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AN A-PRIORI ERROR ANALYSIS FOR DISCONTINUOUS LAGRANGIAN FINITE ELEMENTS APPLIED TO NONCONFORMING DUAL-MIXED FORMULATIONS: POISSON AND STOKES PROBLEMS*

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Abstract. In this paper, we discuss the well-posedness of a mixed discontinuous Galerkin (DG) scheme for the Poisson and Stokes problems in 2D, considering only piecewise Lagrangian finite elements. The complication here lies in the fact that the classical Babuška-Brezzi theory is difficult to verify for low-order finite elements, so we proceed in a non-standard way. First, we prove uniqueness, and then we apply a discrete version of Fredholm's alternative theorem to ensure existence. The a-priori error analysis is done by introducing suitable projections of the exact solution. As a result, we prove that the method is convergent, and, under standard additional regularity assumptions on the exact solution, the optimal rate of convergence of the method is guaranteed.

Key words. discontinuous Galerkin, Lagrange shape functions, a-priori error estimates

AMS subject classifications. 65N30; 65N12; 65N15

1. Introduction.

Let Ω be a bounded and simply-connected domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma := \partial\Omega$, which is decomposed into Γ_D and Γ_N such that $|\Gamma_D| > 0$, $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \Gamma$, and $\mathring{\Gamma}_D \cap \mathring{\Gamma}_N = \emptyset$. Then, given $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$, we look for $u \in H^1(\Omega)$ such that

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{K}\nabla u) &= f && \text{in } \Omega,\\
u &= g_D && \text{on } \Gamma_D,\\
-\boldsymbol{K}\nabla u\cdot\boldsymbol{\nu} &= g_N && \text{on } \Gamma_N,
\end{aligned} \tag{1.1}
$$

where $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$. From here on, $\boldsymbol{K}$ denotes a bounded, measurable, and symmetric tensor living in $[L^\infty(\Omega)]^{2\times 2}$, which describes the material properties. Moreover, we assume that there exist $c_1, c_2 > 0$ such that

$$
c_1|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^{\mathrm{t}}\boldsymbol{K}(x)\boldsymbol{\xi} \le c_2|\boldsymbol{\xi}|^2 \qquad \forall \boldsymbol{\xi}\in\mathbb{R}^2, \forall x\in\Omega,
$$

where $|\cdot|$ denotes the usual Euclidean norm. This property is known as the strong ellipticity of the tensor $\boldsymbol{K}$ and ensures that $\boldsymbol{K}$ is uniformly positive-definite and thus that $\boldsymbol{K}^{-1}$ is well-defined for every $x \in \Omega$.

We follow [3] and introduce the gradient $\boldsymbol{\sigma} := -\boldsymbol{K}\nabla u$ in Ω as an additional unknown. In this way, (1.1) can be reformulated as the following problem in $\bar{\Omega}$:

Find $(\boldsymbol{\sigma}, u)$ in appropriate spaces such that in the distributional sense,

$$
\begin{aligned}
\boldsymbol{K}^{-1}\boldsymbol{\sigma} + \nabla u &= 0 && \text{in } \Omega,\\
\operatorname{div}(\boldsymbol{\sigma}) &= f && \text{in } \Omega,\\
u &= g_D && \text{on } \Gamma_D,\\
\boldsymbol{\sigma}\cdot\boldsymbol{\nu} &= g_N && \text{on } \Gamma_N.
\end{aligned} \tag{1.2}
$$

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In [12], a numerical analysis applying mixed finite element methods to problem (1.2) has been performed, establishing optimal a-priori error estimates. Usually, this approach requires to take into account an $H(\mathrm{div})$-conforming discrete approximation space for the flux if conforming schemes are used (e.g., Raviart-Thomas elements [34] or Brezzi-Douglas-Marini elements [11]). On the other hand, if nonconforming methods are considered, then $H(\mathrm{div})$-like discrete spaces are usually invoked to approximate the flux (see, e.g., [2]). Certainly, the norms provided for these spaces are $H(\mathrm{div})$-like. For example, we point out that problem (1.2) has been already analyzed in [33] using the so-called local discontinuous Galerkin (LDG) method, where the unknowns $\boldsymbol{\sigma}$ and u are approximated in piecewise polynomial spaces that locally belong to H^1 and L^2, respectively. The LDG method has been applied to several elliptic problems, too, such as the Stokes problem [18, 19, 25], the Oseen problem [22], elasticity [26], the Navier-Stokes equation [23], and certain classes of nonlinear elliptic problems (cf. [13, 14, 31]). We point out that in [24], the authors proposed a family of discontinuous Galerkin methods for the incompressible Navier-Stokes equations with the purpose to ensure that the approximation of the velocity is divergence-free. However, the norms considered for the analysis in this paper are the usual ones for the H^1-broken space and L^2 for the pressure unknown. A variant of DG schemes, known as hybridizable discontinuous Galerkin (HDG) methods, has been considered also for solving (1.2) [20] and other elliptic linear problems such as the Stokes problem [21] or elasticity [27], for example. The main characteristic of HDG schemes is that the global discrete linear system can be reduced to another linear system by eliminating the volume unknowns with the introduction of the so-called *local solvers*. As a result, this system is smaller than the global one, and once it is solved, the volume unknowns are recovered by the local solvers. We remark that the unknowns are approximated by piecewise polynomials, considering only L^2-type norms. Our proposal points to the combination of the analysis in $H(\mathrm{div})$-type norms with non-$H(\mathrm{div})$-like finite elements. Up to the authors' knowledge, this has not been studied before.

In the present work, we are interested in approximating the vector unknown $\boldsymbol{\sigma}$ in a suitable discrete space such that it locally belongs to $H(\mathrm{div})$. This motivates us to consider the employment of local (discontinuous) Raviart-Thomas spaces to approximate $\boldsymbol{\sigma}$. This kind of approach has also been applied in previous works [4, 5, 6, 8]. There, the standard definition of numerical fluxes for the LDG scheme has been considered, whose parameters α and $\boldsymbol{\beta}$ are defined such that it is possible to ensure the unique solvability of the discrete scheme as well as the optimal rate of convergence of the method. The corresponding analysis has led us to establish that α and $\boldsymbol{\beta}$ behave as $\mathcal{O}(1/\mathrm{h})$ and $\mathcal{O}(1)$, respectively. Recently, in the framework of the non-conforming pseudo stress-velocity formulation for the Stokes system, in [9] we studied an unusual DG approach which requires a vector numerical flux parameter $\boldsymbol{\beta}$ having the standard behavior $\mathcal{O}(1)$ and two scalar numerical fluxes parameters (α and γ). Here, one of them (γ) behaves as $\mathcal{O}(1/\mathrm{h})$, while the other one ($\alpha$) as $\mathcal{O}(\mathrm{h})$. The well-posedness of the proposed scheme is established using discontinuous Raviart-Thomas finite elements for the pseudo stress unknown and discontinuous polynomials for the velocity one. A particularity of this approach is that when it is tested with continuous functions, we obtain the standard conforming dual-mixed formulation. In this sense, the analysis developed in [9] is an extension of the one described in [15], where it has been proved that the pair of conforming Raviart-Thomas finite elements with discontinuous polynomials is stable for the Stokes system in dual-mixed form. Of course, the DG scheme presented in [9] could be approximated by other finite elements, and this will be the aim of the current work, i.e., in this paper we extend the analysis developed in [9] to other pairs of finite element spaces.

In other words, the analysis described in this paper gives another point of view for applying the LDG method, which shows us that the jumps and average operators used to define the numerical fluxes are enough to control the continuity of the normal component of the flux, avoiding the use of conforming $H(\mathrm{div};\Omega)$ finite elements (such as Raviart-Thomas or BDM ones, for example). In particular, in this work we prove the optimal rate of convergence of the method using the well-known nonconforming Lagrange finite elements to approximate the flux instead of the classical conforming dual-mixed approach (which needs Raviart-Thomas or BDM operators to approximate functions in $H(\mathrm{div};\Omega)$). For instance, when we consider piecewise-linear polynomials to approximate the scalar unknown, we deduce quadratic convergence in the L^2-norm for both unknowns (the scalar and vectorial ones). In this sense, this result can be seen as an extension of the classical dual-mixed approach but now invoking the standard element-wise but discontinuous Lagrange interpolation operator in the process.

The paper is organized as follows. In Section 2, for simplicity, we take into account the analysis described in [9] for the Poisson problem and adapt/generalize it to study the a-priori error analysis for this boundary value problem, considering only finite element spaces of Lagrange type for all unknowns.

In Section 3, we extend the results obtained in the previous section to the Stokes problem, approximating the unknowns by polynomials of suitable degrees. Numerical examples that confirm our theoretical results are shown and discussed in Section 4. Final remarks and conclusions are described in Section 5.

In the rest of the paper we will use the following notation. Given any Hilbert space H, we denote by H^2 the space of vectors of dimension 2 with entries in H and by $H^{2\times 2}$ the space of square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in H^{2\times 2}$, we write, as usual, $\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})$, $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$, and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}$. For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$, we denote by $\boldsymbol{v} \otimes \boldsymbol{w}$ the matrix whose ij-th entry is $v_i w_j$. We also use the standard notation for Sobolev spaces and norms. We define

$$H(\mathrm{div};\ \Omega) := \{\boldsymbol{v} \in [L^2(\Omega)]^2 : \quad \mathrm{div}(\boldsymbol{v}) \in L^2(\Omega)\},$$
$$H = H(\mathbf{div};\ \Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}, \qquad \text{and}$$
$$H_0 := \left\{\boldsymbol{\tau} \in H : \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\}.$$

We recall here, that $\mathbf{div}$ denotes the row-wise divergence operator for square tensors of order 2. Note that $H = H_0 \oplus \mathbb{R}\boldsymbol{I}$, that is, for any $\boldsymbol{\tau} \in H$ there exists a unique pair $(\boldsymbol{\tau}_0, \rho) \in H_0 \times \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \rho\boldsymbol{I}$. In addition, we define the deviator of the tensor $\boldsymbol{\tau} \in H$ by $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}$. We remark that $\mathrm{tr}(\boldsymbol{\tau}^{\mathrm{d}}) = 0$ in Ω, and thus $\boldsymbol{\tau}^{\mathrm{d}} \in H_0$ for any $\boldsymbol{\tau} \in H$. Finally, we use C or c, with or without subscripts, to denote generic constants independent of the discretization parameters, which may take different values at different occurrences.

2. A modified LDG formulation.

In this section, we derive a discrete formulation for the linear model (1.1), applying a discontinuous Galerkin method in divergence form. We begin this section with some definitions and notations that will help us to describe the approach.

2.1. Meshes.

We let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular triangulations of $\bar{\Omega}$ (without hanging nodes) made up of straight-sided triangles T with diameter h_T and unit outward normal to ∂T denoted by $\boldsymbol{\nu}_T$. As usual, the index h is defined as $h := \max_{T\in\mathcal{T}_h} h_T$. Then, given $\mathcal{T}_h$, its edges are defined as follows. An *interior edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent matching elements of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ the list of all interior edges of $\mathcal{T}_h$ (counted only once) in Ω, by $\mathcal{E}_\Gamma$ the list of all boundary edges, respectively, and set $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_\Gamma$, the skeleton inherited by the triangulation $\mathcal{T}_h$. We

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introduce $\mathcal{E}_D$ and $\mathcal{E}_N$ as the list of boundary edges lying on Γ_D and Γ_N, respectively. This implies that $\mathcal{E}_\Gamma = \mathcal{E}_D \cup \mathcal{E}_N$. Moreover, for each $e \in \mathcal{E}$, h_e represents its length. In addition, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, which means that there exists a constant $l > 1$ independent of the mesh size h such that $l^{-1} \le h_T/h_{T'} \le l$ for each pair $T, T' \in \mathcal{T}_h$ sharing an interior edge.

