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ON Q-INTERPOLATION FORMULAE AND THEIR APPLICATIONS*

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Abstract. It is shown that the q-Taylor series corresponding to Jackson's q-difference operator can be generated by the Newton interpolation formulae and the related remainders can be therefore written as the residue of a Newton interpolation formula. The advantage of this approach is that some constraints such as q-integrability in a domain and existence of the q-derivatives of f at zero up to order n are no longer necessary. Two q-quadrature formulae of weighted interpolatory type are derived in this direction and some numerical examples for approximating definite quadratures and solving ordinary differential equations are then given

Key words. q-Taylor series, Jackson's q-difference operator, Newton interpolation formulae, q-quadrature rules of weighted interpolatory type

AMS subject classifications. 65D05, 65D30, 41A05, 41A55

1. Introduction. In recent years, quantum calculus has found extensive applications in physics and numerical analysis, e.g., [6, 8, 17]. Hence, it is a logical request to investigate and obtain q-analogues of classical results within the framework of q-calculus. One of the classical results in numerical analysis is the Taylor interpolation series problem whose q-analogues have been studied in different aspects, e.g., [12, 13, 14].

In [15] Jackson introduced the first type of a q-Taylor series as

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)\ [x-a]_{n,q}, \tag{1.1}$$

in which $q \in (0,1)$, $(q;q)_n = \prod_{k=1}^{n}(1-q^k)$, D_q is the q-difference operator defined by

$$D_q f(x) = \frac{f(x)-f(qx)}{x(1-q)},$$

and

$$[x-a]_{n,q} = \prod_{k=0}^{n-1}(x-aq^k), \quad \text{for} \quad n \geq 1, \qquad [x-a]_{0,q} = 1.$$

Later in 1975, Al-Salam and Verma [2] introduced a second type of q-Taylor series in the interpolatory form

$$f(x) = \sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2} \frac{(1-q)^n}{(q;q)_n} D_q^n f(aq^{-n})\ [a-x]_{n,q}, \tag{1.2}$$

where

$$[a-x]_{n,q} = (-1)^n q^{n(n-1)/2}\, [x-a]_{n,q^{-1}}.$$

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They verified the validity of (1.2) under the assumption of the expandability of f in the form

$$f(x)=\sum_{n=0}^{\infty} c_n\,[a-x]_{n,q}.$$

Recently, Annaby and Mansour [3] have given analytic proofs for both q-Taylor expansions (1.1) and (1.2) (see also [6]) according to the two following theorems.

THEOREM 1.1 ([3]). *Let f be a function defined in $\Omega_R=\{x\in\mathbb{C}:\ |x|<R\}$ and set $[x-a]_{n,q}=G_n(x;a,q)$. If the q-derivatives of f up to order n exist at zero and $D_q^n f$ is q-integrable on Ω_R, then*

$$f(x)=\sum_{k=0}^{n-1}\frac{D_q^k f(a)}{\Gamma_q(k+1)}\,G_k(x;a,q)+\frac{1}{\Gamma_q(n)}\int_a^x G_{n-1}(x;qt,q)\,D_q^n f(t)\,d_qt. \tag{1.3}$$

The proof of this theorem is completely based on the Al-Salam q-Cauchy integral formula [1],

$$\begin{aligned}\int_a^x\int_a^{x_{n-1}}\dots\int_a^{x_1} D_q^n f(t)\,d_qt\,d_qx_1\dots d_qx_n &= \frac{1}{\Gamma_q(n)}\int_a^x G_{n-1}(x;qt,q)\,D_q^n f(t)\,d_qt\\ &= f(x)-\sum_{k=0}^{n-1}\frac{D_q^k f(a)}{\Gamma_q(k+1)}\,G_k(x;a,q).\end{aligned}$$

