)\|Ae_k^\delta\| \\ &\leq 2\|Ae_k^\delta\|. \end{aligned} \tag{4.7}$$

Using (4.7) in (4.6) and the condition of Lemma 3.5, for any $v \in X$, we obtain

$$\begin{aligned} \|A(\alpha_k I + A_k^\delta)^{-1}v\| &\leq \|v\| + K_0\|e_k^\delta\|\|v\| + \frac{2K_1}{\alpha_k}\|Ae_k^\delta\|\|v\| \\ &\leq \|v\| + \frac{2}{c_2}K_0\|e_0\|\|v\| + \frac{2\gamma_1 K_1}{c_2}\|e_0\|\|v\| \\ &\leq \left(1 + \frac{2}{c_2}(K_0 + K_1\gamma_1)\|e_0\|\right)\|v\| \\ &\leq 2\|v\|. \end{aligned} \tag{4.8}$$

Now, including (4.8) in (4.5), we have

$$\begin{aligned} &\|Ae_{k+1}^\delta - Ar_{\alpha_k}(A)e_0\| \\ &\quad\leq \|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| + 2\|F(x_k^\delta) - y^\delta + A_k^\delta(x_k^\delta - x^\dagger)\| \\ &\quad\leq \|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| + (K_0 + K_1)\|e_k^\delta\|\|Ae_k^\delta\| + 2\delta \\ &\quad\leq \|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| + \frac{2(K_0 + K_1)}{c_2}\|e_0\|\|Ae_k^\delta\| + 2\delta. \end{aligned} \tag{4.9}$$

Note that

$$\|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \leq \|A(\alpha_k I + A_k^\delta)^{-1}(A - A_k^\delta)r_{\alpha_k}(A)e_0\|. \tag{4.10}$$

Again making use of (4.8) in (4.10), we have

$$\|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \leq 2\|(A - A_k^\delta)r_{\alpha_k}(A)e_0\|. \tag{4.11}$$

In view of Assumption 2, we have

$$\begin{aligned} \|(A - A_k^\delta)r_{\alpha_k}(A)e_0\| &\leq K_0\|e_k^\delta\|\|Ar_{\alpha_k}(A)e_0\| + K_1\|Ae_k^\delta\|\|r_{\alpha_k}(A)e_0\| \\ &\leq \frac{2}{c_2}K_0\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + K_1\|e_0\|\|Ae_k^\delta\|. \end{aligned} \tag{4.12}$$

Using (4.12) in (4.11), we get

$$\|A(r_{\alpha_k}(A_k^\delta) - r_{\alpha_k}(A))e_0\| \leq \frac{4}{c_2}K_0\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2K_1\|e_0\|\|Ae_k^\delta\|. \tag{4.13}$$

Now from (4.13) and (4.9), we have

$$\begin{aligned} &\|Ae_{k+1}^\delta - Ar_{\alpha_k}(A)e_0\| \\ &\leq \frac{4}{c_2}K_0\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2K_1\|e_0\|\|Ae_k^\delta\| + \frac{2(K_0 + K_1)}{c_2}\|e_0\|\|Ae_k^\delta\| + 2\delta \\ &\leq \left\{\frac{4}{c_2}K_0\|Ar_{\alpha_k}(A)e_0\| + \left(2K_1\left(1 + \frac{1}{c_2}\right) + \frac{2K_0}{c_2}\right)\|Ae_k^\delta\|\right\}\|e_0\| + 2\delta. \end{aligned}$$

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Let $\mathcal{C}_3 = \max\{\frac{4}{c_2}K_0, 2K_1\left(1+\frac{1}{c_2}\right)+\frac{2K_0}{c_2}\}$. Then, we conclude that

$$\|Ae_{k+1}^\delta - Ar_{\alpha_k}(A)e_0\| \leq \mathcal{C}_3\left\{\|Ar_{\alpha_k}(A)e_0\| + \|Ae_k^\delta\|\right\}\|e_0\| + 2\delta. \qquad \square$$

LEMMA 4.2. *Let the assumptions of Lemma 3.5 hold. Let*

$$\max\{\mathcal{C}_3(\mu+1)\|e_0\|, (\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|\} < 1 \quad \textit{and} \quad (K_0+K_1)\|e_0\| < c_2.$$