2.2. Averages and jumps. Here, we define average and jump operators. To this end, we let T and T' be two adjacent elements of $\mathcal{T}_h$ and $\boldsymbol{x}$ be an arbitrary point on the interior edge $e = \partial T \cap \partial T' \in \mathcal{E}_I$. In addition, let q, $\boldsymbol{v}$, and $\boldsymbol{\tau}$ be scalar-, vector-, and matrix-valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(q_T, \boldsymbol{v}_T, \boldsymbol{\tau}_T)$ the restriction of $(q, \boldsymbol{v}, \boldsymbol{\tau})$ to the triangle T and by $(q_{T,e}, \boldsymbol{v}_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(q_T, \boldsymbol{v}_T, \boldsymbol{\tau}_T)$ to the edge e. Then, we define the averages of q, $\boldsymbol{v}$, and $\boldsymbol{\tau}$ on the edge e by:

$$\{q\} := \frac{1}{2}\big(q_{T,e} + q_{T',e}\big), \quad \{\boldsymbol{v}\} := \frac{1}{2}\big(\boldsymbol{v}_{T,e} + \boldsymbol{v}_{T',e}\big), \quad \{\boldsymbol{\tau}\} := \frac{1}{2}\big(\boldsymbol{\tau}_{T,e} + \boldsymbol{\tau}_{T',e}\big).$$

Similarly, the jumps of q, $\boldsymbol{v}$, and $\boldsymbol{\tau}$ on the edge e are given by

$$\begin{aligned} [\![q]\!] &:= q_{T,e}\boldsymbol{\nu}_T + q_{T',e}\boldsymbol{\nu}_{T'}, & [\![\boldsymbol{v}]\!] &:= \boldsymbol{v}_{T,e}\cdot\boldsymbol{\nu}_T + \boldsymbol{v}_{T',e}\cdot\boldsymbol{\nu}_{T'},\\ \underline{[\![\boldsymbol{v}]\!]} &:= \boldsymbol{v}_{T,e}\otimes\boldsymbol{\nu}_T + \boldsymbol{v}_{T',e}\otimes\boldsymbol{\nu}_{T'}, & [\![\boldsymbol{\tau}]\!] &:= \boldsymbol{\tau}_{T,e}\boldsymbol{\nu}_T + \boldsymbol{\tau}_{T',e}\boldsymbol{\nu}_{T'}. \end{aligned}$$

On boundary edges $e \in \mathcal{E}_\Gamma$, we set $\{q\} := q$, $\{\boldsymbol{v}\} := \boldsymbol{v}$, $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$, as well as $[\![q]\!] := q\boldsymbol{\nu}$, $[\![\boldsymbol{v}]\!] := \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}\otimes\boldsymbol{\nu}$, and $[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\boldsymbol{\nu}$. Hereafter, $\mathbf{div}_h$ and ∇_h denote the piecewise divergence and gradient operators, respectively. Associated to these operators, for $\epsilon > 1/2$, we also introduced the broken Sobolev spaces $H^\epsilon(\mathcal{T}_h)$, $H(\mathrm{div};\mathcal{T}_h)$, and $H(\mathbf{div};\mathcal{T}_h)$, which are defined in the standard way, and to short the notation, we set $\boldsymbol{\Sigma} := H(\mathrm{div};\mathcal{T}_h) \cap [H^\epsilon(\Omega)]^2$ and $\underline{\boldsymbol{\Sigma}} := H(\mathbf{div};\mathcal{T}_h) \cap [H^\epsilon(\Omega)]^{2\times 2}$.

2.3. A discontinuous discrete formulation for the Poisson problem. Given a mesh $\mathcal{T}_h$, we proceed as in [33] (or [13]). We want to approximate the exact solution $(\boldsymbol{\sigma}, u)$ of (1.2) by the discrete pair $(\boldsymbol{\sigma}_h, u_h)$ living in an appropriate finite element space $\boldsymbol{\Sigma}_h \times V_h \subset H(\mathrm{div};\mathcal{T}_h) \times L^2(\Omega)$ such that, for each $T \in \mathcal{T}_h$, we have

$$\begin{aligned} \int_T \boldsymbol{K}^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{\tau} - \int_T u_h \operatorname{div}(\boldsymbol{\tau}) + \int_{\partial T} \widehat{u}\boldsymbol{\tau}\cdot\boldsymbol{\nu}_T &= 0 \qquad && \forall \boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,\\ -\int_T \boldsymbol{\sigma}_h\cdot\nabla v + \int_{\partial T} v\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T &= \int_T f v && \forall v\in V_h. \end{aligned} \tag{2.1}$$

Here, the *numerical fluxes* $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on u_h, $\boldsymbol{\sigma}_h$, and the boundary data, are defined in terms of averages and jumps of the discrete unknowns so that some compatibility conditions are satisfied (see [3]).

We are now ready to complete the DG formulation (2.1). Indeed, using the approach from [33] and [16], we define the numerical fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$ as follows:

$$\widehat{u}_{T,e} := \begin{cases} \{u_h\} - [\![u_h]\!]\cdot\boldsymbol{\beta} + \gamma[\![\boldsymbol{\sigma}_h]\!] & \text{if } e\in\mathcal{E}_I,\\ g_D & \text{if } e\in\mathcal{E}_D,\\ u_h + \gamma(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu} - g_N) & \text{if } e\in\mathcal{E}_N, \end{cases} \tag{2.2}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} + [\![\boldsymbol{\sigma}_h]\!]\boldsymbol{\beta} + \alpha[\![u_h]\!] & \text{if } e\in\mathcal{E}_I,\\ \boldsymbol{\sigma}_h + \alpha(u_h - g_D)\boldsymbol{\nu} & \text{if } e\in\mathcal{E}_D,\\ g_N\boldsymbol{\nu} & \text{if } e\in\mathcal{E}_N, \end{cases} \tag{2.3}$$

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where the scalar parameters α and γ as well as the vector parameter $\boldsymbol{\beta}$, to be chosen appropriately, are single valued on each edge $e \in \mathcal{E}$ and are such that they allow us to enforce existence and uniqueness and to deduce the optimal rates of convergence of our approximation. This kind of procedure has been considered in [16], and here we proceed analogously. As a result, we require that $\alpha := \widehat{\alpha}\mathtt{h}$, $\gamma := \widehat{\gamma}/\mathtt{h}$, and $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$ be an arbitrary vector in $\mathbb{R}^2$. Hereafter, $\widehat{\alpha} > 0$ and $\widehat{\gamma} > 0$ are arbitrary but independent of $\mathtt{h}$, where $\mathtt{h}$ is defined by

$$\mathtt{h} := \begin{cases} \max\{h_T, h_{T'}\} & \text{if } e \in \mathcal{E}_I, \\ h_T & \text{if } e \in \mathcal{E}_\Gamma. \end{cases}$$

REMARK 2.1. We point out that, as a consequence of our analysis of existence and uniqueness as well as the corresponding one for the a-priori error, the setting of our parameters α and γ as well as our function $\mathtt{h}$ are different from the ones considered in [16]. These settings will be useful to analyze the error of our scheme in $H(\mathrm{div};\mathcal{T}_h) \times L^2(\Omega)$.

Applying usual and well-known techniques, we arrive at the following discrete dual-mixed discontinuous Galerkin formulation:

Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ *such that*

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau}, u_h) &= G_{DG}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\ b_{DG}(\boldsymbol{\sigma}_h, v) + c_{DG}(u_h, v) &= F_{DG}(v) \qquad \forall\, v \in V_h, \end{aligned} \tag{2.4}$$

where the bilinear forms $a_{DG} : \boldsymbol{\Sigma} \times \boldsymbol{\Sigma} \to \mathbb{R}$, $c_{DG} : H^\epsilon(\mathcal{T}_h) \times H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ *and* $b_{DG} : \boldsymbol{\Sigma} \times H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ *are defined by*

$$\begin{aligned} a_{DG}(\boldsymbol{\rho}, \boldsymbol{\tau}) &:= \int_\Omega \boldsymbol{K}^{-1}\boldsymbol{\rho} \cdot \boldsymbol{\tau} + \int_{\mathcal{E}_I \cup \mathcal{E}_N} \gamma [\![\boldsymbol{\rho}]\!][\![\boldsymbol{\tau}]\!], \\ c_{DG}(w, v) &:= \int_{\mathcal{E}_I \cup \mathcal{E}_D} \alpha [\![v]\!] \cdot [\![w]\!], \\ b_{DG}(\boldsymbol{\tau}, v) &:= \int_\Omega v \,\mathrm{div}_h(\boldsymbol{\tau}) - \int_{\mathcal{E}_I} \left(\{v\} - \boldsymbol{\beta} \cdot [\![v]\!]\right)[\![\boldsymbol{\tau}]\!] - \int_{\mathcal{E}_N} v \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \end{aligned}$$

while the linear functionals $G_{DG} : \boldsymbol{\Sigma} \to \mathbb{R}$ *and* $F_{DG} : H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ *are given by*

$$\begin{aligned} G_{DG}(\boldsymbol{\tau}) &:= -\int_{\mathcal{E}_D} g_D \boldsymbol{\tau} \cdot \boldsymbol{\nu} + \int_{\mathcal{E}_N} \gamma g_N \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \\ F_{DG}(v) &:= \int_\Omega f v + \int_{\mathcal{E}_D} \alpha g_D v - \int_{\mathcal{E}_N} g_N v. \end{aligned}$$

Now, the space $\boldsymbol{\Sigma}_h$ is equipped with the usual norm of $H(\mathrm{div};\mathcal{T}_h)$, which is denoted by $\|\cdot\|_{\boldsymbol{\Sigma}}$ and given by

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} := \left(\|\boldsymbol{K}^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathrm{div}_h(\boldsymbol{\tau})\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{E}_I \cup \mathcal{E}_N}^2\right)^{1/2} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma},$$

while for V_h we introduce its standard L^2-norm. In addition, we define the norm $\|(\cdot,\cdot)\|_{DG} : \boldsymbol{\Sigma} \times L^2(\Omega) \to \mathbb{R}$ by

$$\|(\boldsymbol{\tau}, v)\|_{DG} := \left(\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}^2 + \|v\|_{L^2(\Omega)}^2\right)^{1/2} \quad \forall (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma} \times L^2(\Omega).$$

Finally, we introduce the norm $\|\cdot\|_{\mathbf{\Sigma},0} : \mathbf{\Sigma} \to \mathbb{R}$ given by

$$\|\boldsymbol{\tau}\|_{\mathbf{\Sigma},0} := \Big(\|\boldsymbol{K}^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_N}^2 \Big)^{1/2} \qquad \forall \boldsymbol{\tau} \in \mathbf{\Sigma},$$

which will be helpful for our purposes. The boundedness of the bilinear forms and functionals are reported in the next lemma.

LEMMA 2.2. *There exists $C > 0$, independent of the mesh size, such that*

$$\begin{aligned}
&a_{DG}(\boldsymbol{\tau},\boldsymbol{\zeta}) \le C\|\boldsymbol{\tau}\|_{\mathbf{\Sigma},0}\|\boldsymbol{\zeta}\|_{\mathbf{\Sigma},0} && \forall (\boldsymbol{\tau},\boldsymbol{\zeta}) \in \mathbf{\Sigma}\times\mathbf{\Sigma},\\
&b_{DG}(\boldsymbol{\tau},v) \le C\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|v\|_{L^2(\Omega)} && \forall (\boldsymbol{\tau},v) \in \mathbf{\Sigma}\times V_h,\\
&c_{DG}(v,w) \le C\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} && \forall (v,w) \in V_h\times V_h,\\
&|F_{DG}(v)| \le C\Big(\|f\|_{L^2(\Omega)} + \|\alpha^{1/2} g_D\|_{L^2(\mathcal{E}_D)} \\
&\qquad\qquad + \|\gamma^{1/2} g_N\|_{L^2(\mathcal{E}_N)}\Big)\|v\|_{L^2(\Omega)} && \forall v \in V_h,\\
&|G_{DG}(\boldsymbol{\tau})| \le C\Big(\|\gamma^{1/2} g_D\|_{L^2(\mathcal{E}_D)} + \|\gamma^{1/2} g_N\|_{L^2(\mathcal{E}_N)}\Big)\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}} && \forall \boldsymbol{\tau} \in \mathbf{\Sigma}_h.
\end{aligned}$$

Proof. The proof is analogous to those of Lemmas 3.2 and 3.3 in [9]. We omit further details. □

The well-posedness of problem (2.4) is established in the next theorem.