THEOREM 1.2 ([3]). *Let f be a function defined in Ω_R and a be a non-zero complex number in Ω_R such that the $aq^{-j}(j=1,2,...,n)$ are in Ω_R for some $n\geq 1$. If the q-derivatives of f up to order $n-1$ exist at zero such that $D_q^{n-1}f$ is q-regular at zero, then*

$$\begin{aligned}f(x)=&\sum_{k=0}^{n-1}(-1)^k q^{-k(k-1)/2}\frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)}\,G_k(a;x,q)\\ &+\frac{1}{\Gamma_q(n)}\int_{aq^{-n+1}}^x G_{n-1}(x;qt,q)\,D_q^n f(t)\,d_qt.\end{aligned} \tag{1.4}$$

The proof of this theorem is also based on the modified Al-Salam q-Cauchy integral formula

$$\begin{aligned}&\int_a^x\int_{aq^{-1}}^{x_1}\int_{aq^{-2}}^{x_2}\dots\int_{aq^{-n+1}}^{x_{n-1}} D_q^n f(t)\,d_qt\,d_qx_{n-1}\dots d_qx_1\\ &\qquad=\frac{1}{\Gamma_q(n)}\int_{aq^{-n+1}}^x G_{n-1}(x;qt,q)\,D_q^n f(t)\,d_qt\\ &\qquad=f(x)-\sum_{k=0}^{n-1}(-1)^k q^{-k(k-1)/2}\frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)}\,G_k(a;x,q)\ .\end{aligned}$$

The purpose of this paper is to follow a different approach (not using Al-Salam q-Cauchy integral formulas) to obtain the same as Theorems 1.1 and 1.2, but with two different residue forms. Our approach is based on Newton's divided differences interpolation formula. We show that the sums in formulas (1.3) and (1.4) are indeed two direct consequences of a specific interpolation formula of Newton type and their corresponding remainders must obey the residue of a Newton interpolation formula. The main advantage of this approach is that we no longer consider the given constraints in Theorems 1.1 and 1.2 such as q-integrability on Ω_R and existence of the q-derivatives of f at zero up to order n. Two q-classes of interpolatory quadrature rules can be then computed from our results.

Here we add that the q-Taylor Theorems 1.1 and 1.2 with Lagrangian remainders have been studied in [22].

Another form of the q-Taylor theorem together with its remainder has been considered in [17]. In [21, Section 1.5], Phillips has studied two new interpolation formulas on q-integers.

2. Notations and preliminaries. Let $q \in (0,1)$ and $a \neq 0$. The q-shifted factorial and q-binomial coefficients are respectively defined by

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}) \quad \text{s.t.} \quad (a;q)_0 = 1,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q .$$

The q-binomial theorem [4, 8] gives the explicit expanded form of the q-shifted factorial as

$$\prod_{k=1}^{n}(1-aq^{k-1}) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k. \tag{2.1}$$

The q-Gamma function [9] is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad z \in \mathbb{C}, \quad |\, q\, | < 1,$$

where $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$. In particular we have

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}. \tag{2.2}$$

Let $\mu \in \mathbb{C}$ be fixed. According to [3], a set $A \subseteq \mathbb{C}$ is called a μ-geometric set if for $x \in A$, $\mu\, x \in A$. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{C}$. The q-difference operator is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \qquad x \in A - \{0\}.$$

If $0 \in A$, following [3] we say that f has a q-derivative at zero if

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \qquad x \in A,$$

exists and does not depend on x. The above limit is then denoted by $D_q f(0)$.

For a function f defined on a q-geometric set A, Jackson's q-integration [16] is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \qquad a, b \in A,$$

where

$$\int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n), \qquad x \in A, \tag{2.3}$$

provided that the series in (2.3) converges.

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According to [3], a function f which is defined on a q-geometric set A with $0 \in A$ is said to be q-regular at zero if $\lim_{n\to\infty} f(xq^n) = f(0)$ for every $x \in A$. The rule of q-integration by parts is

$$\int_0^a g(x)\, D_q f(x)\, d_q x = (fg)(a) - \lim_{n\to\infty}(fg)(aq^n) - \int_0^a D_q g(x)\, f(qx)\, d_q x. \tag{2.4}$$

If f, g are q-regular at zero, then $\lim_{n\to\infty}(fg)(aq^n)$ on the right-hand side of (2.4) can be replaced by $(fg)(0)$.

