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INCOMPLETE BETA POLYNOMIALS*

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Abstract. We study some properties of incomplete beta polynomials, in particular, their zero asymptotic distribution. These polynomials satisfy a three-term recurrence relation, so they can be written as the sum of two terms each one leading to their asymptotic behavior in a region.

Key words. incomplete beta function, zero distribution, asymptotics, recurrence relations, potential theory

AMS subject classifications. 33B20, 41A58, 41A80

1. Introduction and main result. Given $\alpha > 0$, we consider the polynomials

$$C_{n,\alpha}(z) = C_n(z) := \sum_{j=0}^{n} \binom{n}{j} \frac{z^j}{\alpha j + 1}, \qquad n \in \mathbb{Z}_{\geq 0}.$$

Then

$$\zeta C_n(-\zeta^\alpha) = \int_0^\zeta (1 - x^\alpha)^n \, dx = \frac{B_{\zeta^{1/\alpha}}(1/\alpha, n+1)}{\alpha} =: \mathbf{B}_n(\zeta), \tag{1.1}$$

where $B_x(p,q)$ is the incomplete beta function

$$B_x(p,q) := \int_0^x t^{p-1}(1-t)^{q-1} \, dt.$$

For definiteness, in (1.1) we consider the main branch of ζ^α in $\mathbb{C} \setminus (-\infty, 0]$ taking the value 0 at 0, and we take as integration contour the line from 0 to ζ or any path of integration Γ from 0 to ζ with $\Gamma \setminus \{0\}$ in the region of holomorphy of ζ^α.

The incomplete beta function is widely used in statistics as a probability measure (see, for instance, [6, Chapter 25]). There are several papers and books about this function (see [5, 14] and the references therein).

The object of this paper is to study the zero distribution and asymptotic properties of the polynomials C_n and the functions $\mathbf{B}_n$ as n goes to infinity. These polynomials satisfy a three-term recurrence relation, so they can be written as the sum of two terms, each one leading to their asymptotic behavior in a region; see formula (3.1). The zero asymptotic behavior of a sequence of polynomials can be linked with either the analytic properties of their limit function (see [3, Jentzsch-Szegő's Theorem]) or with a curve where their asymptotic behavior changes drastically (see [9, Szegő's theorem and related results], [13, 15]).

With the aim of stating our main result, let us set some notations:

$$\mathbf{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}, \qquad \overline{\mathbf{D}}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\},$$
$$\partial\mathbf{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}, \quad \text{and} \quad (\overline{\mathbf{D}}_r(z_0))^c := \{z \in \mathbb{C} : |z - z_0| > r\}.$$

THEOREM 1.1. *Assume that $\alpha \neq 1$. Then, we have*

$$C_n(z) = \frac{\Gamma(1/\alpha)}{\alpha\, n^{1/\alpha}(-z)^{1/\alpha}}(1 + o(1)) \qquad \text{as } n \to \infty, \tag{1.2}$$

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uniformly on compact subsets of $\mathbf{D}_1(-1)$*, where the branch of* $(-z)^{1/\alpha}$ *is taken such that* $1^{1/\alpha} = 1$*. It holds that*

$$C_n(z) = \frac{(1+z)^{n+1}}{\alpha n z}(1 + o(1)) \qquad \text{as } n \to \infty, \tag{1.3}$$

uniformly on compact subsets of $(\overline{\mathbf{D}}_1(-1))^c$*.*

If $0 < \alpha < 1$*, then* (1.3) *also holds uniformly on compact subsets of* $\partial\mathbf{D}_1(-1) \setminus \{0\}$ *and the zeros of* C_n *lie in* $\mathbf{D}_1(-1)$*, while if* $\alpha > 1$*, then* (1.2) *also holds uniformly on compact subsets of* $\partial\mathbf{D}_1(-1) \setminus \{0\}$ *and their zeros lie in* $(\overline{\mathbf{D}}_1(-1))^c$*. They cluster on* $\partial\mathbf{D}_1(-1)$*, and these are the only limit points of the zeros, and their asymptotic distribution is the normalized arc-length on* $\partial\mathbf{D}_1(-1)$*.*

Theorem 1.1 is proved in Section 3. The case $\alpha = 1$ is easy and is included in Remark 3.1. The next section contains several properties of the polynomials C_n such as recurrence relations and zero location. Section 4 includes all results for the incomplete beta polynomials.