THEOREM 2.3. *Under the assumption that $\nabla_h V_h$ is a subspace of $\mathbf{\Sigma}_h$, there exists a unique solution $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{\Sigma}_h \times V_h$ of problem (2.4).*

Proof. Since the discrete system is square, it is enough to verify that the corresponding homogeneous problem has only the trivial solution:

Find $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{\Sigma}_h \times V_h$ such that

$$\begin{aligned}
a_{DG}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau},u_h) &= 0 && \forall\ \boldsymbol{\tau} \in \mathbf{\Sigma}_h,\\
b_{DG}(\boldsymbol{\sigma}_h,v) + c_{DG}(u_h,v) &= 0 && \forall\ v \in V_h.
\end{aligned} \tag{2.5}$$

To this end, we take $\boldsymbol{\tau} = \boldsymbol{\sigma}_h$, $v = u_h$ in (2.5), and after adding these two equations, we deduce that

$$\|\boldsymbol{K}^{-1/2}\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2}^2 + \|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]\|_{L^2(\mathcal{E}_I\cup\mathcal{E}_N)}^2 + \|\alpha^{1/2}[\![u_h]\!]\|_{[L^2(\mathcal{E}_I\cup\mathcal{E}_D)]^2}^2 = 0,$$

which implies that $\boldsymbol{\sigma}_h \in H(\mathrm{div},\Omega)$, $\boldsymbol{\sigma}_h = \mathbf{0}$ in Ω, $u_h \in C(\overline{\Omega})$, and $u_h = 0$ on Γ_D. Now, if the elements of V_h are piecewise constant, then the proof is concluded. Otherwise, since $\nabla_h V_h$ is a subspace of $\mathbf{\Sigma}_h$ and after integrating by parts the first equation of (2.5), we obtain $\nabla_h u_h = \mathbf{0}$ in Ω and thus we deduce $u_h = 0$ in Ω. In other words, problem (2.5) admits only the trivial solution. Therefore, existence is a consequence of a finite-dimensional version of Fredholm's theorem. □

Our next concern is the stability of the scheme (2.4). In order to set the approximation spaces, we denote by $\mathbb{P}_\kappa(T)$ the space of polynomials of degree at most κ on T, for a given integer $\kappa \ge 0$ and for each $T \in \mathcal{T}_h$. Now, in what follows we consider $\mathbf{\Sigma}_h$ and V_h as

$$\begin{aligned}
\mathbf{\Sigma}_h &:= \Big\{ \boldsymbol{\tau} \in [L^2(\Omega)]^2 : \quad \boldsymbol{\tau}\big|_T \in [\mathbb{P}_r(T)]^2 \quad \forall T \in \mathcal{T}_h \Big\},\\
V_h &:= \Big\{ v_h \in L^2(\Omega) : \quad v_h\big|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h \Big\},
\end{aligned}$$

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with $k \geq 0$ and $r \geq 1$. We notice that in order to verify the well-known *mild condition* (i.e., $\nabla_h V_h$ is a subspace of $\mathbf{\Sigma}_h$), we require $k \leq r+1$. This choice of spaces allows us to establish the following discrete $\inf$-$\sup$ condition. This is proven thanks to the ideas given in the proof of [9, Lemma 3.4] and requires the introduction of the local Raviart-Thomas space of order κ (cf. [35]), $RT_\kappa(T) := [\mathbb{P}_\kappa(T)]^2 \oplus \boldsymbol{x}\mathbb{P}_\kappa(T) \subseteq [\mathbb{P}_{\kappa+1}(T)]^2$, and the local Raviart-Thomas interpolation operator $\mathcal{E}_T^\kappa : [H^1(T)]^2 \to RT_\kappa(T)$ such that, given $\boldsymbol{\tau} \in [H^1(T)]^2$, $\mathcal{E}_T^\kappa(\boldsymbol{\tau})$ is the unique element in $RT_\kappa(T)$ satisfying

$$\text{(2.6)} \qquad \begin{aligned} \int_e \mathcal{E}_T^\kappa(\boldsymbol{\tau}) \cdot \boldsymbol{\nu}_T q &= \int_e \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T q \qquad \forall e \in \mathcal{E} \cap \partial T,\ \forall q \in \mathbb{P}_\kappa(e), \text{if } \kappa \geq 0, \\ \int_T \mathcal{E}_T^\kappa(\boldsymbol{\tau}) \cdot \boldsymbol{\rho} &= \int_T \boldsymbol{\tau} \cdot \boldsymbol{\rho} \qquad \forall T \in \mathcal{T}_h,\ \forall \boldsymbol{\rho} \in [\mathbb{P}_{\kappa-1}(T)]^2, \text{if } \kappa \geq 1. \end{aligned}$$

Then, we introduce the global Raviart-Thomas interpolation operator (see [35])

$$\mathcal{E}_h^\kappa : [H^1(\Omega)]^2 \to \mathbf{\Sigma}_h^{\text{RT}}, \quad \text{with} \quad \mathbf{\Sigma}_h^{\text{RT}} := \{\boldsymbol{\rho} \in H(\text{div};\Omega) : \boldsymbol{\rho}|_T \in RT_\kappa(T) \quad \forall T \in \mathcal{T}_h\},$$

such that, given $\boldsymbol{\tau} \in [H^1(\Omega)]^2$, $\mathcal{E}_h^\kappa(\boldsymbol{\tau})|_T := \mathcal{E}_T^\kappa(\boldsymbol{\tau}|_T)$, for all $T \in \mathcal{T}_h$. It is important to emphasize that $\mathcal{E}_h^\kappa$ can be also defined from $[H^s(\Omega)]^2 \cap H(\text{div};\Omega)$ onto $\mathbf{\Sigma}_h^{\text{RT}}$ for any $s \in (0,1]$ (cf. Theorem 3.16 in [30]).

To obtain the a-priori error estimates for the scheme (2.4), we need the following lemmas, which establish local approximation properties of piecewise polynomial approximations.

LEMMA 2.4. *Let $\mathcal{T}_h$ be an element of a shape-regular triangulation family $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a nonnegative integer m, let $\Pi_T^m : L^2(T) \to \mathbb{P}_m(T)$ be the linear and bounded operator given by the $L^2(T)$-orthogonal projection, which satisfies $\Pi_T^m(p) = p$ for all $p \in \mathbb{P}_m(T)$. Then there exists $C > 0$, independent of the mesh size, such that for each s, t satisfying $0 \leq s \leq m+1$ and $0 \leq s < t$, there holds*

$$|(I - \Pi_T^m)(w)|_{s,T} \leq C h_T^{\min\{t,m+1\}-s} \|w\|_{t,T} \quad \forall w \in H^t(T),$$

and for each $t > 1/2$, there holds

$$|(I - \Pi_T^m)(w)|_{0,\partial T} \leq C h_T^{\min\{t,m+1\}-1/2} \|w\|_{t,T} \quad \forall w \in H^t(T).$$

Proof. We refer to [17, 29]. □

LEMMA 2.5. *Let $\mathcal{T}_h$ be an element of a shape-regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a nonnegative integer k, the local interpolation operator $\mathcal{E}_T^k$ satisfies de Rham's commutative diagram: $\text{div}(\mathcal{E}_T^k(\boldsymbol{\tau})) = \Pi_T^k(\text{div}(\boldsymbol{\tau}))$ for all $\boldsymbol{\tau} \in [H^1(T)]^2$. Moreover, there exists $C > 0$, independent of the mesh size but depending on integers $l > 0$ and $s \geq 0$, such that for all $\boldsymbol{\tau} \in [H^l(T)]^2$ with $\text{div}(\boldsymbol{\tau}) \in H^s(T)$ there hold*

$$\|\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau})\|_{[L^2(T)]^2} \leq C h_T^l |\boldsymbol{\tau}|_{[H^l(T)]^2} \quad 1 \leq l \leq k+1,$$

and

$$\|\text{div}(\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau}))\|_{L^2(T)} \leq C h_T^s \|\text{div}(\boldsymbol{\tau})\|_{H^s(T)} \quad 0 \leq s \leq k+1.$$

Proof. We refer to Propositions 2.5.1 and 2.5.3 in [12]. □

LEMMA 2.6. *Let V_h a finite-dimensional subspace of $L^2(\Omega)$ such that V_h is a subspace of $\mathbb{P}_{r-1}(\mathcal{T}_h)$. Then for all $v \in V_h$, there exists $\tilde{c} > 0$, independent of the mesh size, such that*

$$\sup_{\boldsymbol{\tau} \in \mathbf{\Sigma}_h \setminus \{0\}} \frac{b_{DG}(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}} \geq \tilde{c} \|v\|_{L^2(\Omega)}.$$

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Proof. We take $v \in V_h$ and define the auxiliary problem: Find $w \in H^1_{\Gamma_D}(\Omega)$ such that, in the distributional sense, $-\nabla\cdot(\boldsymbol{K}\nabla w) = v$ in Ω, $w = 0$ on Γ_D, and $\boldsymbol{K}\nabla w\cdot\boldsymbol{\nu} = 0$ on Γ_N. This allows us to set $\boldsymbol{\zeta} := -\boldsymbol{K}\nabla w$ in Ω, and since $\mathrm{div}(\boldsymbol{\zeta}) = v$ in Ω, we conclude that $\boldsymbol{\zeta} \in \boldsymbol{\Sigma}$ and there exists $C > 0$, independent of the mesh size, such that $\|\boldsymbol{K}^{-1/2}\boldsymbol{\zeta}\|_{0,\Omega} \le C\|v\|_{0,\Omega}$. In addition, we notice that $\boldsymbol{\zeta}\cdot\boldsymbol{\nu} = 0$ on Γ_N. Now, we introduce $\boldsymbol{\zeta}_h := \mathcal{E}_h^{r-1}(\boldsymbol{\zeta}) \in \boldsymbol{\Sigma}_h \cap H(\mathrm{div};\Omega)$, which means that $\mathrm{div}(\boldsymbol{\zeta}_h) = v$, $\|\gamma^{1/2}[\![\boldsymbol{\zeta}_h]\!]\|_{0,\mathcal{E}_I} = 0$, and $\boldsymbol{\zeta}_h\cdot\boldsymbol{\nu} = 0$ on Γ_N. These relations imply that

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\setminus\{\mathbf{0}\}} \frac{b_{DG}(\boldsymbol{\tau},v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} \ge \frac{b_{DG}(\boldsymbol{\zeta}_h,v)}{\|\boldsymbol{\zeta}_h\|_{\boldsymbol{\Sigma}}} \ge \frac{1}{\sqrt{C^2+1}}\frac{\|v\|^2_{0,\Omega}}{\|v\|_{0,\Omega}} = \frac{1}{\sqrt{C^2+1}}\|v\|_{0,\Omega},$$

which ends the proof. ∎

Now, we introduce the DG norm for our approach as

$$|||(\boldsymbol{\tau},v)|||_{DG} := \left(\|\boldsymbol{K}^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega} + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{E}_I\cup\mathcal{E}_N} + \|v\|^2_{0,\Omega}\right)^{1/2} \quad \forall(\boldsymbol{\tau},v)\in\boldsymbol{\Sigma}\times L^2(\Omega).$$

Through the rest of this section, we denote by $(\boldsymbol{\sigma},u)$ and $(\boldsymbol{\sigma}_h,u_h)$ the unique solutions of (1.2) and (2.4), respectively. The strategy we propose will be to obtain error estimates for $|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)|||_{DG}$. In addition, we also derive error estimates for $\|\mathrm{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}$.

The optimal rate of convergence of $|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)|||_{DG}$ is reported in the following result, whose proof requires the assumption that $\mathrm{div}_h\,\boldsymbol{\Sigma}_h$ is a subspace of V_h. This is satisfied when $r \le k+1$, which together with the *mild condition* let us conclude that $r = k+1$.

THEOREM 2.7. *Assume that $\boldsymbol{\sigma}|_T \in [H^t(T)]^2$ and $u|_T \in H^{1+t}(T)$ with $t > 1/2$, for any $T\in\mathcal{T}_h$. Then there exists $C_{\mathrm{err}} > 0$, independent of the mesh size, such that*

$$|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)|||^2_{DG} \le C_{\mathrm{err}}\sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}}\left\{\|\boldsymbol{\sigma}\|^2_{[H^t(T)]^2} + \|u\|^2_{H^{1+t}(T)}\right\}.$$

Proof. First, we notice that our discrete scheme (2.4) is consistent with the exact solution $(\boldsymbol{\sigma},u)$ of (1.2). This means that

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau},u-u_h) &= 0 \qquad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,\\ b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,v) + c_{DG}(u-u_h,v) &= 0 \qquad \forall\,v\in V_h. \end{aligned} \tag{2.7}$$

Now, let $\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_h$ and $\Pi_{V_h}u\in V_h$ be suitable projections of $\boldsymbol{\sigma}$ and u, respectively. By the triangle inequality, we have

$$|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)|||_{DG} \le |||(\boldsymbol{\sigma}-\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma},u-\Pi_{V_h}u)|||_{DG} + |||(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\Pi_{V_h}u-u_h)|||_{DG}. \tag{2.8}$$

Our aim is to bound $|||(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\Pi_{V_h}u-u_h)|||_{DG}$. To this end, we consider $\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}$ as the conforming Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ onto $\boldsymbol{\Sigma}_h\cap H(\mathrm{div};\Omega)$, that is, $\Pi_{\boldsymbol{\Sigma}_h} := \mathcal{E}_h^{r-1}$, while $\Pi_{V_h}u$ is the well-known L^2-orthogonal projection of u onto V_h. We also introduce the pair of projection of the error $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)$ given by

$$(e_h^{\boldsymbol{\sigma}},e_h^u) := (\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\Pi_{V_h}u-u_h)\in\boldsymbol{\Sigma}_h\times V_h.$$