Then,

$$\|Ae_k^\delta\| \leq \mathcal{C}_4\left\{\|Ar_{\alpha_k}(A)e_0\| + \delta\right\}, \tag{4.14}$$

$$\|Ar_{\alpha_k}(A)e_0\| \leq \mathcal{C}_6\left\{\|F(x_k^\delta) - y^\delta\| + \delta\right\}, \tag{4.15}$$

for $0 \leq k \leq k_*^\delta$, *where*

$$\mathcal{C}_4 = \max\left\{\frac{(\mu+1)(1+\mathcal{C}_3)\|e_0\|}{1-\mathcal{C}_3(1+\mu)\|e_0\|}, \frac{2(\mu+1)}{1-\mathcal{C}_3(1+\mu)\|e_0\|}\right\} \quad \textit{and}$$

$$\mathcal{C}_6 = \left(1+\frac{c_2}{c_2-(K_0+K_1)\|e_0\|}\right)\mathcal{C}_5 \quad \textit{with}$$

$$\mathcal{C}_5 = \max\left\{\frac{1}{1-(\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|}, \frac{2+\mathcal{C}_3\mathcal{C}_4\|e_0\|}{1-(\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|}\right\}$$

and $\mathcal{C}_3$ *is as in Lemma 3.5.*

Proof. From (4.2), it follows that

$$\begin{aligned}\|Ae_{k+1}^\delta\| &\leq (1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2\delta + \mathcal{C}_3\|e_0\|\|Ae_k^\delta\| \\ &\leq (1+\mathcal{C}_3)\|Ar_{\alpha_k}(A)e_0\| + 2\delta + \mathcal{C}_3\|e_0\|\|Ae_k^\delta\|.\end{aligned}$$

Denoting $\sigma_k := (1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2\delta$ and $\eta_k := \frac{\|Ae_k^\delta\|}{(1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_k}(A)e_0\|+2\delta}$, we get

$$\eta_{k+1} \leq \frac{\sigma_k}{\sigma_{k+1}} + \mathcal{C}_3\|e_0\|\frac{\sigma_k}{\sigma_{k+1}}\eta_k.$$

Using (1.5), we can easily show that

$$\|Ar_{\alpha_k}(A)e_0\| \leq \mu\|Ar_{\alpha_{k+1}}(A)e_0\|.$$

Now, note that

$$\frac{\sigma_k}{\sigma_{k+1}} = \frac{(1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2\delta}{(1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_{k+1}}(A)e_0\| + 2\delta} \leq \mu + 1.$$

Thus, we have

$$\eta_{k+1} \leq 1 + \mu + \mathcal{C}_3\|e_0\|(1+\mu)\eta_k.$$

From the scaling condition $\|A\| \leq \alpha_0$, we have

$$\begin{aligned}\|Ae_0\| \leq 2\|Ar_{\alpha_0}(A)e_0\| &\leq \frac{1+\mu}{1-\mathcal{C}_3\|e_0\|(1+\mu)}\|Ar_{\alpha_0}(A)e_0\| \\ &\leq \frac{1+\mu}{1-\mathcal{C}_3\|e_0\|(1+\mu)}\left(\|Ar_{\alpha_0}(A)e_0\| + 2\delta\right).\end{aligned}$$

Hence, we find $\eta_0 \leq \frac{1+\mu}{1-\mathcal{C}_3\|e_0\|(1+\mu)}$. Now, applying the induction hypothesis, we get

$$\eta_k \leq \frac{1+\mu}{1-\mathcal{C}_3\|e_0\|(1+\mu)},$$

for all $0 \leq k \leq k_*^\delta$. Thus,

$$\|Ae_k^\delta\| \leq \frac{1+\mu}{1-\mathcal{C}_3(1+\mu)\|e_0\|}\left\{(1+\mathcal{C}_3)\|e_0\|\|Ar_{\alpha_k}(A)e_0\| + 2\delta\right\}.$$