2. Auxiliary results. The incomplete beta function can be written in terms of hypergeometric functions, and it satisfies some recurrence relations; see [14, p. 288–313]. We have the following result for C_n:

LEMMA 2.1.

1. *It holds that*

$$(\alpha n + 1)C_n(z) - \alpha n C_{n-1}(z) = (1+z)^n, \qquad n \geq 1. \tag{2.1}$$

2. *For* $n \geq 1$*,*

$$\begin{aligned}(\alpha n + \alpha + 1)C_{n+1}(z) &- ((\alpha n + 1)(1+z) \\ &+ \alpha(n+1))C_n(z) + \alpha n(1+z)C_{n-1}(z) = 0,\end{aligned} \tag{2.2}$$

   where $C_0(z) = 1$ *and* $C_1(z) = 1 + \frac{z}{\alpha+1}$*.*
3. *For* $n \geq 1$*, we have* $C_n(z) = P_n(1+z)$*, where* $P_n(\zeta) = a_{0,n,\alpha} + a_{1,n,\alpha}\zeta + \ldots + a_{n,n,\alpha}\zeta^n$ *is a polynomial of degree* n *with positive coefficients and* $a_{n,n,\alpha} = \frac{1}{\alpha n+1}$*. If* $0 < \alpha < 1$*, then the coefficients of* P_n *are increasing, i.e.,* $0 < a_{j,n,\alpha} < a_{j+1,n,\alpha}$*, for* $j = 0, 1, \ldots, n-1$*. On the other hand, if* $1 < \alpha$*, then the coefficients of* P_n *are decreasing, i.e.,* $a_{j,n,\alpha} > a_{j+1,n,\alpha} > 0$*, for* $j = 0, 1, \ldots, n-1$*.*
4. *If* $0 < \alpha < 1$*, then the zeros of* C_n *lie in* $\mathbf{D}_1(-1)$ *and for* $\alpha > 1$ *in* $(\overline{\mathbf{D}}_{-1}(1))^c$*.*
5. *If* $C_{n-1}(z_0) = 0$*, then* $C_n(z_0) \neq 0$*. Moreover,*

$$C_n(z) + \alpha z C_n'(z) = (1+z)^n, \tag{2.3}$$

   and the zeros of C_n *are simple.*

Proof. Statement 1 follows easily by comparing coefficients, and 1 yields 2 straightforwardly. Next, we prove 3. Let us assume that $0 < \alpha < 1$; the other case is similar. The polynomial $C_n(z)$ has degree n and

$$C_1(z) = 1 + \frac{z}{\alpha+1} = P_1(z+1),$$

where $P_1(\zeta) = \frac{\alpha}{\alpha+1} + \frac{\zeta}{\alpha+1}$. Thus, statement 3 holds for $n = 1$. We assume that 3 is true for some n, and it will then be verified for $n+1$. According to (2.1),

$$(\alpha n + \alpha + 1)P_{n+1}(\zeta) = (\alpha n + \alpha)P_n(\zeta) + \zeta^{n+1},$$

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so, by the induction hypothesis, the coefficient of ζ^{n+1} in the polynomial P_{n+1} is $\frac{1}{\alpha n+\alpha+1}$, and $a_{n,n+1,\alpha} = \frac{\alpha n+\alpha}{(\alpha n+1)(\alpha n+\alpha+1)} < \frac{1}{\alpha n+\alpha+1} = a_{n+1,n+1,\alpha}$. Thus, 3 follows by induction.