In what follows, we first use the definition of the bilinear forms $a_{DG}(\cdot,\cdot)$ and $c_{DG}(\cdot,\cdot)$. Hence, by adding and subtracting the exact solution and after some algebraic manipulations, we deduce that

$$\begin{aligned}\|\boldsymbol{K}^{-1/2}e_h^{\boldsymbol{\sigma}}\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_N}^2 + \|\alpha^{1/2}[\![e_h^u]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_D}^2 \\ = a_{DG}(e_h^{\boldsymbol{\sigma}}, e_h^{\boldsymbol{\sigma}}) + c_{DG}(e_h^u, e_h^u) \\ = a_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}}) + a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, e_h^{\boldsymbol{\sigma}}) \\ + c_{DG}(\Pi_{V_h}u - u, e_h^u) + c_{DG}(u - u_h, e_h^u).\end{aligned}$$

Now, invoking (2.7), we have

$$\begin{aligned}a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, e_h^{\boldsymbol{\sigma}}) &= b_{DG}(e_h^{\boldsymbol{\sigma}}, u - u_h) \\ &= b_{DG}(e_h^{\boldsymbol{\sigma}}, u - \Pi_{V_h}u) + b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi_{V_h}u - u_h)\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}c_{DG}(u - u_h, e_h^u) &= -b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, e_h^u) \\ &= b_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^u) - b_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, e_h^u).\end{aligned} \tag{2.10}$$

Then, taking into account (2.9) and (2.10), we derive from (2.9) that

$$\begin{aligned}\|\boldsymbol{K}^{-1/2}e_h^{\boldsymbol{\sigma}}\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_N}^2 + \|\alpha^{1/2}[\![e_h^u]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_D}^2 \\ = a_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}}) - b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi_{V_h}u - u) \\ + b_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^u) + c_{DG}(\Pi_{V_h}u - u, e_h^u).\end{aligned} \tag{2.11}$$

Now, we aim to bound each one of the four terms on the right-hand side. To this end, we observe that $[\![\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}]\!] = 0$ on $\mathcal{E}_I$, and then, after applying the Cauchy-Schwarz inequality, we deduce (recall that $r = k+1$ and $\Pi_{\boldsymbol{\Sigma}_h} := \mathcal{E}_h^{r-1}$)

$$\begin{aligned}|a_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}})| \lesssim \|\boldsymbol{K}^{-1/2}(\boldsymbol{\sigma} - \Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma})\|_{0,\Omega}\|\boldsymbol{K}^{-1/2}e_h^{\boldsymbol{\sigma}}\|_{0,\Omega} \\ + \|\mathcal{E}_h^{k+1}(\boldsymbol{\sigma}) - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega}\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_N},\end{aligned} \tag{2.12}$$

$$|c_{DG}(\Pi_{V_h}u - u, e_h^u)| \le \|\alpha^{1/2}[\![\Pi_{V_h}u - u]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_D}\|\alpha^{1/2}[\![e_h^u]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_D}. \tag{2.13}$$

In addition, since $\mathrm{div}_h\,\boldsymbol{\Sigma}_h$ is a subspace of V_h and $\Pi_{V_h}u$ is the L^2-orthogonal projection of u onto V_h, the definition of $b_{DG}(\cdot,\cdot)$ implies that

$$\begin{aligned}|b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi_{V_h}u - u)| \le \left|\int_{\mathcal{E}_I}\gamma^{-1/2}\big(\{\Pi_{V_h}u - u\} - \boldsymbol{\beta}\cdot[\![\Pi_{V_h}u - u]\!]\big)\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\right| \\ + \left|\int_{\mathcal{E}_N}\gamma^{-1/2}(\Pi_{V_h}u - u)\gamma^{1/2}e_h^{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}\right|.\end{aligned}$$

Then, applying the Cauchy-Schwarz inequality, we derive that

$$\begin{aligned}&|b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi_{V_h}u - u)| \\ &\le \left(\|\gamma^{-1/2}\{\Pi_{V_h}u - u\}\|_{0,\mathcal{E}_I} + ||\boldsymbol{\beta}||_{\infty,\mathcal{E}_I}\,\|\gamma^{-1/2}[\![\Pi_{V_h}u - u]\!]\|_{0,\mathcal{E}_I}\right)\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I} \\ &\qquad + \|\gamma^{-1/2}(\Pi_{V_h}u - u)\|_{0,\mathcal{E}_N}\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_N}.\end{aligned} \tag{2.14}$$

For the remaining term, we take into consideration identity (2.6) and the fact that $\mathrm{div}(\mathcal{E}_h^k(\boldsymbol{\sigma})) = \Pi_{V_h}(\mathrm{div}(\boldsymbol{\sigma})) \in V_h$ to obtain that

$$b_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^u) = 0. \tag{2.15}$$

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Thus, applying (2.12)–(2.15) in (2.11), we conclude that there exists $C_* > 0$, independent of the mesh size, such that

$$\begin{aligned} &\|\boldsymbol{K}^{-1/2} e_h^{\boldsymbol{\sigma}}\|_{0,\Omega} + \|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_N} + \|\alpha^{1/2}[\![e_h^u]\!]\|_{0,\mathcal{E}_I\cup\mathcal{E}_D} \\ &\le C_* \Big(\|\boldsymbol{K}^{-1/2}(\boldsymbol{\sigma} - \Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma})\|_{0,\Omega} + \|\mathcal{E}_h^{k+1}(\boldsymbol{\sigma}) - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega} \\ &\quad + \|\alpha^{1/2}\{\Pi_{V_h}u - u\}\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}[\![u - \Pi_{V_h}u]\!]\|_{0,\mathcal{E}} \Big). \end{aligned} \tag{2.16}$$

Now, we focus on bounding $\|e_h^u\|_{0,\Omega}$. To this end, we first notice that for any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$

$$b_{DG}(\boldsymbol{\tau}, e_h^u) = b_{DG}(\boldsymbol{\tau}, \Pi_{V_h}u - u) + b_{DG}(\boldsymbol{\tau}, u - u_h).$$

Thanks to the first equation in (2.7), we obtain that

$$b_{DG}(\boldsymbol{\tau}, e_h^u) = b_{DG}(\boldsymbol{\tau}, \Pi_{V_h}u - u) - a_{DG}(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}) + a_{DG}(e_h^{\boldsymbol{\sigma}}, \boldsymbol{\tau}).$$

Taking into account the inf-sup condition given by Lemma 2.6 and bounding each term in the bilinear forms a_{DG} and b_{DG}, we estimate

$$\begin{aligned} \tilde{c}\|e_h^u\|_{0,\Omega} &\le \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\setminus\{0\}} \frac{b_{DG}(\boldsymbol{\tau}, e_h^u)}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} \\ &\le \hat{c}\Big(\|u - \Pi_{V_h}u\|_{0,\Omega} + \|\alpha^{1/2}\{u - \Pi_{V_h}u\}\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}[\![u - \Pi_{V_h}u]\!]\|_{0,\mathcal{E}} \\ &\quad + \|\boldsymbol{K}^{-1/2}(\boldsymbol{\sigma} - \Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma})\|_{0,\Omega} + \|\mathcal{E}_h^{k+1}(\boldsymbol{\sigma}) - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega} \Big), \end{aligned} \tag{2.17}$$

where we have also taken into account the bound for $e_h^{\boldsymbol{\sigma}}$ given in (2.16). Finally, the conclusion follows from (2.8), (2.16), (2.17), and the well-known approximation results of the projection operators $\mathcal{E}_h^k(\boldsymbol{\sigma})$ and $\Pi_{V_h}u$ that we have introduced in Lemmas 2.4 and 2.5. □

The error estimate for $\|\operatorname{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ is presented in the next theorem.

THEOREM 2.8. *Assume that $\boldsymbol{\sigma}|_T \in [H^t(T)]^2$, $\operatorname{div}(\boldsymbol{\sigma}) \in H^t(T)$, and $u|_T \in H^{1+t}(T)$, with $t > 1/2$, for each $T \in \mathcal{T}_h$. Then there exists $C_2 > 0$, independent of the mesh size, such that*

$$\begin{aligned} &\|\operatorname{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}^2 \\ &\qquad \le C_2 \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{[H^t(T)]^2}^2 + \|\operatorname{div}(\boldsymbol{\sigma})\|_{H^t(T)}^2 + \|u\|_{H^{1+t}(T)}^2 \Big\}. \end{aligned}$$

Proof. First, we denote again by $\mathcal{E}_h^k(\boldsymbol{\sigma})$ the conforming Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k onto $\boldsymbol{\Sigma}_h \cap H(\operatorname{div};\Omega)$. Then, applying the triangle inequality, we deduce

$$\|\operatorname{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \le \|\operatorname{div}_h(\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\operatorname{div}_h(\mathcal{E}_h^k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h)\|_{0,\Omega}.$$

A straightforward application of Lemma 2.5 implies that

$$\|\operatorname{div}_h(\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma}))\|_{0,\Omega} \le c \sum_{T\in\mathcal{T}_h} h_T^t \|\operatorname{div}(\boldsymbol{\sigma})\|_{t,T}.$$

For the second term, we set $\hat{e}_h^{\boldsymbol{\sigma}} := \mathcal{E}_h^k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h$. Then, given any $v \in V_h$, we have

$$\int_\Omega \operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v = \int_\Omega \operatorname{div}_h(\mathcal{E}_h^k(\boldsymbol{\sigma}))v - \int_\Omega \operatorname{div}_h(\boldsymbol{\sigma}_h)v.$$

Since $\displaystyle\int_\Omega \operatorname{div}_h(\mathcal{E}_h^k(\boldsymbol{\sigma}))v = \int_{\mathcal{T}_h} \Pi_{V_h}(\operatorname{div}(\boldsymbol{\sigma}))v = \int_\Omega \operatorname{div}(\boldsymbol{\sigma})v$, we deduce that

$$\begin{aligned}\int_\Omega \operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v &= \int_\Omega \operatorname{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)v\\ &= b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, v) + \int_{\mathcal{E}_I} (\{v\} - [\![v]\!]\cdot\boldsymbol{\beta})[\![\boldsymbol{\sigma}-\boldsymbol{\sigma}_h]\!] + \int_{\mathcal{E}_N} v(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{\nu}.\end{aligned} \tag{2.18}$$

Furthermore, using (2.7) we notice that

$$b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, v) = -c_{DG}(u-u_h, v) = c_{DG}(\Pi_{V_h}u - u, v) - c_{DG}(\Pi_{V_h}u - u_h, v).$$

Hence, replacing this identity in (2.18), we obtain

$$\begin{aligned}\int_\Omega \operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v &= c_{DG}(\Pi_{V_h}u - u, v) - c_{DG}(\Pi_{V_h}u - u_h, v)\\ &\quad + \int_{\mathcal{E}_I} (\{v\} - [\![v]\!]\cdot\boldsymbol{\beta})[\![\boldsymbol{\sigma}-\boldsymbol{\sigma}_h]\!] + \int_{\mathcal{E}_N} v\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{\nu}.\end{aligned} \tag{2.19}$$

In this way, bounding each term on the right-hand side in (2.19) and taking into account (2.16), we have

$$\begin{aligned}\|\operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})\|_{0,\Omega} &\le \sup_{v\in V_h\setminus\{0\}} \|v\|_{0,\Omega}^{-1}\int_\Omega \operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v\\ &\le C\Big(\|\boldsymbol{K}^{-1/2}(\boldsymbol{\sigma}-\mathcal{E}_h^k(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\mathcal{E}_h^{k+1}(\boldsymbol{\sigma}) - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega}\\ &\quad + ||\gamma^{1/2}[\![\boldsymbol{\sigma}-\boldsymbol{\sigma}_h]\!]||_{0,\mathcal{E}_I\cup\mathcal{E}_N}\\ &\quad + \|\alpha^{1/2}\{\Pi_{V_h}u - u\}\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}[\![\Pi_{V_h}u - u]\!]\|_{0,\mathcal{E}}\Big).\end{aligned}$$

Finally, the conclusion follows from Lemmas 2.4 and 2.5 and Theorem 2.7. ◻

REMARK 2.9. In summary, the analysis developed in this section shows us that $[\mathbb{P}_{k+1}(\mathcal{T}_h)]^2 \times \mathbb{P}_k(\mathcal{T}_h)$, with $k \ge 0$, define a set of stable pairs for the dual-mixed DG approach (2.4).