Alternately, one can write

$$\|Ae_k^\delta\| \leq \mathcal{C}_4\left\{\|Ar_{\alpha_k}(A)e_0\| + \delta\right\},$$

where $\mathcal{C}_4 = \max\left\{\frac{(1+\mu)(1+\mathcal{C}_3)\|e_0\|}{1-\mathcal{C}_3(1+\mu)\|e_0\|}, \frac{2(\mu+1)}{1-\mathcal{C}_3(1+\mu)\|e_0\|}\right\}$. Now again from (4.2), it follows that

$$(1-\mathcal{C}_3\|e_0\|)\|Ar_{\alpha_k}(A)e_0\| \leq \|Ae_{k+1}^\delta\| + \mathcal{C}_3\|e_0\|\|Ae_k^\delta\| + 2\delta.$$

Combining this with (4.14), we get

$$(1-\mathcal{C}_3\|e_0\|)\|Ar_{\alpha_k}(A)e_0\| \leq \|Ae_{k+1}^\delta\| + \mathcal{C}_3\|e_0\|\{\mathcal{C}_4\left(\|Ar_{\alpha_k}(A)e_0\| + \delta\right)\} + 2\delta.$$

Hence,

$$(1-(1+\mathcal{C}_4)\mathcal{C}_3\|e_0\|)\|Ar_{\alpha_k}(A)e_0\| \leq \|Ae_{k+1}^\delta\| + (\mathcal{C}_3\mathcal{C}_4\|e_0\| + 2)\delta.$$

Equivalently,

$$\|Ar_{\alpha_k}(A)e_0\| \leq \frac{1}{1-(\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|}\left\{\|Ae_{k+1}^\delta\| + (\mathcal{C}_3\mathcal{C}_4\|e_0\| + 2)\delta\right\}.$$

Hence, we get

$$\|Ar_{\alpha_k}(A)e_0\| \leq \mathcal{C}_5(\|Ae_{k+1}^\delta\| + \delta), \tag{4.16}$$

where $\mathcal{C}_5 = \max\left\{\frac{1}{1-(\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|}, \frac{2+\mathcal{C}_3\mathcal{C}_4\|e_0\|}{1-(\mathcal{C}_4+1)\mathcal{C}_3\|e_0\|}\right\}$.

Now, using Assumption 2 and Lemma 3.5, we obtain

$$\begin{aligned}\|Ae_{k+1}^\delta\| &\leq \|F(x_{k+1}^\delta) - y^\delta\| + \|F(x_{k+1}^\delta) - y - Ae_{k+1}^\delta\| + \delta \\ &\leq \frac{K_0+K_1}{2}\|e_{k+1}^\delta\|\|Ae_{k+1}^\delta\| + \|F(x_{k+1}^\delta) - y^\delta\| + \delta \\ &\leq \frac{K_0+K_1}{c_2}\|e_0\|\|Ae_{k+1}^\delta\| + \|F(x_{k+1}^\delta) - y^\delta\| + \delta.\end{aligned}$$

Thus,

$$\left(1 - \frac{(K_0+K_1)\|e_0\|}{c_2}\right)\|Ae_{k+1}^\delta\| \leq \|F(x_{k+1}^\delta) - y^\delta\| + \delta.$$

Hence,

$$\|Ae_{k+1}^\delta\| \leq \frac{c_2}{c_2-(K_0+K_1)\|e_0\|}(\|F(x_{k+1}^\delta) - y^\delta\| + \delta). \tag{4.17}$$

Combining (4.16) and (4.17), we get

$$\|Ar_{\alpha_k}(A)e_0\| \leq \mathcal{C}_5 \left\{ \frac{c_2}{c_2 - (K_0 + K_1)\|e_0\|} (\|F(x_{k+1}^\delta) - y^\delta\| + \delta) + \delta \right\}.$$

Denoting $\mathcal{C}_6 = \left(1 + \frac{c_2}{c_2 - (K_0+K_1)\|e_0\|}\right)\mathcal{C}_5$, we obtain

$$\|Ar_{\alpha_k}(A)e_0\| \leq \mathcal{C}_6(\|F(x_{k+1}^\delta) - y^\delta\| + \delta).$$