For $0 < R \leq \infty$, if Ω_R denotes again the disc $\{x \in \mathbb{C} : \ |\,x\,| < R\}$, then the following theorem is an analogue of the fundamental theorem of calculus whose proof is straightforward.

THEOREM 2.1. *Let $f : \Omega_R \to \mathbb{C}$ be q-regular at zero and $\theta \in \Omega_R$ be fixed. Define*

$$F(x) = \int_\theta^x f(t)\, d_q t, \qquad x \in \Omega_R.$$

Then the function F is q-regular at zero, $D_q F(x)$ exists for any $x \in \Omega_R$ and $D_q F(x) = f(x)$ for any $x \in \Omega_R$. Conversely, if $a, b \in \Omega_R$ then

$$\int_a^b D_q f(t)\, d_q t = f(b) - f(a).$$

The function f is q-integrable on Ω_R if $\displaystyle\int_0^x |\,f(t)\,|\; d_q t$ exists for all $x \in \Omega_R$.

3. *q*-Taylor series via Newton interpolation formulae. Let us consider the Newton interpolation formula [5] at the arbitrary nodes $\{x_k\}_{k=0}^{n-1} \in [a, b]$ as

$$\begin{aligned} f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + ... \\ + b_{n-1}(x - x_0)(x - x_1)...(x - x_{n-2}) + R_n(f; x), \end{aligned} \tag{3.1}$$

where $b_0 = f[x_0]$, $b_1 = f[x_0, x_1]$, $\ldots$, and $b_{n-1} = f[x_0, x_1, ..., x_{n-1}]$, respectively, denote divided differences [7] defined by

$$f[x_0, x_1, ..., x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{Q'_{n+1}(x_k)},$$

in which

$$Q_{n+1}(x) = (x - x_0)...(x - x_n) = \prod_{i=0}^{n}(x - x_i),$$

and $R_n(f; x)$ is the corresponding error in the form

$$R_n(f; x) = f[x_0, x_1, ..., x_{n-1}, x] \prod_{k=0}^{n-1}(x - x_k).$$

It is clear that if $f \in C^n[a, b]$, the above error can be expressed as

$$R_n(f; x) = \frac{f^{(n)}(\xi(x))}{n\,!} \prod_{k=0}^{n-1}(x - x_k)\,,$$

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in which $\xi(x)$ is strictly between the smallest and largest of the points $x_0, x_1, ..., x_{n-1}, x$. The following theorem now gives another representation for the residue of Theorem 1.1.

THEOREM 3.1. *Let f be a function defined in Ω_R. If the q-derivatives of f up to order n exist at zero and $\Gamma_q(k+1)$ is defined by (2.2), then for $a \neq 0$ and $q > 0$ we have*

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)}\, G_k(x;a,q) + r_n^*(f;x), \tag{3.2}$$

in which

$$r_n^*(f;x) = f[a, aq, \ldots, aq^{n-1}, x] \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)\,(n-k-1)/2} x^k. \tag{3.3}$$

Moreover, if $f \in C^n[\alpha, \beta]$ such that α is no larger than the smallest of the points $\{aq^k\}_{k=0}^{n-1}$ and β no smaller than the largest of the points $\{aq^k\}_{k=0}^{n-1}$, then the residue (3.3) changes to

$$r_n^*(f;x) = \frac{f^{(n)}(\xi(x))}{n\,!} \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)\,(n-k-1)/2} x^k, \qquad \xi(x) \in (\alpha, \beta).$$

Proof. Replace $\{x_k\}_{k=0}^{n-1} = \{aq^k\}_{k=0}^{n-1}$ in (3.1) to obtain

$$\begin{aligned} f(x) = f[a] + f[a, aq]\,(x-a) + f[a, aq, aq^2](x-a)(x-aq) + \ldots \\ + f[a, aq, \ldots, aq^{n-1}]\,(x-a)(x-aq)\ldots(x-aq^{n-2}) \\ + f[a, aq, \ldots, aq^{n-1}, x]\,(x-a)(x-aq)\ldots(x-aq^{n-1}), \end{aligned}$$

which is equivalent to

$$f(x) = \sum_{k=0}^{n-1} f[a, aq, \ldots, aq^k]\, G_k(x;a,q) + f[a, aq, \ldots, aq^{n-1}; x]\, G_n(x;a,q).$$