Now, we obtain 4. It holds that[1]

$$(1-\zeta)P_n(\zeta) = a_{0,n,\alpha} + (a_{1,n,\alpha} - a_{0,n,\alpha})\zeta + \ldots + (a_{n,n,\alpha} - a_{n-1,n,\alpha})\zeta^n - a_{n,n,\alpha}\zeta^{n+1},$$

so, if $P_n(\zeta_0) = 0$ and $|\zeta_0| > 1$, then

$$a_{n,n,\alpha} = a_{0,n,\alpha}\frac{1}{\zeta_0^{n+1}} + (a_{1,n,\alpha} - a_{0,n,\alpha})\frac{1}{\zeta_0^n} + \ldots + (a_{n-1,n,\alpha} - a_{n,n,\alpha})\frac{1}{\zeta_0}, \tag{2.4}$$

which yields

$$\begin{aligned} a_{n,n,\alpha} &< a_{0,n,\alpha} + (a_{1,n,\alpha} - a_{0,n,\alpha}) \\ &\qquad + (a_{2,n,\alpha} - a_{1,n,\alpha}) + \ldots + (a_{n,n,\alpha} - a_{n-1,n,\alpha}) \\ &= a_{n,n,\alpha}, \end{aligned} \tag{2.5}$$

which is impossible. So the zeros of P_n are in $\overline{\mathbf{D}}_1(0)$, which is equivalent to the zeros of C_n lying in $\overline{\mathbf{D}}_1(-1)$. Moreover, observe that $C_n(1) \neq 0$ because of $a_{j,n,\alpha} > 0$ for all j, and $C_n(-1) \neq 0$ from $C_0(z) = 1 \neq 0$ and (2.1). Likewise, if $C_n(\zeta_0) = 0$ and $|\zeta_0| = 1$, $\zeta_0 \neq \pm 1$, then $\Re(\zeta_0) < 1$, and taking the real part of (2.4), we get again the contradiction in (2.5). So, $C_n(z) \neq 0$ in $\partial\mathbf{D}_1(-1)$, and this completes the proof of 4.

If $C_n(z_0) = 0$, then $z_0 \notin (\partial\mathbf{D}_1(-1))$, and conclusion 5 also follows from (2.1). Taking the derivative in (1.1), we get (2.3) which implies that the zeros of C_n are simple. □

REMARK 2.2. Figure 2.1 displays the result of some numerical experiments with a visualization of the zeros of C_n for several values of n and $\alpha = 1/2$ and $3/2$.

LEMMA 2.3. *Let* $z \in (\overline{\mathbf{D}}_1(-1) \setminus \{0\})$. *Then*

1. $\lim_{n\to\infty} C_n(z) = 0$,
2.
$$\sum_{n=1}^{\infty} \frac{C_n(z)}{n} = \alpha - \log(-z),$$

where $\log(\cdot)$ *is the main branch of the logarithm function.*

The above statements hold uniformly on compact subsets of $\overline{\mathbf{D}}_1(-1) \setminus \{0\}$. *If* $z = -1$, *we get*

$$\sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1+\frac{1}{\alpha})} = \frac{\alpha^2}{\Gamma(1/\alpha)}.$$

Moreover, we have

$$\begin{aligned} |C_n(-1)| &\le \max\{|C_n(z)| : z \in \overline{\mathbf{D}}_1(-1)\} \le 1, \\ C_n(-1) &\sim \frac{\Gamma(1/\alpha)}{\alpha\, n^{1/\alpha}} \qquad \textit{as } n \to \infty. \end{aligned} \tag{2.6}$$

Proof. By the dominated convergence theorem, statement 1 follows from (1.1). Since $C_0(z) = 1$ and $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$, $|x| \le 1, x \neq 1$, summing up in (2.1) yields statement 2. The first inequality in (2.6) is obviously true; the second inequality is a consequence of (1.1) since $|1+z| \le 1$ is equivalent to $|1-\zeta^\alpha| \le 1$ with $z = -\zeta^\alpha$ and

$$|\zeta C_n(-\zeta^\alpha)| = |\int_0^\zeta (1-x^\alpha)^n\, dx| \le \int_0^\zeta |1-x^\alpha|^n\, |dx| \le |\zeta|.$$

[1] See Eneström-Kakeya's theorem [10, pp. 255, 271].