REMARK 2.10. Let $\rho > 0$ be the density, $\mathbf{g}$ the gravity vector, g_c a conversion constant, φ the volumetric flow rate source or sink, and ψ the normal component of the velocity field on the boundary such that the data φ and ψ satisfy the compatibility constraint $\int_\Omega \varphi = \int_\Gamma \psi$. Denoting by $\boldsymbol{f} := -\frac{\rho}{g_c}\mathbf{g}$, we have a version of the Darcy problem: Find the Darcy velocity vector $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ and the pressure $p : \Omega \to \mathbb{R}$ such that

$$\left\{\begin{aligned} \boldsymbol{v} + \boldsymbol{\mathcal{K}}\nabla p &= \boldsymbol{f} && \text{in } \Omega,\\ \operatorname{div}(\boldsymbol{v}) &= \varphi && \text{in } \Omega,\\ \boldsymbol{v}\cdot\boldsymbol{\nu} &= \psi && \text{on } \Gamma,\end{aligned}\right. \tag{2.20}$$

where $\boldsymbol{\mathcal{K}} \in [L^\infty(\Omega)]^{2\times 2}$ in general is a given symmetric and uniformly positive-definite matrix-valued function. However, in many applications it is assumed that the medium isisotropic.

This allows us to set $\mathcal{K} := \frac{\kappa}{\mu}\boldsymbol{I}$, where $\kappa > 0$ and $\mu > 0$ denote, respectively, the permeability and the viscosity of the porous medium and $\boldsymbol{I}$ being the identity matrix. Then it is not difficult to see that the treatment of the model problem (1.1) is similar to a Poisson problem with Neumann boundary conditions, when considering a dual-mixed formulation. Therefore, the results in Section 2 can be extended to the Darcy flow problem (2.20) in a natural way, once the analysis of the Poisson problem with mixed boundary conditions is extended to Neumann boundary conditions.

3. The Stokes system. In this section, we concentrate our efforts on the extension of the results developed in Section 2 to the incompressible Stokes problem: Given the source terms $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\boldsymbol{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\boldsymbol{u}$ and the pressure p that satisfy

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega,\\
\operatorname{div}(\boldsymbol{u}) &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma,
\end{aligned}
\tag{3.1}
$$

where $\nu > 0$ represents the viscosity of the fluid. Hereafter, Ω is a bounded and simply connected domain in the plane with polygonal boundary Γ.

Next, we proceed to write (3.1) as a linear system of first order. To this aim, we proceed as in [9] (cf. Section 2) and introduce the pseudo stress $\boldsymbol{\sigma} := \nu\nabla\boldsymbol{u} - p\boldsymbol{I}$ in Ω. This allows us to eliminate the pressure in (3.1) since it is not difficult to deduce $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$. In order to ensure the uniqueness of the solution of (3.1), we require that $p \in L^2_0(\Omega)$, which is equivalent to ask that $\boldsymbol{\sigma}$ lives in $\boldsymbol{\Sigma}_0 := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\}$. Then, we arrive at the dual-mixed formulation:

Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{\Sigma}_0 \times [H^1(\Omega)]^2$ *such that*

$$
\begin{aligned}
\boldsymbol{\sigma}^{\mathrm{d}} &= \nu\nabla\boldsymbol{u} && \text{in } \Omega,\\
\mathbf{div}(\boldsymbol{\sigma}) &= -\boldsymbol{f} && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma.
\end{aligned}
\tag{3.2}
$$

In order to approximate the solution of problem (3.2), we consider the DG scheme introduced and analyzed in [9]. To this end, we introduce the discrete spaces $\boldsymbol{\Sigma}_h$, $\boldsymbol{\Sigma}_{h,0}$, and $\boldsymbol{V}_h$ as follows:

$$
\begin{aligned}
\boldsymbol{\Sigma}_h &:= \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\tau}\big|_T \in [\mathbb{P}_r(T)]^{2\times 2} \quad \forall T \in \mathcal{T}_h\Big\},\\
\boldsymbol{\Sigma}_{h,0} &:= \Big\{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\Big\},\\
\boldsymbol{V}_h &:= \Big\{\boldsymbol{v} \in [L^2(\Omega)]^2 : \quad \boldsymbol{v}\big|_T \in [\mathbb{P}_k(T)]^2 \quad \forall T \in \mathcal{T}_h\Big\},
\end{aligned}
$$

with $k \geq 0$ and $r \geq 1$. Then, problem (3.2) reads:

Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *such that*

$$
\begin{aligned}
a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u}_h) &= G_{DG}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0},\\
-b_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h, \boldsymbol{v}) &= F_{DG}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h.
\end{aligned}
\tag{3.3}
$$

Here the bilinear forms $a_{DG} : \underline{\mathbf{\Sigma}} \times \underline{\mathbf{\Sigma}} \to \mathbb{R}$, $c_{DG} : [H^\epsilon(\mathcal{T}_h)]^2 \times [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$, and $b_{DG} : \underline{\mathbf{\Sigma}} \times [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$ are defined by

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}, \boldsymbol{\tau}) &:= \frac{1}{\nu} \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_{\mathcal{E}_I} \gamma [\![\boldsymbol{\sigma}]\!] \cdot [\![\boldsymbol{\tau}]\!], \\ c_{DG}(\boldsymbol{w}, \boldsymbol{v}) &:= \int_{\mathcal{E}} \alpha \underline{[\![\boldsymbol{v}]\!]} : \underline{[\![\boldsymbol{w}]\!]}, \\ b_{DG}(\boldsymbol{\tau}, \boldsymbol{v}) &:= \int_\Omega \boldsymbol{v} \cdot \mathbf{div}_h(\boldsymbol{\tau}) - \int_{\mathcal{E}_I} \left(\{\boldsymbol{v}\} + \underline{[\![\boldsymbol{v}]\!]} \boldsymbol{\beta} \right) \cdot [\![\boldsymbol{\tau}]\!], \end{aligned}$$

while the functionals $G_{DG} : \underline{\mathbf{\Sigma}} \to \mathbb{R}$ and $F_{DG} : [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$ are given by

$$G_{DG}(\boldsymbol{\tau}) := \int_{\mathcal{E}_\Gamma} \boldsymbol{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu} \qquad \text{and} \qquad F_{DG}(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\mathcal{E}_\Gamma} \alpha(\boldsymbol{g} \otimes \boldsymbol{\nu}) : (\boldsymbol{v} \otimes \boldsymbol{\nu}).$$

We point out that $\epsilon > 1/2$, and the parameters α, γ, and $\boldsymbol{\beta}$ introduced here to define the numerical fluxes are at our disposal. Indeed, they will be defined as in Section 2. Now, considering $\underline{\mathbf{\Sigma}}_0 := \{\boldsymbol{\tau} \in \underline{\mathbf{\Sigma}} : \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0\}$, we introduce the seminorm

$$|(\boldsymbol{\tau}, \boldsymbol{v})|_{\underline{DG}} := \left(\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\gamma^{1/2} [\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\boldsymbol{v}\|^2_{[L^2(\Omega)]^2} \right)^{1/2} \qquad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \underline{\mathbf{\Sigma}}_0 \times [L^2(\Omega)]^2,$$

and the norm

$$\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{\underline{DG}} := \left(|(\boldsymbol{\tau}, \boldsymbol{v})|^2_{\underline{DG}} + \|\,\mathbf{div}_h(\boldsymbol{\tau})\|^2_{0,\Omega} \right)^{1/2} \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \underline{\mathbf{\Sigma}}_0 \times [L^2(\Omega)]^2.$$

REMARK 3.1. At this point, thanks to [7, Lemma 3.1] (see also [9, Lemma 3.10]), we note that the norm $\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{\underline{DG}}$ is equivalent on $\underline{\mathbf{\Sigma}}_0 \times [L^2(\Omega)]^2$ to the standard one defined by

$$|||(\boldsymbol{\tau}, \boldsymbol{v})|||_{\underline{DG}} := \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\,\mathbf{div}_h(\boldsymbol{\tau})\|^2_{0,\Omega} + \|\gamma^{1/2} [\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\boldsymbol{v}\|^2_{[L^2(\Omega)]^2} \right)^{1/2}.$$

Now with the aim to ensure existence and uniqueness, hereafter we assume that $\nabla_h \boldsymbol{V}_h$ is a subspace of $\mathbf{\Sigma}_h$. Then, the proof of the well-posedness of (3.3), under this assumption, is very similar to the one developed in [9, Section 3].

THEOREM 3.2. *Under the assumption that $\boldsymbol{\nabla}_h \boldsymbol{V}_h$ is a subspace of $\mathbf{\Sigma}_h$, problem* (3.3) *has one and only one solution.*

Proof. Since the linear system (3.3) is square, it is enough to show that the corresponding homogeneous system has only the trivial solution:

Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$ such that

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u}_h) &= 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{\Sigma}_{h,0}, \\ -b_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h, \boldsymbol{v}) &= 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h. \end{aligned} \tag{3.4}$$

To this end, we replace $\boldsymbol{\tau} := \boldsymbol{\sigma}_h$ and $\boldsymbol{v} := \boldsymbol{u}_h$ in (3.4) and, after summing the equations, we deduce

$$\frac{1}{\nu} \|\boldsymbol{\sigma}_h^{\mathrm{d}}\|^2_{0,\Omega} + \|\gamma^{1/2} [\![\boldsymbol{\sigma}_h]\!]\|^2_{0,\mathcal{E}_I} + \|\alpha^{1/2} \underline{[\![\boldsymbol{u}_h]\!]}\|^2_{0,\mathcal{E}} = 0.$$

This lets us to infer that

$$\begin{aligned} \boldsymbol{\sigma}_h^{\mathrm{d}} = \mathbf{0} \quad &\text{in} \quad \Omega \qquad &&\Leftrightarrow \qquad \boldsymbol{\sigma}_h = \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\boldsymbol{I}, \\ [\![\boldsymbol{\sigma}_h]\!] = \mathbf{0} \quad &\text{on} \quad \mathcal{E}_I \qquad &&\Leftrightarrow \qquad \boldsymbol{\sigma}_h \in H(\mathbf{div};\Omega), \\ \underline{[\![\boldsymbol{u}_h]\!]} = \mathbf{0} \quad &\text{on} \quad \mathcal{E} \qquad &&\Leftrightarrow \qquad \Big(\boldsymbol{u}_h \in C(\overline{\Omega}) \wedge \boldsymbol{u}_h = \mathbf{0} \quad \text{on} \quad \mathcal{E}_\Gamma\Big). \end{aligned} \tag{3.5}$$

Then, system (3.4) is reduced to

$$\int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u}_h : \boldsymbol{\tau} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}, \tag{3.6}$$

$$\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h) \cdot \mathrm{div}(\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{3.7}$$

Since $\boldsymbol{\nabla}_h \boldsymbol{u}_h \in \boldsymbol{\Sigma}_h$ and

$$\int_\Omega \mathrm{tr}(\boldsymbol{\nabla}_h \boldsymbol{u}_h) = \int_\Omega \mathrm{div}_h(\boldsymbol{u}_h) = \sum_{T\in\mathcal{T}_h} \int_T \mathrm{div}(\boldsymbol{u}_h) = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \boldsymbol{u}_h \cdot \boldsymbol{\nu} = \int_{\mathcal{E}_\Gamma} \boldsymbol{u}_h \cdot \boldsymbol{\nu} = 0,$$

we conclude that $\tilde{\boldsymbol{\tau}} := \boldsymbol{\nabla}_h \boldsymbol{u}_h \in \boldsymbol{\Sigma}_{h,0}$. Then, after testing (3.6) with $\tilde{\boldsymbol{\tau}}$, we obtain $\boldsymbol{\nabla}_h \boldsymbol{u}_h = \mathbf{0}$ in Ω, which together with (3.5) implies that $\boldsymbol{u}_h \in [\mathbb{P}_0(\overline{\Omega})]^2$. Since $\boldsymbol{u}_h = \mathbf{0}$ on Γ, it is concluded that $\boldsymbol{u}_h = \mathbf{0}_{\boldsymbol{V}_h}$. Now, since $\boldsymbol{\sigma}_h \in H(\mathbf{div};\Omega)$, there exists a unique $\boldsymbol{w} \in [H_0^1(\Omega)]^2$ such that $\mathrm{div}(\boldsymbol{w}) = \mathrm{tr}(\boldsymbol{\sigma}_h)$. Therefore, setting $\boldsymbol{w}_h$ as the piecewise, local Raviart-Thomas projection of $\boldsymbol{w}$ of order k (i.e., $\boldsymbol{w}_h|_T := \mathcal{E}_T^k(\boldsymbol{w})$ for each $T \in \mathcal{T}_h$) and replacing it in (3.7), we derive that $\mathrm{tr}(\boldsymbol{\sigma}_h) = 0$ in Ω. Thus $\boldsymbol{\sigma}_h = \mathbf{0}$, and we conclude the proof. □

To ensure the stability of the discrete scheme, we require that $\mathbf{div}_h\, \boldsymbol{\Sigma}_h$ is a subspace of $\boldsymbol{V}_h$. This condition, together with the *mild condition* required in Theorem 3.2 allow us to conclude that they are valid when $r = k+1$. This leads us to deal with the pair of approximation spaces $[\mathbb{P}_{k+1}(\mathcal{T}_h)]^{2\times 2} \times [\mathbb{P}_k(\mathcal{T}_h)]^2$ as for the Poisson problem in Section 2.