Therefore we must respectively prove

$$G_n(x;a,q) = \prod_{i=1}^{n} (x - aq^{i-1}) = \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)\,(n-k-1)/2} x^k \tag{3.4}$$

and

$$f[a, aq, \ldots, aq^k] = \frac{D_q^k f(a)}{\Gamma_q(k+1)}. \tag{3.5}$$

Note that Theorem 1.5.1 from [21] contains a connection between divided differences and q-differences and consequently a close relationship with (3.5).

To prove (3.4), we substitute $a \to x/a$ and $q \to 1/q$ in the q-binomial identity (2.1) to obtain

$$\left(\frac{x}{a}; \frac{1}{q}\right)_n = \prod_{i=1}^{n} \left(1 - \frac{x}{aq^{i-1}}\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} q^{-k(k-1)/2} a^{-k} x^k,$$

which results in

$$\prod_{i=1}^{n} (x - aq^{i-1}) = (-a)^n q^{\frac{n(n-1)}{2}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} q^{-k(k-1)/2} a^{-k} x^k. \tag{3.6}$$

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On the other hand, since

$$\begin{bmatrix} n \\ k \end{bmatrix}_{1/q} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q ,$$

the polynomial (3.6) leads to the geometric node polynomial (3.4).

To prove (3.5), we first refer to the following identity [9, 11] indicating that the mth q-derivative D_q^m of a function f can be explicitly expressed in terms of its values at the points $\{aq^j\}_{j=0}^m$:

$$D_q^k f(a) = (1-q)^{-k} a^{-k} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(j+1-2k)/2} f(aq^j). \tag{3.7}$$

The identity (3.7) is valid for every a in $\Omega_R - \{0\}$ and $k = 1, 2, \ldots, m$. Consider the following general identity [5, 7] relative to divided differences

$$f[a, aq, \ldots, aq^k] = \sum_{j=0}^{k} \frac{f(aq^j)}{G'_{k+1}(aq^j; a, q)}, \tag{3.8}$$

in which $G'_{k+1}(aq^j; a, q)$ is the derivative of the polynomial $G_{k+1}(x; a, q)$ at $x = aq^j$. By comparing (3.5), (3.7) and (3.8) we must have

$$\frac{(-1)^j}{(q;q)_k a^k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{j(j+1-2k)}{2}} = \frac{1}{G'_{k+1}(aq^j; a, q)} = \frac{1}{\prod_{\substack{i=0 \\ i\neq j}}^{k} (aq^j - aq^i)}. \tag{3.9}$$

After simplifying (3.9) we obtain

$$(-1)^j q^{\frac{j(j+1-2k)}{2}} \prod_{\substack{i=0 \\ i\neq j}}^{k} (q^j - q^i) = (q;q)_j \, (q;q)_{k-j} ,$$

which completes the proof of (3.5). □

Using the explicit form of (3.4) it can be concluded from (3.9) that

$$\begin{aligned} G'_{m+1}(aq^j; a, q) &= (-a)^m \sum_{k=0}^{m} (-1)^k (k+1) \begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q q^{\frac{(m-k)\,(m-k-1)}{2} + jk} \\ &= \frac{(q;q)_m a^m (-1)^j}{\begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{j(j+1-2m)}{2}}} , \end{aligned}$$

or equivalently by

$$\sum_{k=0}^{m} (-1)^{m-k} (k+1) \begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q q^{\frac{(m-k)\,(m-k-1)}{2} + jk} = \prod_{\substack{i=0 \\ i\neq j}}^{m} (q^j - q^i) ,$$

for any $j = 0, 1, \ldots, m$.