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On other hand, taking $z = 1$ in (1.1), we get

$$C_n(-1) = \frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}(1-t)^{(n+1)-1}\,dt = \frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\alpha})},$$

hence from Stirling's asymptotic formula it follows readily that $C_n(-1) \sim \frac{\Gamma(1/\alpha)}{\alpha\, n^{1/\alpha}}$ as $n \to \infty$. $\square$

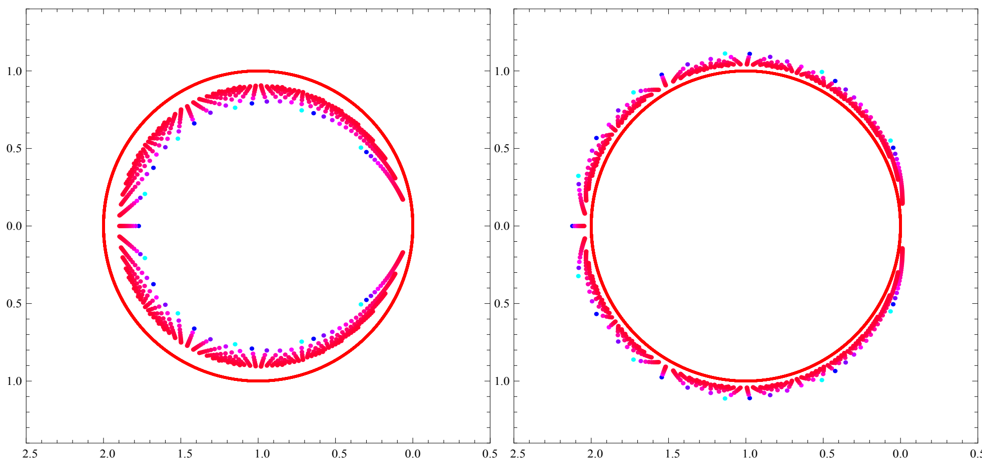

FIG. 2.1. *Left: the zeros of* C_n *for* $n = 10, 11, \ldots, 40$ *with* $\alpha = 1/2$. *Right: the zeros of* C_n *for* $n = 10, 11, \ldots, 40$ *with* $\alpha = 3/2$. *The zeros are colored from cyan to magenta according to the degree of the polynomials varying from* 10 *to* 40. *In red we indicate the circumference* $|1 + z| = 1$.

3. Proof of Theorem 1.1. Let us assume that $\alpha > 1$; the other case is similar. The change of variable $\frac{t^2}{n} = \frac{x^\alpha - \zeta^\alpha}{1-\zeta^\alpha}$ in (1.1) yields

$$\begin{aligned}\zeta C_n(-\zeta^\alpha) &= C_n(1) + \int_1^\zeta (1-x^\alpha)^n\,dx \\ &= \frac{\Gamma(n+1)\Gamma(1/\alpha)}{\alpha\Gamma(n+1+1/\alpha)} \\ &\qquad - \frac{2(1-\zeta^\alpha)^{n+1}}{\alpha n}\int_0^{\sqrt{n}}(1-\frac{t^2}{n})^n \frac{t}{((1-\zeta^\alpha)\frac{t^2}{n}+\zeta^\alpha)^{(\alpha-1)/\alpha}}\,dt.\end{aligned} \tag{3.1}$$

Then, from the dominated convergence theorem, it readily follows that (1.2) holds uniformly on compact subsets of $\mathbf{D}_1(-1)$ and on compact subsets of $\partial\mathbf{D}_1(-1)\setminus\{0\}$, and also that (1.3) holds true. See also (4.1).