Using the relation $\|Ar_{\alpha_{k+1}}(A)e_0\| \leq \|Ar_{\alpha_k}(A)e_0\|$, we get

$$\|Ar_{\alpha_{k+1}}(A)e_0\| \leq \mathcal{C}_6 \left\{ \|F(x_{k+1}^\delta) - y^\delta\| + \delta \right\},$$

for $0 < k \leq k_*^\delta$. Using Assumption 2 and the fact that $0 < c_2 < 1$, we find

$$\|Ae_0\| \leq \frac{1}{c_2 - (K_0 + K_1)\|e_0\|}(\|F(x_0) - y^\delta\| + \delta) \leq \mathcal{C}_6(\|F(x_0) - y^\delta\| + \delta).$$

Therefore, by $\|Ar_{\alpha_0}(A)e_0\| \leq \|Ae_0\|$, we note that (4.15) holds for $k = 0$. ∎

In the following result, we obtain an estimate for $\|e_k^\delta\|$ for all $0 \leq k \leq k_*^\delta$.

LEMMA 4.3. *Let Assumption 3 and the assumptions of Lemma 3.5, Lemma 4.1, and Lemma 4.2 be satisfied. Let k_*^δ be the integer chosen according to the stopping rule (3.14), and let $\|e_0\|\mathcal{C}_8(\mu + 1)^2 < 1$. Then,*

$$\|e_k^\delta\| \leq \mathcal{C}_{10} \left\{ \|v\| \left(\frac{\|F(x_k^\delta) - y^\delta\| + \delta}{\|v\|} \right)^{p/p+1} + \frac{\|F(x_k^\delta) - y^\delta\| + \delta}{\alpha_k} \right\},$$

for $0 \leq k \leq k_*^\delta$, where

$$\mathcal{C}_{10} = \max\left\{ \tilde{c}\,\mathcal{C}_9 {\mathcal{C}_6}^{p/p+1}, \mathcal{C}_9(1 + \mathcal{C}_6) \right\} \quad \textit{with} \quad \mathcal{C}_9 = \frac{\mathcal{C}_8(\mu + 1)^2}{1 - \|e_0\|\mathcal{C}_8(\mu + 1)^2} \quad \textit{and}$$

$$\mathcal{C}_8 = \max\{\mathcal{C}_7, \mathcal{C}_2\} \quad \textit{with} \quad C_7 = 1 + \mathcal{C}_4\mathcal{C}_2\|e_0\|.$$

Here, $\mathcal{C}_2$ is as in Lemma 4.1, and $\mathcal{C}_6$ and $\mathcal{C}_4$ are as in Lemma 4.2.

Proof. From (4.1) and (4.14), it follows that

$$\begin{aligned}
\|e_{k+1}^\delta\| &\leq \|r_{\alpha_k}(A)e_0\| + \mathcal{C}_2\|e_0\| \left\{ \|e_k^\delta\| + \frac{\mathcal{C}_4(\|Ar_{\alpha_k}(A)e_0\| + \delta)}{\alpha_k} \right\} + \frac{\delta}{\alpha_k} \\
&\leq \|r_{\alpha_k}(A)e_0\| + \mathcal{C}_2\|e_0\|\|e_k^\delta\| + \frac{\mathcal{C}_2\mathcal{C}_4\|e_0\|\|Ar_{\alpha_k}(A)e_0\|}{\alpha_k} + (1 + \mathcal{C}_4\mathcal{C}_2\|e_0\|)\frac{\delta}{\alpha_k} \\
&\leq \|r_{\alpha_k}(A)e_0\| + \mathcal{C}_7\frac{\|Ar_{\alpha_k}(A)e_0\| + \delta}{\alpha_k} + \mathcal{C}_2\|e_0\|\|e_k^\delta\|,
\end{aligned}$$

where $C_7 = 1 + \mathcal{C}_4\mathcal{C}_2\|e_0\|$. Denoting $\mathcal{C}_8 := \max\{\mathcal{C}_7, \mathcal{C}_2\}$, we get

$$\|e_{k+1}^\delta\| \leq \mathcal{C}_8(\theta_k + \|e_0\|\|e_k^\delta\|), \tag{4.18}$$

where $\theta_k = \|r_{\alpha_k}(A)e_0\| + \frac{\|Ar_{\alpha_k}(A)e_0\| + \delta}{\alpha_k}$. Using (1.5), it is easily seen that the inequality $\theta_k \leq (\mu + 1)^2\theta_{k+1}$ holds. Diving both sides of equation (4.18) by θ_{k+1}, and denoting $\ell_k = \frac{\|e_k^\delta\|}{\theta_k}$, we get

$$\ell_{k+1} \leq \mathcal{C}_8(\mu + 1)^2 + \|e_0\|\mathcal{C}_8(\mu + 1)^2\ell_k.$$