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Similarly, corresponding to Theorem 1.2, the following theorem gives another representation for the residue of relation (1.4).

THEOREM 3.2. *Let f be a function defined in Ω_R. If the q-derivatives of f up to order n exist at zero and $\Gamma_q(k+1)$ be defined by (2.2), then for $a \neq 0$ and $q > 0$ we have*

$$f(x) = \sum_{k=0}^{n-1} (-1)^k q^{-k(k-1)/2} \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)} \, G_k(a; x, q) + r_n^{**}(f; x), \tag{3.10}$$

in which

$$r_n^{**}(f; x) = f[a, aq^{-1}, \ldots, aq^{-(n-1)}, x] \, \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-k)\,(n+k-1)/2} x^k. \tag{3.11}$$

Moreover, if $f \in C^n[\alpha^*, \beta^*]$ such that α^* is no larger than the smallest of the points $\{aq^{-k}\}_{k=0}^{n-1}$ and β^* no smaller than the largest of the points $\{aq^{-k}\}_{k=0}^{n-1}$, then the residue (3.11) changes to

$$r_n^{**}(f; x) = \frac{f^{(n)}(\xi^*(x))}{n\,!} \, \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-k)\,(n+k-1)/2} x^k, \qquad \xi^*(x) \in (\alpha^*, \beta^*).$$

Proof. The proof is similar to the proof of Theorem 3.1 if we take $\{x_k\}_{k=0}^{n-1} = \{aq^{-k}\}_{k=0}^{n-1}$ and substitute them into (3.1) to get

$$f(x) = \sum_{k=0}^{n-1} f[a, aq^{-1}, \ldots, aq^{-k}] \, G_k(x; a, q^{-1}) + f[a, aq^{-1}, \ldots, aq^{-(n-1)}; x] \, G_n(x; a, q^{-1}).$$

On the other hand since

$$\begin{aligned} G_n(x; a, q^{-1}) &= \prod_{i=1}^{n} (x - aq^{-(i-1)}) = \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} q^{-(n-k)\,(n-k-1)/2} x^k \\ &= \sum_{k=0}^{n} (-a)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-k)\,(n+k-1)/2} x^k = (-1)^n q^{-n(n-1)/2} G_n(a; x, q)\,, \end{aligned}$$

and

$$f[a, aq^{-1}, \ldots, aq^{-k}] = \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)},$$

so the proof is complete. $\square$

4. Convergence analysis. First by using the explicit form of $G_k(x; a, q)$ in (3.4), the relation (3.2) can be written as

$$\begin{aligned} f(x) - r_n^*(f; x) &= \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)} \left(\sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)\,(k-j-1)/2} x^j \right) \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-k} \frac{(-a)^j q^{j\,(j-1)/2} D_q^{j+k} f(a)}{\Gamma_q(j+k+1)} \begin{bmatrix} j+k \\ k \end{bmatrix}_q \right) x^k \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-k} (-a)^j q^{\frac{j\,(j-1)}{2}} \frac{D_q^{j+k} f(a)}{\Gamma_q(j+1)} \right) \frac{x^k}{\Gamma_q(k+1)}. \end{aligned} \tag{4.1}$$

It is clear that if $\lim_{n\to\infty} r_n^*(f;x) = 0$ in (4.1), then f has the q-Taylor expansion

$$(4.2)\qquad \begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{D_q^k f(a)}{\Gamma_q(k+1)}\, G_k(x;a,q) \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} (-a)^j q^{\frac{j\,(j-1)}{2}} \frac{D_q^{j+k} f(a)}{\Gamma_q(j+1)} \right) \frac{x^k}{\Gamma_q(k+1)}, \end{aligned}$$

and any linear operator can be directly employed on the both sides of (4.2); see, e.g., [18, 19] for more details.