From now on, $(\boldsymbol{\sigma}, \boldsymbol{u})$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ will be the unique solutions of (3.2) and (3.3), respectively.

THEOREM 3.3. *Assume in addition that* $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$ *and* $\boldsymbol{u}|_T \in [H^1(T)]^2$ *with* $t > 1/2$, *for all* $T \in \mathcal{T}_h$. *Then we have*

$$|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)|^2_{\underline{DG}} \le C_{\text{err}} \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} + \|\boldsymbol{u}\|^2_{[H^1(T)]^2} \Big\},$$

where $C_{\text{err}} > 0$ *is independent of* h.

Proof. First, we notice that our discrete scheme (3.3) is consistent, i.e, if $(\boldsymbol{\sigma}, \boldsymbol{u})$ is the exact solution of (3.2), then

$$\left\{ \begin{array}{ll} a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u} - \boldsymbol{u}_h) = 0 & \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}, \\ -b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = 0 & \forall\, \boldsymbol{v} \in \boldsymbol{V}_h. \end{array} \right. \tag{3.8}$$

Let $\Pi\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{h,0}$ and $\Pi\boldsymbol{u} \in \boldsymbol{V}_h$ be suitable projections of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$, respectively. By the triangle inequality, we have

$$|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)|_{\underline{DG}} \le |(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}, \boldsymbol{u} - \Pi\boldsymbol{u})|_{\underline{DG}} + |(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi\boldsymbol{u} - \boldsymbol{u}_h)|_{\underline{DG}}. \tag{3.9}$$

Our aim is to bound $|(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi\boldsymbol{u} - \boldsymbol{u}_h)|_{\underline{DG}}$. To this end, we let $\Pi\boldsymbol{\sigma}$ be the L^2-orthogonal projection of $\boldsymbol{\sigma}$ onto $\boldsymbol{\Sigma}_h \cap [C(\overline{\Omega})]^{2\times 2}$, while $\Pi \boldsymbol{u}$ denotes the L^2-projection of $\boldsymbol{u}$ onto $\boldsymbol{V}_h$. We also introduce $(e_h^{\boldsymbol{\sigma}}, e_h^{\boldsymbol{u}}) := (\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi\boldsymbol{u} - \boldsymbol{u}_h)$. We notice that

$$\int_\Omega \operatorname{tr}(e_h^{\boldsymbol{\sigma}}) = \int_\Omega e_h^{\boldsymbol{\sigma}} : \boldsymbol{I} = \int_\Omega (\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \boldsymbol{I} = \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \boldsymbol{I} = \int_\Omega \operatorname{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = 0,$$

and thus, $e_h^{\boldsymbol{\sigma}} \in \boldsymbol{\Sigma}_{h,0}$.

Next, we test (3.8) with $(\boldsymbol{\tau}, \boldsymbol{v}) := (e_h^{\boldsymbol{\sigma}}, e_h^{\boldsymbol{u}})$. After adding all the equations, we deduce

$$\begin{aligned}
&\nu^{-1}\|\big(e_h^{\boldsymbol{\sigma}}\big)^{\mathrm{d}}\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I}^2 + \|\alpha^{1/2}\underline{[\![e_h^{\boldsymbol{u}}]\!]}\|_{0,\mathcal{E}}^2 \\
&\quad = a_{DG}(e_h^{\boldsymbol{\sigma}}, e_h^{\boldsymbol{\sigma}}) + c_{DG}(e_h^{\boldsymbol{u}}, e_h^{\boldsymbol{u}}) \\
&\quad = a_{DG}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}}) + b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi\boldsymbol{u} - \boldsymbol{u}) \\
&\qquad\qquad - b_{DG}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{u}}) + c_{DG}(\Pi\boldsymbol{u} - \boldsymbol{u}, e_h^{\boldsymbol{u}}).
\end{aligned} \tag{3.10}$$

Now we bound each term on the right-hand side of (3.10) and derive that

$$\begin{aligned}
|a_{DG}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}})| &\le \frac{1}{\nu}\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega}\|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{0,\Omega}, \\
|c_{DG}(\Pi\boldsymbol{u} - \boldsymbol{u}, e_h^{\boldsymbol{u}})| &\le \|\alpha^{1/2}[\![\Pi\boldsymbol{u} - \boldsymbol{u}]\!]\|_{0,\mathcal{E}}\|\alpha^{1/2}[\![e_h^{\boldsymbol{u}}]\!]\|_{0,\mathcal{E}}.
\end{aligned}$$

In addition, using that $\mathbf{div}(\boldsymbol{\Sigma}_h)$ is a subspace of $\boldsymbol{V}_h$ and $\Pi\boldsymbol{u}$ is the L^2-orthogonal projection of $\boldsymbol{u}$, from the definition of b_{DG}, we deduce that

$$\begin{aligned}
|b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi\boldsymbol{u} - \boldsymbol{u})| &= \left|\int_{\mathcal{E}_I} \gamma^{-1/2}\big(\{\Pi\boldsymbol{u} - \boldsymbol{u}\} - \boldsymbol{\beta}\cdot[\![\Pi\boldsymbol{u} - \boldsymbol{u}]\!]\big)\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\right| \\
&\le c\left(\|\gamma^{-1/2}\{\Pi\boldsymbol{u} - \boldsymbol{u}\}\|_{0,\mathcal{E}_I} + \|\gamma^{-1/2}[\![\Pi\boldsymbol{u} - \boldsymbol{u}]\!]\|_{0,\mathcal{E}_I}\right)\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I}.
\end{aligned}$$

For the last term, we introduce the conforming Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k, $\mathcal{E}_h^k(\boldsymbol{\sigma})$. Then, we have $\mathbf{div}(\mathcal{E}_h^k(\boldsymbol{\sigma})) \in \boldsymbol{V}_h$ and $[\![\mathcal{E}_h^k(\boldsymbol{\sigma})]\!] = 0$ on $\mathcal{E}_I$ (see [28, Section 3]). Denoting $\tilde{e}_h^{\boldsymbol{\sigma}} := \Pi\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma})$ and using (3.8), we note that

$$\begin{aligned}
b_{DG}&(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, e_h^{\boldsymbol{u}}) = \int_\Omega \mathbf{div}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma})\cdot e_h^{\boldsymbol{u}} = \int_\Omega \mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot e_h^{\boldsymbol{u}} \\
&= \int_\Omega \mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot(\Pi\boldsymbol{u} - \boldsymbol{u}_h) = \int_\Omega \mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot(\boldsymbol{u} - \boldsymbol{u}_h) \\
&= -a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{e}_h^{\boldsymbol{\sigma}}) = -\frac{1}{\nu}\int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^{\mathrm{d}} : (\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathrm{d}} = -\frac{1}{\nu}\int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : (\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathrm{d}} \\
&= -\frac{1}{\nu}\int_\Omega (\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : (\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathrm{d}} = -\frac{1}{\nu}\int_\Omega (\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^{\mathrm{d}} : (\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathrm{d}} = -a_{DG}(e_h^{\boldsymbol{\sigma}}, \tilde{e}_h^{\boldsymbol{\sigma}}) \\
&\le \frac{1}{\nu}\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega}\|(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega} \le \frac{1}{\nu}\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega}\|\tilde{e}_h^{\boldsymbol{\sigma}}\|_{0,\Omega} \\
&\le \frac{1}{\nu}\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega}\left(\|\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{0,\Omega}\right).
\end{aligned}$$

In this way, we deduce that there exists $C_* > 0$, independent of the mesh size, such that

$$
\begin{aligned}
&\frac{1}{\sqrt{\nu}}\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega} + \|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}\underline{[\![e_h^{\boldsymbol{u}}]\!]}\|_{0,\mathcal{E}} \\
&\quad \leq C_*\Big(\|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{0,\Omega} + \|\gamma^{-1/2}\{\Pi\boldsymbol{u} - \boldsymbol{u}\}\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}\underline{[\![\boldsymbol{u} - \Pi\boldsymbol{u}]\!]}\|_{0,\mathcal{E}} \\
&\qquad + \|\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega}\Big).
\end{aligned} \tag{3.11}
$$

On the other hand, concerning $\|e_h^{\boldsymbol{u}}\|_{0,\Omega}$, we first notice that for any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}$,

$$
b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}}) = b_{DG}(\boldsymbol{\tau}, \Pi\boldsymbol{u} - \boldsymbol{u}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u} - \boldsymbol{u}_h),
$$

and using the first equation in (3.8), we deduce that

$$
b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}}) = b_{DG}(\boldsymbol{\tau}, \Pi\boldsymbol{u} - \boldsymbol{u}) + a_{DG}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}) - a_{DG}(e_h^{\boldsymbol{\sigma}}, \boldsymbol{\tau}).
$$

Thanks to an inf-sup condition analogous to that in [9, Lemma 3.4] (whose proof should be quite similar to the one of Lemma 2.6) and bounding each term in the bilinear forms a_{DG} and b_{DG}, we estimate

$$
\begin{aligned}
\tilde{c}\|e_h^{\boldsymbol{u}}\|_{0,\Omega} &\leq \sup_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}\setminus\{\boldsymbol{\theta}\}} \frac{b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}})}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} \\
&\leq \hat{c}\Big(\|\gamma^{-1/2}\{\Pi\boldsymbol{u} - \boldsymbol{u}\}\|_{0,\mathcal{E}_I} + \|\alpha^{1/2}\underline{[\![\boldsymbol{u} - \Pi\boldsymbol{u}]\!]}\|_{0,\mathcal{E}} \\
&\qquad + \|\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{0,\Omega}\Big),
\end{aligned}
$$

where we have also taken into account the bound for $\|(e_h^{\boldsymbol{\sigma}})^{\mathrm{d}}\|_{0,\Omega}$ given in (3.11).

Finally, the conclusion follows from (3.9) and the well-known approximation results for the projection operators $\Pi\boldsymbol{\sigma}$, $\mathcal{E}_h^k(\boldsymbol{\sigma})$, and $\Pi\boldsymbol{u}$. □

The L^2-error of $\mathbf{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$ is presented in the next theorem.

THEOREM 3.4. *Assume that* $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^t(T)]^2$, *and* $\boldsymbol{u}|_T \in [H^{1+t}(T)]^2$, *with* $t > \frac{1}{2}$, *for each* $T \in \mathcal{T}_h$. *Then there exists* $C_2 > 0$, *independent of the mesh size, such that*

$$
\begin{aligned}
&\|\mathbf{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}^2 \\
&\quad \leq C_2 \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{[H^t(T)]^{2\times 2}}^2 + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^t(T)]^2}^2 + \|\boldsymbol{u}\|_{[H^{1+t}(T)]^2}^2 \Big\}.
\end{aligned}
$$

Proof. First, we denote again by $\mathcal{E}_h^k(\boldsymbol{\sigma})$ the continuous Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k. Then, applying the triangle inequality, we deduce that

$$
\|\mathbf{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \leq \|\mathbf{div}(\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\mathbf{div}_h(\mathcal{E}_h^k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h)\|_{0,\Omega}.
$$

A straightforward application of the local interpolation property implies that

$$
\|\mathbf{div}_h(\boldsymbol{\sigma} - \mathcal{E}_h^k(\boldsymbol{\sigma}))\|_{0,\Omega} \leq c \sum_{T\in\mathcal{T}_h} h_T^t \|\mathbf{div}(\boldsymbol{\sigma})\|_{t,T}.
$$

For the second term, we set $\hat{e}_h^{\boldsymbol{\sigma}} := \mathcal{E}_h^k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h$. Let $\boldsymbol{v} \in \boldsymbol{V}_h$. Then we have

$$
\int_\Omega \mathbf{div}_h(\hat{e}_h^{\boldsymbol{\sigma}}) \cdot \boldsymbol{v} = \int_\Omega \mathbf{div}(\mathcal{E}_h^k(\boldsymbol{\sigma})) \cdot \boldsymbol{v} - \int_\Omega \mathbf{div}_h(\boldsymbol{\sigma}_h) \cdot \boldsymbol{v}.
$$