Let $\nu[C_n] := \frac{1}{n}\sum_{z:C_n(z)=0}\delta_z$ be the zero counting measure of C_n. Define the monic polynomial $\widehat{C}_n(z) := (\alpha n+1)C_n(z)$. Let μ be a weak-$*$ limit of $\nu[C_n]$, i.e., there exists a subsequence $(\nu[C_{n_j}])$ such that $\lim_{j\to\infty}\int f(\zeta)\,d\nu[C_{n_j}](\zeta) = \int f(\zeta)\,d\mu(\zeta)$ for all continuous function f in $\mathbb{C}$ with compact support. To simplify the notation, instead of n_j we write n. We are going to prove that μ is the equilibrium measure of $\partial\mathbf{D}_1(-1)$, i.e., the normalized arc-length on $\partial\mathbf{D}_1(-1)$.

By (2.6), $\lim_{n\to\infty}(\max\{|C_n(z)| : |z|\le 1\})^{1/n} = 1$, hence according to [3, Lemma 3.1], at most $o(n)$ of the zeros of C_n could tend to infinity. Thus, by (1.2) and (1.3), we have

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$\operatorname{supp}\mu \subset \partial\mathbf{D}_1(-1)$, where $\operatorname{supp}\mu$ denotes the support of the measure μ. Also, it is well known that $\operatorname{cap}(\partial\mathbf{D}_1(-1)) = 1$; see [11, Theorem 5.2.5]. By (2.6) and the principle of descent ([12, Theorem 6.8, p. 70]), we have[2]

$$V(\mu, z) \le 0 = \lim_{n\to\infty} V(\nu[C_n], z) = \lim_{n\to\infty} \frac{1}{n}\log\frac{1}{|\widehat{C}_n(z)|}, \qquad z \in \partial\mathbf{D}_1(-1).$$

Since μ is a probability measure, $I(\mu) = \int V(\mu, z)\, d\mu(z) \le 0 = \log(\operatorname{cap}(\partial\mathbf{D}_1(-1)))$. But the equilibrium measure of $\partial\mathbf{D}_1(-1)$, $\mu_{\partial\mathbf{D}_1(-1)}$, is the unique measure which minimizes the energy between the probability measures with support in $\partial\mathbf{D}_1(-1)$, and $I(\mu_{\partial\mathbf{D}_1(-1)}) = \log(\operatorname{cap}(\partial\mathbf{D}_1(-1))) = 0$ ([12, Theorem 1.3, p. 27]). Therefore, we have $\mu = \mu_{\partial\mathbf{D}_1(-1)}$ and

$$\operatorname*{w-lim}_{n\to\infty} \nu[C_n] = \mu_{\partial\mathbf{D}_1(-1)}.$$

It is well known that the equilibrium measure for a circle is the normalized arc-length on $\partial\mathbf{D}_1(-1)$. Therefore, the zeros of $\{C_n\}$ are dense in $\partial\mathbf{D}_1(-1)$, these are the only limit points of the zeros, and their asymptotic distribution is the normalized arc-length on $\partial\mathbf{D}_1(-1)$. □

REMARK 3.1.

1. Now we outline another way to obtain (1.3). From (2.2),

$$C_{n+1}(z) = \frac{(\alpha n+1)(1+z) + \alpha(n+1)}{\alpha n + \alpha + 1} C_n(z) - \frac{\alpha n(1+z)}{\alpha n + \alpha + 1} C_{n-1}(z),$$

   which is a three-term recurrence relation with $C_0(z) = 1$ and $C_1(z) = 1 + \frac{z}{\alpha+1}$. Moreover, the coefficients have the limits

$$\lim_{n\to\infty} \frac{(\alpha n+1)(1+z) + \alpha(n+1)}{\alpha n + \alpha + 1} = 2 + z, \qquad \lim_{n\to\infty} \frac{\alpha n(1+z)}{\alpha n + \alpha + 1} = 1 + z,$$