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The scaling condition $\|A\| \le \alpha_0$ implies $\|e_0\| \le 2\|r_{\alpha_0}(A)e_0\| \le 2\theta_0$. Therefore, we get

$$\ell_0 \le 2 \le \frac{\mathcal{C}_8(\mu+1)^2}{1-\|e_0\|\mathcal{C}_8(\mu+1)^2}.$$

Thus, using induction, we can conclude that

$$\ell_k \le \frac{\mathcal{C}_8(\mu+1)^2}{1-\|e_0\|\mathcal{C}_8(\mu+1)^2},$$

for $0 \le k \le k_*^\delta$. Thus, we have

$$\begin{aligned} \|e_k^\delta\| &\le \frac{\mathcal{C}_8(\mu+1)^2}{1-\|e_0\|\mathcal{C}_8(\mu+1)^2}\theta_k \\ &= \mathcal{C}_9\left\{\|r_{\alpha_k}(A)e_0\| + \frac{\|Ar_{\alpha_k}(A)e_0\|+\delta}{\alpha_k}\right\}, \qquad \text{for } 0 \le k \le k_*^\delta, \end{aligned} \tag{4.19}$$

where $\mathcal{C}_9 = \frac{\mathcal{C}_8(\mu+1)^2}{1-\|e_0\|\mathcal{C}_8(\mu+1)^2}$.

Now, we estimate $\|r_{\alpha_k}(A)e_0\|$. In view of Assumption 3 and Proposition 3.1, we have

$$\begin{aligned} \|r_{\alpha_k}(A)e_0\| &= \|r_{\alpha_k}(A)A^p v\| = \|A^p r_{\alpha_k}(A)v\| \\ &\le \tilde{c}\|A^{p+1}r_{\alpha_k}(A)v\|^{p/p+1}\|r_{\alpha_k}(A)v\|^{1/p+1} \\ &\le \tilde{c}\|Ar_{\alpha_k}(A)e_0\|^{p/p+1}\|r_{\alpha_k}(A)\|^{1/p+1}\|v\|^{1/p+1} \\ &\le \tilde{c}\|v\|^{1/p+1}\|Ar_{\alpha_k}(A)e_0\|^{p/p+1}. \end{aligned}$$

Using (4.15), we get

$$\|r_{\alpha_k}(A)e_0\| \le \tilde{c}\|v\|^{1/p+1}\left(\mathcal{C}_6\left\{\|F(x_k^\delta)-y^\delta\|+\delta\right\}\right)^{p/p+1}. \tag{4.20}$$

Therefore, from (4.19), (4.15), and (4.20) we obtain

$$\begin{aligned} \|e_k^\delta\| &\le \mathcal{C}_9\left(\tilde{c}\|v\|^{1/p+1}\left(\mathcal{C}_6\left\{\|F(x_k^\delta)-y^\delta\|+\delta\right\}\right)^{p/p+1} + \frac{\mathcal{C}_6\left\{\|F(x_k^\delta)-y^\delta\|+\delta\right\}+\delta}{\alpha_k}\right) \\ &\le \mathcal{C}_{10}\left\{\|v\|\left(\frac{\|F(x_k^\delta)-y^\delta\|+\delta}{\|v\|}\right)^{p/p+1} + \frac{\|F(x_k^\delta)-y^\delta\|+\delta}{\alpha_k}\right\}, \end{aligned}$$

where $\mathcal{C}_{10} = \max\left\{\tilde{c}\,\mathcal{C}_9{\mathcal{C}_6}^{p/p+1}, \mathcal{C}_9(1+\mathcal{C}_6)\right\}$. Hence, we conclude that

$$\|e_k^\delta\| \le \mathcal{C}_{10}\left\{\|v\|\left(\frac{\|F(x_k^\delta)-y^\delta\|+\delta}{\|v\|}\right)^{p/p+1} + \frac{\|F(x_k^\delta)-y^\delta\|+\delta}{\alpha_k}\right\},$$

for $0 \le k \le k_*^\delta$. $\square$

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5. Main result. Now we prove main result of the paper.