Since $\displaystyle\int_\Omega \mathbf{div}(\mathcal{E}_h^k(\boldsymbol{\sigma}))\cdot\boldsymbol{v} = \int_{\mathcal{T}_h} \Pi_k(\mathbf{div}(\boldsymbol{\sigma}))\cdot\boldsymbol{v} = \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\boldsymbol{v}$, we deduce

$$\begin{aligned}\int_\Omega \mathbf{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})\cdot\boldsymbol{v} &= \int_\Omega \mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{v}\\ &= b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{v}) - \int_{\mathcal{E}_I}(\{\boldsymbol{v}\}+\underline{[\![\boldsymbol{v}]\!]}\,\boldsymbol{\beta})\cdot[\![\boldsymbol{\sigma}_h]\!]\,.\end{aligned}$$

Furthermore, using (3.8) we note that

$$b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{v}) = c_{DG}(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) = c_{DG}(\Pi\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) + c_{DG}(\boldsymbol{u}-\Pi\boldsymbol{u},\boldsymbol{v}).$$

Hence, replacing this identity in the above equality, we obtain

$$\begin{aligned}\int_\Omega \mathbf{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})\cdot\boldsymbol{v} &= b_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{v}) - \int_{\mathcal{E}_I}(\{\boldsymbol{v}\}+\underline{[\![\boldsymbol{v}]\!]}\,\boldsymbol{\beta})\cdot[\![\boldsymbol{\sigma}_h]\!]\\ &= c_{DG}(\Pi\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) + c_{DG}(\boldsymbol{u}-\Pi\boldsymbol{u},\boldsymbol{v}) - \int_{\mathcal{E}_I}(\{\boldsymbol{v}\}+\underline{[\![\boldsymbol{v}]\!]}\,\boldsymbol{\beta})\cdot[\![\boldsymbol{\sigma}_h]\!]\\ &= c_{DG}(e_h^{\boldsymbol{u}},\boldsymbol{v}) + c_{DG}(\boldsymbol{u}-\Pi\boldsymbol{u},\boldsymbol{v}) - \int_{\mathcal{E}_I}(\{\boldsymbol{v}\}+\underline{[\![\boldsymbol{v}]\!]}\,\boldsymbol{\beta})\cdot[\![\boldsymbol{\sigma}_h]\!]\,.\end{aligned}$$

In this way, bounding each term of the bilinear forms and using (3.11), we deduce that

$$\begin{aligned}\|\,\mathbf{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})\|_{0,\Omega} &\le \sup_{\boldsymbol{v}\in\boldsymbol{V}_h\setminus\{0\}} \frac{\displaystyle\int_\Omega \mathbf{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|_{0,\Omega}}\\ &\le C\Big(\|\gamma^{-1/2}\{\Pi\boldsymbol{u}-\boldsymbol{u}\}\|_{0,\mathcal{E}_I} + \|\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}\|_{0,\Omega} + \|\gamma^{1/2}\,[\![\boldsymbol{\sigma_h}]\!]\|_{0,\mathcal{E}_I}\\ &\qquad + \|\alpha^{1/2}[\![\boldsymbol{u}-\Pi\boldsymbol{u}]\!]\|_{0,\mathcal{E}} + \|\boldsymbol{\sigma}-\mathcal{E}_h^k(\boldsymbol{\sigma})\|_{0,\Omega}\Big).\end{aligned}$$

Thus, considering that $\gamma = \mathcal{O}(\mathsf{h}^{-1})$ and $\alpha = \mathcal{O}(\mathsf{h})$, the proof follows by applying Theorem 3.3 as well as well-known approximation properties in Sobolev spaces. □

REMARK 3.5. In summary, the analysis developed in this section allows us to consider the set of pairs $[\mathbb{P}_{k+1}(\mathcal{T}_h)]^{2\times2}\times[\mathbb{P}_k(\mathcal{T}_h)]^2$, with $k\ge 0$, since each one of them is stable for the dual-mixed DG approach (3.3) of the Stokes system.

REMARK 3.6. Consider the Stokes problem with mixed boundary conditions: Given the source terms $\boldsymbol{f}\in[L^2(\Omega)]^2$ and $\boldsymbol{g}_N\in[H^{-1/2}(\Gamma_N)]^2$, we look for the velocity $\boldsymbol{u}$ and the pressure p that satisfy

$$\begin{aligned}-\nu\Delta\boldsymbol{u}+\nabla p &= \boldsymbol{f} && \text{in } \Omega,\\ \operatorname{div}(\boldsymbol{u}) &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D,\\ 2\nu\epsilon(\boldsymbol{u})\boldsymbol{n}-p\boldsymbol{I} &= \boldsymbol{g}_N && \text{on } \Gamma_N,\end{aligned}$$

with Γ_D and Γ_N being a partition of $\partial\Omega$ and $\boldsymbol{n}$ the outward unit normal to $\partial\Omega$. Due to the Neumann boundary condition, it is not a good idea to introduce the pseudo stress as an auxiliary unknown. However, by introducing the symmetric new variable $\boldsymbol{\tau} := 2\nu\epsilon(\boldsymbol{u}) - p\boldsymbol{I}$, we are able to eliminate the pressure since $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})$ in Ω. Thus, eliminating the pressure, we arrive at the first-order system:

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Find $(\boldsymbol{\tau}, \boldsymbol{u}) \in \boldsymbol{\Sigma}_0 \times [H^1(\Omega)]^2$ *such that*

$$\begin{aligned} \boldsymbol{\tau}^{\mathrm{d}} &= 2\nu\epsilon(\boldsymbol{u}) && \text{in } \Omega,\\ \mathbf{div}(\boldsymbol{\tau}) &= -\boldsymbol{f} && \text{in } \Omega,\\ \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma_D\\ \boldsymbol{\tau}\boldsymbol{n} &= \boldsymbol{g}_N && \text{on } \Gamma_N, \end{aligned}$$

where $\boldsymbol{\Sigma}_0 := \{\boldsymbol{\zeta} \in H(\mathbf{div} : \Omega) : \boldsymbol{\zeta}^{\mathrm{t}} = \boldsymbol{\zeta} \text{ in } \Omega \text{ and } \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0\}$.

Therefore, we think that our method is appropriate for approximating this problem. It is easy to impose the symmetry of $\boldsymbol{\tau}$ strongly in the discrete space. Now, since our intention here is to exhibit a simple extension to the Stokes problem from the Poisson problem and to avoid a lengthy article, we omit the analysis of this mixed-boundary condition for the Stokes problem in this paper. This will be addressed in a separate work.

REMARK 3.7. The extension of the analysis described in this paper to 3 spatial dimension is quite natural. Just replace *edge* by *face*, and so on.

4. Numerical examples. In this section we present several examples illustrating our results for the Poisson problem (cf. (2.5)) and the Stokes problem (cf. (3.3)). All the numerical results given below have been obtained using a MATLAB code. The errors on each triangle are computed with a 7-point quadrature rule. We consider, for both problems, the lowest polynomial approximation spaces: $[\mathbb{P}_1(\mathcal{T}_h)]^2 - \mathbb{P}_0(\mathcal{T}_h)$ and $[\mathbb{P}_1(\mathcal{T}_h)]^{2\times 2} - [\mathbb{P}_0(\mathcal{T}_h)]^2$ for the Poisson and Stokes problem, respectively (which means that in this case $k = 0$). Concerning the parameters that define both discrete schemes, we set $\boldsymbol{\beta} := (1,1)^t$, $\gamma := 1/\mathtt{h}$, and $\alpha := \mathtt{h}$.

4.1. Numerical examples for the Poisson equation. Here, we first introduce some useful notations for the errors and the experimental rates of convergence. Let N be the number of degrees of freedom, and define

$$\boldsymbol{e}_0(u) := \|u - u_h\|_{0,\Omega}, \qquad \boldsymbol{e}_0(\boldsymbol{\sigma}) := \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{0,\Omega} + \|\gamma^{1/2}[\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]\|^2_{0,\mathcal{E}_I\cup\mathcal{E}_N}\right)^{1/2},$$
$$\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma}) := \|\,\mathrm{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}, \qquad \boldsymbol{e} := \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)\|_{DG}.$$

We point out that in this case we have $N = 7 \times \mathrm{card}(\mathcal{T}_h)$. Considering that in 2D, h behaves as $N^{-1/2}$, we set the so-called experimental rate of convergence of the global error $\boldsymbol{e}$ as

$$r := -2\frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')},$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ denote the corresponding errors at two consecutive triangulations with the number of degrees of freedom N and N', respectively. The experimental rates of convergence for the other errors are defined in an analogous way.

We present three examples. In all of them, we consider $\boldsymbol{K} := \boldsymbol{I}$. Their domain Ω as well as their corresponding exact solution u are given in Table 4.1. In Example 1, problem (2.5) is solved, considering Dirichlet boundary ($|\Gamma_N| = 0$) and mixed boundary conditions with $\Gamma_N := [0,1] \times \{1\} \cup \{0\} \times [0,1]$. We notice that the exact solution is a smooth function. Thus, we expect that the rate of convergence for the global error $\boldsymbol{e}$ is close to 1 as well as for $\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma})$, since we are using the lowest order of the discrete approximation space for each unknown. The results shown in Tables 4.3 and 4.4 are in agreement with this. The exact solutions for Examples 2 and 3 are the same, are given in polar coordinates, and live in $H^{1+2/3}(\Omega)$, since their gradients have a singularity at the origin. For these examples, we consider $\Gamma_N := \{0\} \times [-1,0] \cup [0,1] \times \{0\}$ and $\Gamma_D := \partial\Omega\backslash\Gamma_N$. We point out that the results

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TABLE 4.1
Examples considered for the Poisson problem with Dirichlet and mixed boundary conditions.

EXAMPLE	Ω	$u(x_1, x_2)$
1	$(0,1)^2$	$\sin(\pi x_1)\sin(\pi x_2)$
2	$(-1,1)^2 \backslash [0,1]\times[-1,0]$	$r^{2/3}\sin\left(\frac{2}{3}\theta\right)$
3	$\{(x_1,x_2): x_1^2 + x_2^2 = 1\}\backslash[0,1]\times[-1,0]$	$r^{2/3}\sin\left(\frac{2}{3}\theta\right)$

for Example 3 are not covered by the current work since the corresponding domain does not have a polygonal boundary. The rates of convergence for each one of the introduced errors behaves as $\mathcal{O}(h)$, as predicted by the theorems since in these cases $\operatorname{div}(\boldsymbol{\sigma}) = 0$. They are shown in Tables 4.5 and 4.6.

4.2. Numerical examples for the Stokes system. We first note that since the search of a suitable basis of $\boldsymbol{\Sigma}_{h,0}$ is very difficult, we introduce the zero mean value condition of the trace of elements of $\boldsymbol{\Sigma}_h$ with the help of a Lagrange multiplier. This allows us to establish the following result.

THEOREM 4.1. *Consider the problem:*
Find$(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \lambda) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ such that

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u}_h) + \lambda \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) &= G_{DG}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\ -b_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h, \boldsymbol{v}) &= F_{DG}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h, \\ \mu \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) &= 0 \qquad\quad\ \forall\, \mu \in \mathbb{R}. \end{aligned} \tag{4.1}$$

Then, we have

1. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \lambda) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *is a solution of* (4.1)*, then* $\lambda = 0$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *is a solution of* (3.3)*.*
2. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *is a solution of* (3.3)*, then* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *is a solution of* (4.1)*.*

We proceed to implement (4.1). As for the Poisson problem, we need to introduce some useful notations for the errors and the experimental rates of convergence. We let N be the number of degrees of freedom, which in our case corresponds to $N = 14 \times \operatorname{card}(\mathcal{T}_h) + 1$. We also introduce

$$\boldsymbol{e}_0(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}, \qquad \boldsymbol{e}_0(\boldsymbol{\sigma}) := \left(\|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,\Omega}^2 + \|\gamma^{1/2}[\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]\|_{0,\mathcal{E}_I}^2\right)^{1/2},$$
$$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma}) := \|\,\mathbf{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}, \qquad \boldsymbol{e} := |(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|_{\underline{DG}}.$$

The so-called experimental rate of convergence of the seminorm of the total error $\boldsymbol{e}$ is computed by

$$r := -2\frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')},$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ denote the corresponding errors at two consecutive triangulations with the number of degrees of freedom N and N', respectively. The experimental rates of convergence for the other errors are defined in an analogous way.

TABLE 4.2
Examples considered for the Stokes system.