Hence, for the convergence of the series expansion (4.2), which is exactly equivalent to the condition $\lim_{n\to\infty} r_n^*(f;x) = 0$, we can use the ratio test by taking

$$u_k(x) = \frac{D_q^k f(a)}{\Gamma_q(k+1)}\, G_k(x;a,q),$$

such that we have

$$\frac{G_{k+1}(x;a,q)}{G_k(x;a,q)} = x - aq^k, \qquad \frac{\Gamma_q(k+1)}{\Gamma_q(k+2)} = \frac{1-q}{1-q^{k+1}},$$

and if we let

$$\lim_{k\to\infty} \left| \frac{D_q^{k+1} f(a)}{D_q^k f(a)} \right| = R_q(a),$$

then

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| = R_q(a) \lim_{k\to\infty} \left| \frac{1-q}{1-q^{k+1}(x-aq^k)} \right| < 1,$$

implies, for $0 < q < 1$, that

$$|x| < \frac{1}{(1-q)R_q(a)}.$$

PROPOSITION 4.1. *The series* (4.2) *is convergent for* $x \in \left(-\frac{1}{(1-q)R_q(a)}, \frac{1}{(1-q)R_q(a)}\right)$, *which directly depends on the value* $R_q(a)$.

This proposition similarly holds for Theorem 3.2.

REMARK 4.2. Since the formula (3.2) is exact for all elements of the monomial basis $\{x^r\}_{r=0}^{n-1}$, i.e., $r_n^*(\{x^r\}_{r=0}^{n-1};x) = 0$, the expansion (4.2) can be reduced to a finite sum. In other words, by noting that

$$D_q^m x^r = \begin{cases} \frac{(q;q)_r}{(q;q)_{r-m}(1-q)^m}\, x^{r-m} & r \geq m\,, \\ 0 & r < m, \end{cases}$$

substituting $f(x) = x^r$ in (3.2) finally yields

$$x^r = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q G_k(x;a,q),\ (r = 0, 1, \cdots, n-1).$$

5. Applications. In this section, we provide two applications for the interpolatory Theorems 3.1 and 3.2. First, two q-quadrature rules of forward and backward type are derived for approximating some definite integrals. Then they are applied for numerically solving a nonlinear class of ordinary differential equations. Our computations in this sense are done by the MAPLE software.

5.1. Application of Theorems 3.1 and 3.2 in q-quadrature rules. Keeping in mind that $r_n^*(f;x) = r_n^{**}(f;x) = 0$ for $f(x) = \{x^r\}_{r=0}^{n-1}$, one of the applications of Theorems 3.1 and 3.2 is to establish two backward and forward classes of weighted quadrature rules with geometric nodes; see also [20]. In other words, by multiplying the positive weight function $w(x)$ on both sides of (3.2) and then integrating from $[\alpha, \beta]$ we can obtain a backward rule as

$$\int_\alpha^\beta w(x)f(x)dx = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)} \int_\alpha^\beta w(x)G_k(x;a,q)dx + \int_\alpha^\beta w(x)r_n^*(f;x)dx, \tag{5.1}$$

where $0 < q < 1$, $a \neq 0$ and

$$\int_\alpha^\beta w(x)\,G_k(x;a,q)\,dx = \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)\,(k-j-1)/2} \int_\alpha^\beta w(x)\,x^j\,dx.$$

Also, by noting (3.7), (5.1) can be simplified as

$$\int_\alpha^\beta w(x)\,f(x)\,dx = \sum_{k=0}^{n-1} w_k(a,q)\,f(aq^k) + \int_\alpha^\beta w(x)\,r_n^*(f;x)\,dx\,, \tag{5.2}$$

where

$$w_k(a,q) = \frac{(-1)^k q^{\frac{k\,(k+1)}{2}}}{(q;q)_k} \sum_{j=k}^{n-1} \frac{\left(\sum_{i=0}^{j} (-a)^{-i} \begin{bmatrix} j \\ i \end{bmatrix}_q q^{\frac{(j-i)\,(j-i-1)}{2}} \int_\alpha^\beta w(x)\,x^i dx\right)}{(-1)^j (q;q)_{j-k} q^{k\,j}}.$$