   Thus, the characteristic polynomial of the limit recurrence relation is

$$p(\lambda) = \lambda^2 - (2+z)\lambda + (1+z),$$

   whose roots are $\lambda_1(z) = 1$ and $\lambda_2(z) = 1 + z$. Given a point $z \in (\overline{\mathbf{D}}_1(-1))^c = \{z \in \mathbb{C} : |\lambda_1(z)| < |\lambda_2(z)|\}$, by Poincaré's theorem [8] (see also [1, 2, 4]), either the sequence $\{C_n(z) : n \in \mathbb{Z}_{\ge 0}\}$ is the zero sequence for n large enough or $\lim_{n\to\infty} \frac{C_{n+1}(z)}{C_n(z)} = \lambda_2(z)$ uniformly on compact subsets of $(\overline{\mathbf{D}}_1(-1))^c$. The former cannot occur because of item 5 in Lemma 2.1. The proof of (1.3) follows from (2.1). Moreover, if $\alpha \ne 1$, we have $\lim_{n\to\infty} \frac{C_{n+1}(z)}{C_n(z)} = \begin{cases} (1+z) & \text{if } 0 < \alpha < 1, \\ 1 & \text{if } \alpha > 1, \end{cases}$ uniformly on compact subsets of $\partial\mathbf{D}_1(-1) \setminus \{0\}$.
2. If $\alpha = 1$, then

$$C_n(z) = \frac{(1+z)^{n+1} - 1}{(n+1)z},$$

   and the statements about their asymptotic behavior in $\mathbb{C} \setminus \partial\mathbf{D}_1(-1)$ are trivially true. The zeros of C_n lie on $\partial\mathbf{D}_1(-1)$.

[2] Hereafter, if ν is a positive measure with compact support in the complex plane, then its logarithm potential and energy are denoted by $V(\nu, z) := \int \log\frac{1}{|z-x|}\, d\nu(x)$ and $I(\nu) := \int V(\nu, z)\, d\nu(z)$.

Moreover, the recurrence relation

$$f_{n+1} = (z+2)f_n - (z+1)f_{n-1}, \qquad n = 0, 1, \ldots. \tag{3.2}$$

with initial conditions $f_0 = 1,\ f_{-1} = 0$, has the solution

$$u_n(z) = \sum_{j=0}^{n} (z+1)^j = \frac{(z+1)^{n+1} - 1}{z}.$$

On the other hand, if the initial conditions are $f_0 = 0,\ f_{-1} = 1$, then the solution of (3.2) is

$$v_n(z) = -\sum_{j=1}^{n} (z+1)^j = -(z+1)\frac{(z+1)^n - 1}{z}.$$

By the Euler-Wallis theorem for continued fractions (see, for example, [7, p. 8]), we have partial continued fractions:

$$\mathbf{K}_{k=1}^{n}\left(\frac{-(z+1)}{(z+2)}\right) := \frac{v_n(z)}{u_n(z)}$$

and

$$\mathbf{K}_{k=1}^{\infty}\left(\frac{-(z+1)}{(z+2)}\right) := \lim_{n\to\infty}\frac{v_n}{u_n} = \begin{cases} -(z+1) & \text{if } z \in (\mathbf{D}_1(-1))^c, \\ -1 & \text{if } z \in \mathbf{D}_1(-1). \end{cases}$$

4. Results for incomplete beta polynomials. Let $\mathcal{I}_\alpha := \{z \in \mathbb{C} : |1 - z^\alpha| < 1\}$, $\partial\mathcal{I}_\alpha := \{z \in \mathbb{C} : |1 - z^\alpha| = 1\}$, $\overline{\mathcal{I}}_\alpha := \mathcal{I}_\alpha \cup \partial\mathcal{I}_\alpha$, and $\mathcal{E}_\alpha := \{z \in \mathbb{C} : |1 - z^\alpha| > 1\}$. As a consequence of Theorem 1.1 and (1.1), we obtain:

COROLLARY 4.1. *Assume that* $\alpha \neq 1$. *We have*

$$\mathbf{B}_n(z) = \frac{\Gamma(1/\alpha)}{\alpha\, n^{1/\alpha}}(1 + o(1)) \tag{4.1}$$

uniformly on compact subsets of $\mathcal{I}_\alpha$. *Moreover, it holds that*

$$\mathbf{B}_n(z) = -\frac{(1 - z^\alpha)^{n+1}}{\alpha n z^{\alpha-1}}(1 + o(1)), \tag{4.2}$$

uniformly on compact subsets of $\mathcal{E}_\alpha$.

If $0 < \alpha < 1$, *then* (4.2) *holds uniformly on compact subsets of* $\partial\mathcal{I}_\alpha \setminus \{0\}$, *the zeros of* $\mathbf{B}_n$ *different from zero lie in* $\mathcal{I}_\alpha$, *and if* $\alpha > 1$, *then* (4.1) *holds uniformly on compact subsets of* $\partial\mathcal{I}_\alpha \setminus \{0\}$ *and the zeros of* $\mathbf{B}_n$ *different from zero lie in* $\mathcal{E}_\alpha$. *They cluster on* $\partial\mathcal{I}_\alpha$, *these are the only limit points of the zeros, and their asymptotic distribution is the measure which is the pre-image of the normalized arc-length on* $\partial\mathbf{D}_1(-1)$ *under the mapping* $\varphi_\alpha(z) = -z^\alpha$.

As an illustration, Figures 4.1 and 4.2 display the zeros of $\mathbf{B}_n$ for several values of n and α. From Lemma 2.1, we have

COROLLARY 4.2.

1. It holds that

$$(\alpha n + 1)\mathbf{B}_n(z) - \alpha n\mathbf{B}_{n-1}(z) = z(1 - z^\alpha)^n, \qquad n \geq 0.$$

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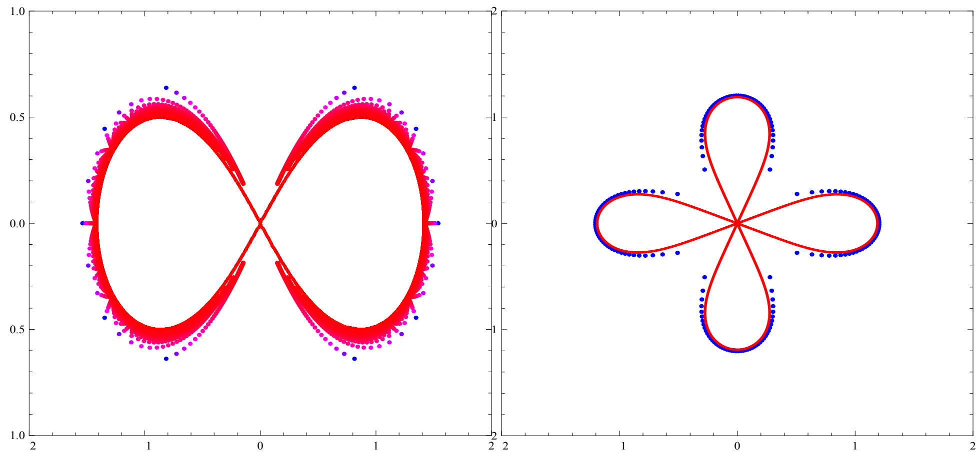

FIG. 4.1. *Left: the zeros of* $\mathbf{B}_n$*, for* $n = 5, 6, \ldots, 100$*,* $\alpha = 2$*, and in red the curve* $|1 - z^2| = 1$*. Right: the zeros of* $\mathbf{B}_{50}$ *for* $\alpha = 4$ *and the curve* $|1 - z^4| = 1$*. Points are colored from cyan to magenta according to the degree of the polynomials.*