EXAMPLE	Ω	$u(x_1,x_2)$	$p(x_1,x_2)$
1	$]-1,1[^2$	$\begin{bmatrix} -e^{x_1}(x_2\cos(x_2)+\sin(x_2)) \\ e^{x_1}x_2\sin(x_2) \end{bmatrix}$	$2e^{x_1}\sin(x_2)$
2	$]-1/2,3/2[\times]0,2[$	$\begin{bmatrix} 1-e^{\lambda x_1}\cos(2\pi x_2) \\ \frac{\lambda}{2\pi}e^{\lambda x_1}\sin(2\pi x_2) \end{bmatrix}$	$-\frac{1}{2}e^{2\lambda x_1}-\bar{p}$

TABLE 4.3
Errors and experimental rates of convergence for Example 1 (Poisson, Dirichlet boundary condition).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma})$	$r_{\mathrm{div}}(\boldsymbol{\sigma})$
28	0.3452	——	1.4007	——	1.4426	——	6.1890	——
112	0.1535	1.1696	0.6084	1.2030	0.6275	1.2011	2.9646	1.0618
448	0.0752	1.0285	0.3294	0.8852	0.3379	0.8930	1.5725	0.9148
1792	0.0364	1.0488	0.1714	0.9424	0.1752	0.9473	0.8064	0.9634
7168	0.0179	1.0193	0.0875	0.9705	0.0893	0.9725	0.4072	0.9857
28672	0.0089	1.0057	0.0442	0.9850	0.0451	0.9858	0.2045	0.9938
114688	0.0045	1.0015	0.0222	0.9924	0.0227	0.9928	0.1025	0.9971

TABLE 4.4
Errors and experimental rates of convergence for Example 1 (Poisson, mixed boundary conditions).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma})$	$r_{\mathrm{div}}(\boldsymbol{\sigma})$
28	0.4343	——	2.0196	——	2.0658	——	6.1907	——
112	0.1619	1.4236	0.7428	1.4430	0.7603	1.4421	2.9562	1.0663
448	0.0770	1.0730	0.3644	1.0275	0.3724	1.0295	1.5716	0.9115
1792	0.0366	1.0711	0.1803	1.0155	0.1839	1.0178	0.8064	0.9627
7168	0.0180	1.0270	0.0897	1.0070	0.0915	1.0078	0.4072	0.9856
28672	0.0089	1.0078	0.0447	1.0031	0.0456	1.0032	0.2045	0.9938
114688	0.0045	1.0020	0.0224	1.0014	0.0228	1.0014	0.1025	0.9971

We consider two smooth examples. Their domains Ω as well as their corresponding exact solutions $(\boldsymbol{u}, p)$ are given in Table 4.2. Concerning Example 1, we resume our results in Table 4.1, where the total error and their components go to zero as $\mathcal{O}(h)$. This is in agreement with our expectations. In addition, we observe that the L^2 norms of the stress error $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)$ and of the pressure $(p-p_h)$ have higher rates of convergence: $\mathcal{O}(h^2)$.

Example 2 is taken from [32] where the parameter λ is given by

$$\lambda := -\frac{8\pi^2}{\nu^{-1}+\sqrt{\nu^{-2}+16\pi^2}}.$$

It will help us to test the robustness of our method for different values of the viscosity, $\nu \in \{1, 0.1, 0.059\}$. Numerical results for each one of these values are shown in Tables 4.8, 4.9, and 4.10. From these tables, we realize that the individual error $\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$ is more dominant than the seminorm $\boldsymbol{e}$, and thus it will determine the behavior $\mathcal{O}(h)$ of the total error $\|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \boldsymbol{u}-\boldsymbol{u}_h)\|_{\underline{\boldsymbol{\Sigma}}}$, as established in Theorems 3.3 and 3.4. We also notice that the errors $\boldsymbol{e}_0(\boldsymbol{\sigma})$ and $\boldsymbol{e}_0(p)$ show a better behavior than expected: $\mathcal{O}(h^2)$, for each of the considered values of ν here.

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TABLE 4.5
Errors and experimental rates of convergence for Example 2 (Poisson, mixed boundary conditions).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\text{div}}(\boldsymbol{\sigma})$	$r_{\text{div}}(\boldsymbol{\sigma})$
42	0.5179	——	0.7932	——	0.9473	——	0.4292	——
168	0.2951	0.8117	0.5646	0.4904	0.6371	0.5724	0.6960	——
672	0.1248	1.2414	0.3289	0.7796	0.3518	0.8567	0.4552	0.6125
2688	0.0487	1.3563	0.1749	0.9112	0.1816	0.9543	0.2237	1.0249
10752	0.0212	1.2003	0.0907	0.9466	0.0932	0.9622	0.1034	1.1129
43008	0.0101	1.0665	0.0470	0.9491	0.0481	0.9547	0.0484	1.0956
172032	0.0050	1.0168	0.0245	0.9371	0.0251	0.9405	0.0232	1.0616

TABLE 4.6
Errors and experimental rates of convergence for Example 3 (Poisson, mixed boundary conditions).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\text{div}}(\boldsymbol{\sigma})$	$r_{\text{div}}(\boldsymbol{\sigma})$
42	0.2989	——	0.5479	——	0.6241	——	0.4167	——
168	0.1440	1.0531	0.3439	0.6721	0.3728	0.7434	0.5597	——
672	0.0605	1.2503	0.1898	0.8571	0.1993	0.9039	0.3553	0.6558
2688	0.0270	1.1656	0.1009	0.9118	0.1044	0.9318	0.1862	0.9319
10752	0.0130	1.0568	0.0534	0.9176	0.0550	0.9261	0.0933	0.9966
43008	0.0064	1.0138	0.0286	0.9024	0.0293	0.9082	0.0464	1.0067
172032	0.0032	1.0027	0.0156	0.8750	0.0159	0.8806	0.0231	1.0056

5. Final comments and conclusions. In this paper, we have first extended the techniques of [9] to the case of Lagrangian finite elements to approximate each unknown for a mixed discontinuous Galerkin formulation of the Poisson equation with mixed boundary conditions. We have proved that the method is stable and converges with the optimal rate of convergence if reasonable additional regularity of the exact solution is assumed. To this aim, we need to redefine the definition of the *numerical fluxes* (2.2)–(2.3) in order to establish Theorems 2.7 and 2.8. The main relevance is that we have been able to obtain a discrete approximation of local $H(\text{div})$-functions using the standard discontinuous polynomial space instead of the conforming Raviart-Thomas space. The results shown in Tables 4.3, 4.4, 4.5, and 4.6, for $[\mathbb{P}_1(\mathcal{T}_h)]^2 \times \mathbb{P}_0(\mathcal{T}_h)$ as approximation spaces, are in agreement with the conclusions of the a-priori error analysis that we have established.

Next, we have extended the approach to solve a Stokes system. We recall that in [9] we have analyzed a pseudo stress-velocity mixed discontinuous formulation considering the pair $[RT_k(\mathcal{T}_h)]^2 \times [\mathbb{P}_k(\mathcal{T}_h)]^2$ with $k \geq 0$. We have proved that this family of approximation spaces is stable. This can be seen as a generalization of the scheme studied earlier in [15]. We point out that here we have developed an a-priori error analysis for an unusual nonconforming dual-mixed variational formulation for the Poisson and Stokes problem, considering piecewise polynomial approximation spaces for each unknown. In this sense, we have circumvented the well-known de Rham's commutative diagram when a local subspace of $H(\text{div})$ is used, proving optimal convergence of the method in an unusual way. We would like to emphasize that in this paper we have proved that the pair $[\mathbb{P}_{k+1}(\mathcal{T}_h)]^{2\times 2} \times [\mathbb{P}_k(\mathcal{T}_h)]^2$, with $k \geq 0$, is stable (in a pseudo-stress velocity formulation). In particular, for this nonconforming scheme, surprisingly we have proved that $[\mathbb{P}_1(\mathcal{T}_h)]^{2\times 2} \times [\mathbb{P}_0(\mathcal{T}_h)]^2$ is a stable pair for the Stokes problem, whereas for the corresponding conforming scheme it is well known that the pair $[\mathbb{P}_1(\Omega)]^{2\times 2} \times [\mathbb{P}_0(\mathcal{T}_h)]^2$ is not stable, and it needs some stabilization procedure in order to use it. Tables 4.7, 4.8, 4.9, and 4.10 show that the method converges for each case with order

TABLE 4.7
Errors and experimental rates of convergence for Example 1 (Stokes), for $\nu = 1$.

N	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
29	2.3999	——	4.6090	——	4.5215	——
113	1.2536	0.9550	3.5506	0.3836	7.6949	——
449	0.6395	0.9757	2.3368	0.6064	4.8937	0.6561
1793	0.3215	0.9935	1.3147	0.8309	2.4012	1.0284
7169	0.1610	0.9979	0.6921	0.9260	1.1352	1.0811
28673	0.0805	0.9994	0.3544	0.9655	0.5449	1.0589
114689	0.0403	0.9999	0.1793	0.9834	0.2661	1.0340
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
29	4.1739	——	2.4102	——		
113	4.0573	0.0417	2.7151	——		
449	2.0405	0.9963	1.4032	0.9569		
1793	0.6270	1.7044	0.4343	1.6941		
7169	0.1672	1.9077	0.1161	1.9032		
28673	0.0428	1.9657	0.0298	1.9633		
114689	0.0108	1.9852	0.0075	1.9839		

TABLE 4.8
Errors and experimental rates of convergence for Example 2 (Stokes), for $\nu = 1$.

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	19.6523	——	81.5980	——	469.6606	——
225	10.1306	0.9652	65.5374	0.3193	380.7820	0.3056
897	4.9803	1.0269	28.3148	1.2137	301.9265	0.3356
3585	2.5719	0.9540	13.1790	1.1040	187.0733	0.6910
14337	1.3105	0.9728	6.4181	1.0382	99.9140	0.9050
57345	0.6585	0.9930	3.2116	0.9989	50.8263	0.9751
229377	0.3296	0.9982	1.6119	0.9946	25.5173	0.9941
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	307.4529	——	211.3465	——		
225	81.5107	1.9338	37.0654	2.5357		
897	36.2827	1.1705	19.3759	0.9381		
3585	12.1414	1.5803	7.1110	1.4470		
14337	3.1585	1.9429	1.8808	1.9190		
57345	0.7931	1.9936	0.4745	1.9869		
229377	0.1979	2.0025	0.1186	2.0004		

$\mathcal{O}(h)$, as predicted by the theory that we have developed here. On the other hand, the results in these tables give us numerical evidence that the L^2-error of the pseudo stress and the pressure behave as $\mathcal{O}(h^2)$. This could be the subject of future work.

Finally, it is important to remark that the analysis has been performed under suitable additional regularity assumptions, which give us good hope for the corresponding a-posteriori error estimate, which will be reported in a separate work. Furthermore, the fact that the same formulation works for the Stokes and Darcy problem should simplify its coupling, therefore this topic will be explored in another work.

TABLE 4.9
Errors and experimental rates of convergence for Example 2 (Stokes), for $\nu = 0.1$.

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	4.9442	—–	7.2237	—–	27.6457	—–
225	2.6894	0.8869	3.8397	0.9205	17.7333	0.6468
897	1.4207	0.9229	3.8946	—–	18.3476	—–
3585	0.6790	1.0657	2.7499	0.5024	10.6941	0.7792
14337	0.3372	1.0101	1.5354	0.8409	5.4225	0.9799
57345	0.1665	1.0184	0.7939	0.9516	2.7213	0.9947
229377	0.0828	1.0077	0.4009	0.9856	1.3583	1.0025
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	14.9810	—–	10.5309	—–		
225	3.5498	2.0974	2.3011	2.2154		
897	2.3914	0.5713	1.6136	0.5133		
3585	1.2373	0.9512	0.8460	0.9320		
14337	0.3613	1.7761	0.2458	1.7838		
57345	0.0936	1.9493	0.0633	1.9575		
229377	0.0235	1.9916	0.0159	1.9954		

TABLE 4.10
Errors and experimental rates of convergence for Example 2 (Stokes), for $\nu = 0.059$.

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	3.2095	—–	4.3081	—–	13.2613	—–
225	1.8459	0.8057	2.2457	0.9489	7.0122	0.9281
897	1.0290	0.8452	2.4596	—-	10.2963	—–
3585	0.5340	0.9469	1.7508	0.4906	5.8493	0.8163
14337	0.2585	1.0467	1.0116	0.7916	3.0870	0.9222
57345	0.1189	1.1200	0.5325	0.9259	1.6129	0.9366
229377	0.0569	1.0640	0.2704	0.9774	0.8179	0.9796
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	6.7667	—–	4.7644	—–		
225	2.5420	1.4261	1.7484	1.4602		
897	1.5787	0.6889	1.0874	0.6868		
3585	0.7923	0.9953	0.5423	1.0042		
14337	0.2446	1.6959	0.1647	1.7200		
57345	0.0653	1.9063	0.0433	1.9266		
229377	0.0166	1.9749	0.0110	1.9834		

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