According to (5.2), the error term is represented as

$$E_n^*[f] = \int_\alpha^\beta w(x)\,f(x)\,dx - \sum_{k=0}^{n-1} w_k(a,q)\,f(aq^k) = \int_\alpha^\beta w(x)\,r_n^*(f;x)\,dx,$$

and if $f \in C^n[\alpha,\beta]$, it takes the form

$$E_n^*[f] = \frac{1}{n!} \int_\alpha^\beta f^{(n)}(\xi(x))G_n(x;a,q)w(x)\,dx,$$

where $\xi(x) \in [\alpha,\beta]$. Moreover, if $\left|f^{(n)}(x)\right| \leq M_n^*$ for any $x \in [\alpha,\beta]$, then we have

$$|E_n^*[f]| \leq \frac{M_n^*}{n!} \int_\alpha^\beta |G_n(x;a,q)|\,w(x)\,dx. \tag{5.3}$$

Similarly, a forward rule for $q > 1$ and $a \neq 0$ can be obtained by using (3.10) as follows

$$\int_{\alpha^*}^{\beta^*} w(x)\,f(x)\,dx = \sum_{k=0}^{n-1} w_k^*(a,q)\,f(aq^{-k}) + \int_{\alpha^*}^{\beta^*} w(x)\,r_n^{**}(f;x)\,dx\,, \tag{5.4}$$

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where

$$w_k^*(a,q) = \frac{(-1)^k q^{\frac{k\,(k-1)}{2}}}{(q;q)_k} \sum_{j=k}^{n-1} \frac{\left(\sum_{i=0}^{j} (-a)^{-i} \begin{bmatrix} j \\ i \end{bmatrix}_q q^{\frac{(j-i)\,(-j-i+1)}{2}} \int_{\alpha^*}^{\beta^*} w(x)\, x^i dx\right)}{(q;q)_{j-k} q^{-j(j+1)/2}},$$

and the corresponding error term is represented as

$$E_n^{**}[f] = \int_{\alpha^*}^{\beta^*} w(x)\, f(x)\, dx - \sum_{k=0}^{n-1} w_k^*(a,q)\, f(aq^{-k}) = \int_{\alpha^*}^{\beta^*} w(x)\, r_n^{**}(f;x)\, dx.$$

Again if $f \in C^n[\alpha^*, \beta^*]$, $E_n^{**}[f]$ takes the form

$$E_n^{**}[f] = \frac{1}{n!} \int_{\alpha^*}^{\beta^*} f^{(n)}(\xi^*(x)) G_n(x; a, q^{-1}) w(x)\, dx,$$

where $\xi^*(x) \in [\alpha^*, \beta^*]$. Also, if $\left|f^{(n)}(x)\right| \leq M_n^{**}$ for any $x \in [\alpha^*, \beta^*]$, then

$$|E_n^{**}[f]| \leq \frac{M_n^{**}}{n!} \int_{\alpha^*}^{\beta^*} \left|G_n(x; a, q^{-1})\right| w(x)\, dx. \tag{5.5}$$

REMARK 5.1. Let

$$\int_{\alpha}^{\beta} |G_n(x;a,q)|\, w(x)\, dx \overset{0<q<1}{=} A_n(q) \quad \text{and} \quad \int_{\alpha^*}^{\beta^*} \left|G_n(x;a,q^{-1})\right| w(x)\, dx \overset{q\geq 1}{=} B_n(q).$$

It is clear that minimizing $A_n(q)$ and $B_n(q)$ completely depends on the weight function $w(x)$ and the parameter q. Also note that by minimizing two upper bounds (5.3) and (5.5) one may obtain an optimized value for the free parameter q. Let us study some numerical examples for this criteria.