THEOREM 5.1. *Let the assumptions of Lemma 4.3 be satisfied, and let the stopping index k_* be chosen according to the stopping rule (2.1). Then,*

$$\|x_{k_*}^\delta - x^\dagger\| \le \xi\|v\|\left(\frac{\delta_* + \delta}{\|v\|}\right)^{\frac{p}{p+1}},$$

where $\xi = \mathcal{C}_{10}\left(1 + \frac{\tau\mu(1+\kappa^{-1})}{\mathcal{C}_{11}^{\frac{p}{p+1}}}\right)$ *with* $\mathcal{C}_{10}$ *as in Lemma 4.3 and* $\mathcal{C}_{11} = \frac{1}{((K_0+K_1)\|e_0\|+1)\mathcal{C}_4}$, $\delta = \|y - y^\delta\|$, *and* $\delta_* = \|F(x_{k_*}^\delta) - y^\delta\|$.

Proof. Lemma 3.4 ensures that $k_* \le k_*^\delta$. Taking $k = k_*$ in Lemma 4.3 and using the definition of $\theta(k, y^\delta)$, we obtain

$$\begin{aligned}
\|x_{k_*}^\delta - x^\dagger\| &\le \mathcal{C}_{10}\left\{\|v\|\left(\frac{\|F(x_{k_*}^\delta) - y^\delta\| + \delta}{\|v\|}\right)^{p/p+1} + \frac{\|F(x_{k_*}^\delta) - y^\delta\| + \delta}{\alpha_{k_*}}\right\}\\
&\le \mathcal{C}_{10}\left\{\|v\|\left(\frac{\delta_* + \delta}{\|v\|}\right)^{p/p+1} + \theta(k_*, y^\delta)^{\frac{1}{2}} + \frac{\theta(k_*, y^\delta)^{\frac{1}{2}}}{\kappa}\right\}\\
&\le \mathcal{C}_{10}\left\{\|v\|\left(\frac{\delta_* + \delta}{\|v\|}\right)^{p/p+1} + (1+\kappa^{-1})\theta(k_*, y^\delta)^{\frac{1}{2}}\right\}. \qquad (5.1)
\end{aligned}$$

Now, we estimate $\theta(k_*, y^\delta)$. Let k_δ be the integer satisfying

$$\|F(x_{k_\delta}^\delta) - y^\delta\| \le \tau\delta < \|F(x_k^\delta) - y^\delta\|, \qquad 0 \le k < k_\delta,$$

where $\tau = \max\{\mathcal{C}, 2 + \mathcal{C}_4((K_0 + K_1)\|e_0\| + 1)\}$ with constants $\mathcal{C}$ and $\mathcal{C}_4$ as in Lemma 3.3 and Lemma 4.2, respectively. Lemma 3.3 ensures that k_δ is well-defined and $k_\delta \le \hat{k} \le k_*^\delta$. Using Assumption 2 and Lemma 3.2, we note that

$$\begin{aligned}
\|F(x_{k_\delta-1}^\delta) - y^\delta\| &\le \|F(x_{k_\delta-1}^\delta) - y - Ae_{k_\delta-1}^\delta\| + \delta + \|Ae_{k_\delta-1}^\delta\|\\
&\le \frac{K_0 + K_1}{2}\|e_{k_\delta-1}^\delta\|\|Ae_{k_\delta-1}^\delta\| + \delta + \|Ae_{k_\delta-1}^\delta\|\\
&\le ((K_0 + K_1)\|e_0\| + 1)\,\|Ae_{k_\delta-1}^\delta\| + \delta.
\end{aligned}$$

Making use of the inequality $\tau\delta \le \|F(x_{k_\delta-1}) - y^\delta\|$ and (4.14) we get

$$\begin{aligned}
\tau\delta &\le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\left\{\|Ar_{\alpha_k}(A)e_0\| + \delta\right\} + \delta\\
&\le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|Ar_{\alpha_k}(A)e_0\| + \{((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4 + 1\}\delta.
\end{aligned}$$