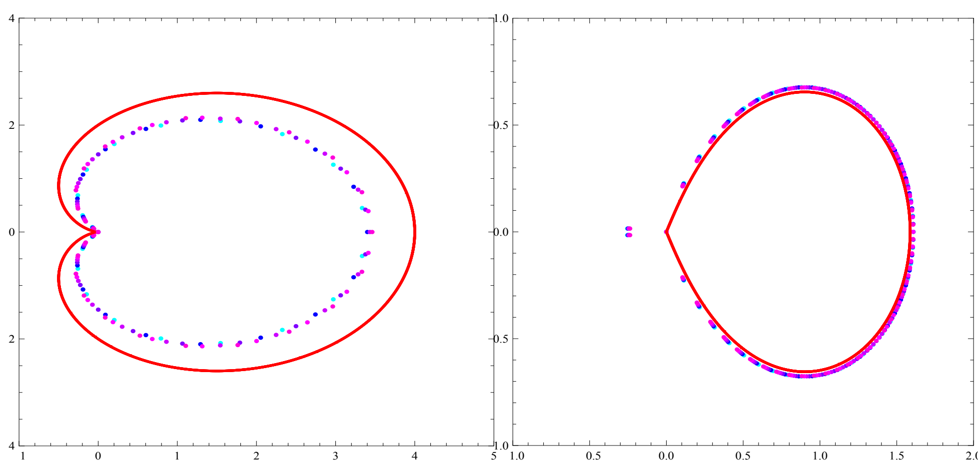

FIG. 4.2. *Left: the zeros of* $\mathbf{B}_n$*, for* $n = 20, 21, \ldots, 25$*,* $\alpha = 1/2$*, and the curve* $|1 - z^{1/2}| = 1$*. Right: the zeros of* $\mathbf{B}_n$*, for* $n = 45, 46, \ldots, 50$*,* $\alpha = 3/2$*, and the curve* $|1 - z^{3/2}| = 1$*. Points are colored from cyan to magenta according to the degree of the polynomials.*

2. *For* $n \geq 1$*,*

$$(\alpha n + \alpha + 1)\mathbf{B}_{n+1}(z) - ((\alpha n + 1)(1 - z^\alpha) + \alpha(n+1))\mathbf{B}_n(z) + \alpha n(1 - z^\alpha)\mathbf{B}_{n-1}(z) = 0,$$

where $\mathbf{B}_0(z) = z$ *and* $\mathbf{B}_1(z) = z - \frac{z^{\alpha+1}}{\alpha+1}$.

3. *For* $n \geq 1$*, we have* $\mathbf{B}_n(z) = zQ_n(1 - z^\alpha)$*, where* $Q_n(\zeta) = a_{0,n,\alpha} + a_{1,n,\alpha}\zeta + \ldots + a_{n,n,\alpha}\zeta^n$ *is a polynomial of degree* n *with positive coefficients. If* $0 < \alpha < 1$*, then the coefficients of* Q_n *are increasing, i.e.,* $0 < a_{j,n,\alpha} < a_{j+1,n,\alpha}$*, for* $j = 0, 1, \ldots, n-1$ *and* $a_{n,n,\alpha} = \frac{1}{\alpha n+1}$*. On the other hand, if* $1 < \alpha$*, then the coefficients*

of Q_n are decreasing, i.e., $a_{j,n,\alpha} > a_{j+1,n,\alpha} > 0$, for $j = 0, 1, \ldots, \alpha n - 1$ and $a_{n,n,\alpha} = \frac{1}{\alpha n+1}$.

4. *If $\mathbf{B}_{n-1}(z_0) = 0$ and $z_0 \neq 0$, then $\mathbf{B}_n(z_0) \neq 0$. Their zeros different from zero are simple.*

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