As $\tau \ge 2 + \mathcal{C}_4((K_0 + K_1)\|e_0\| + 1)$, we conclude from the above inequality that

$$\delta \le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|Ar_{\alpha_k}(A)e_0\|.$$

Using the source condition as given in Assumption 3, it follows that

$$\begin{aligned}
\delta &\le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|Ar_{\alpha_k}(A)A^p v\|\\
&\le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|A^{p+1}r_{\alpha_k}Av\|\\
&\le ((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|v\|(\alpha_{k_\delta-1})^{p+1}.
\end{aligned}$$

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Therefore,

$$(\alpha_{k_\delta - 1})^{p+1} \geq \frac{\delta}{((K_0 + K_1)\|e_0\| + 1)\mathcal{C}_4\|v\|}.$$

Let us denote $\mathcal{C}_{11} = \frac{1}{((K_0+K_1)\|e_0\|+1)\mathcal{C}_4}$. We get $\alpha_{k_\delta-1}^{p+1} \geq \frac{\mathcal{C}_{11}\delta}{\|v\|}$, which implies

$$\alpha_{k_\delta - 1} \geq \left(\frac{\mathcal{C}_{11}}{\|v\|}\right)^{\frac{1}{p+1}} \delta^{\frac{1}{p+1}}.$$

Thus, we have

$$\frac{\delta}{\alpha_{k_\delta-1}} \leq \|v\| \left(\frac{\delta}{\mathcal{C}_{11}\|v\|}\right)^{\frac{p}{p+1}}. \tag{5.2}$$

Using (1.5) and (5.2) with the fact that $0 < \mathcal{C}_{11} < 1$, we find

$$\frac{\delta}{\alpha_{k_\delta}} \leq \frac{\mu\delta}{\alpha_{k_\delta-1}} \leq \mu\|v\| \left(\frac{\delta}{\mathcal{C}_{11}\|v\|}\right)^{\frac{p}{p+1}} \leq \frac{\mu\|v\|}{\mathcal{C}_{11}^{\frac{p}{p+1}}} \left(\frac{\delta+\delta_*}{\|v\|}\right)^{\frac{p}{p+1}}.$$

Thus, by the definition of $\theta(k_*, y^\delta)$, we have

$$\theta(k_*, y^\delta) \leq \theta(k_\delta, y^\delta) = \frac{\|F(x_{k_\delta}^\delta) - y^\delta\|^2}{\alpha_{k_\delta}^2} \leq \frac{\tau^2\delta^2}{\alpha_{k_\delta}^2} \leq \frac{\tau^2\mu^2\|v\|^2}{\mathcal{C}_{11}^{\frac{2p}{p+1}}} \left(\frac{\delta+\delta_*}{\|v\|}\right)^{\frac{2p}{p+1}}. \tag{5.3}$$

Combining (5.1) and (5.3) we get

$$\|x_{k_*}^\delta - x^\dagger\| \leq \mathcal{C}_{10} \left\{ \|v\| \left(\frac{\delta_* + \delta}{\|\nu\|}\right)^{p/p+1} + (1+\kappa^{-1}) \frac{\tau\mu\|v\|}{\mathcal{C}_{11}^{\frac{p}{p+1}}} \left(\frac{\delta+\delta_*}{\|v\|}\right)^{\frac{p}{p+1}} \right\}.$$

Denoting $\xi = \mathcal{C}_{10}\left(1 + \frac{\tau\mu(1+\kappa^{-1})}{\mathcal{C}_{11}^{\frac{p}{p+1}}}\right)$, we arrive at

$$\|x_{k_*}^\delta - x^\dagger\| \leq \xi\|v\| \left(\frac{\delta_*+\delta}{\|v\|}\right)^{\frac{p}{p+1}}. \qquad \square$$

REMARK 5.2. The obtained error bound in Theorem 5.1 is order optimal if the quantity $\delta_* = \|F(x_{k_*}^\delta) - y^\delta\|$ is of the order of the noise level δ, and in that case, the error estimate obtained in Theorem 5.1 is of the same order as the one obtained in [22] under the stopping rule (1.7).

Acknowledgement. The authors are grateful to the National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Government of India, for the financial support through the research project grant No 2/11/42/2017/NBHM(R.P.)/ R& D II/ 534 dated January 10, 2018.

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