EXAMPLE 5.2. Let $w(x) = 1$ and $f(x) = \sin x$ be defined on $[0, 1]$. Therefore

$$\int_0^1 \sin x\, dx = 0.459697694131860282.$$

Thanks to relations (5.3) and (5.5) for, e.g., $n = 2$ and $a = 1$, we obtain

$$A_2(q) = \int_0^1 |G_2(x;1,q)|\; dx \overset{0<q<1}{=} \frac{1}{6} - \frac{1}{2}\, q + q^2 - \frac{1}{3}\, q^3,$$
$$B_2(q) = \int_0^1 \left|G_2(x;1,q^{-1})\right|\; dx \overset{q\geq 1}{=} \frac{-2 + 6\, q - 3\, q^2 + q^3}{6q^3}.$$

It is easy to verify that there are two possible values $q_1 = 2.475686517795720716$ and $q_2 = 0.403928362016678476$ such that $A_2(q_1) \simeq 0$ and $B_2(q_2) \simeq 0$. However, due to the fact that we must have $q_1 \in (0, 1)$ and $q_2 > 1$, these values cannot be accepted.

On the other hand, $A_2(q)$ and $B_2(q)$ can be locally minimized so that we have

$$\tfrac{d}{dq} A_2(q) = 0 \Rightarrow q = 0.29289321881345247560 \vee q = 1.70710678118654752440.$$
$$\tfrac{d}{dq} B_2(q) = 0 \Rightarrow q = 0.58578643762690495120 \vee q = 3.41421356237309504880.$$

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One can easily check that the eligible points are respectively $q_1^l = 0.292893218813452475$ and $q_2^l = 3.41421356237309504$ so that we have

$$A_2(q_1^l) = 0.0976310729378174918 \quad \text{and} \quad B_2(q_2^l) = 0.0976310729378174918.$$

In this way, numerical results given in Table 5.1 show that both values q_1^l and q_2^l may be appropriate choices with respect to other values of q by noting that both of them have the least upper error bounds.

TABLE 5.1
Numerical results obtained by the formulas (5.2) and (5.4) for Example 5.2.

	The backward q-quadrature rule (5.2)			The forward q-quadrature rule (5.4)	
q	Upper Error Bound	Absolute Error	q	Upper Error Bound	Absolute Error
0.19	5.2517737042e-02	2.1073678356e-02	3.11	4.9093254557e-02	5.4583672030e-03
q_1^l	4.8815536469e-02	9.4371034810e-03	3.21	4.8931958964e-02	6.7376474350e-03
0.39	5.2184403708e-02	3.6728237213e-03	3.31	4.8843034373e-02	7.9262575080e-03
0.49	6.1624338759e-02	1.8199666522e-02	q_2^l	4.8815536469e-02	9.0332753730e-03
0.59	7.6135341622e-02	3.4078215194e-02	3.51	4.8840192338e-02	1.0066627474e-02
0.69	9.4717412297e-02	5.1234899408e-02	3.61	4.8909122624e-02	1.1033263279e-02
0.79	1.1637055078e-01	6.9588261175e-02	3.71	4.9015614269e-02	1.1939299338e-02
0.89	1.4009475708e-01	8.9049484853e-02	3.81	4.9153933816e-02	1.2790138841e-02
0.99	1.6489003119e-01	1.0952298031e-01	3.91	4.9319173401e-02	1.3590571368e-02

Note that the numerical results of Table 5.1 can be compared with the 2-point Newton-Cotes (classical trapezoid) rule as:

$$\int_0^1 \sin x \, dx \approx \frac{1}{2}\left(\sin(0) + \sin(1)\right) \approx 4.2073549240\text{e-}01,$$

whose absolute error is 3.8962201700e-02.

EXAMPLE 5.3. Let $w(x) = x$ and $f(x) = e^x$ be defined on $[0,1]$. Then $\int_0^1 xe^x \, dx = 1$. Thanks to the relations (5.3) and (5.5) for, e.g., $n = 3$ and $a = 1$ we obtain

$$\begin{aligned} A_3(q) &= \int_0^1 |G_3(x;1,q)|\, x \, dx \overset{0<q<1}{=} \frac{1}{20} - \frac{1}{12}\, q - \frac{1}{12}\, q^2 + \frac{1}{6}\, q^3 + \frac{1}{6}\, q^4 - \frac{13}{30}\, q^5 + \frac{1}{6}\, q^6 \\ &\quad + \frac{1}{3}\, q^7 - \frac{1}{6}\, q^8 - \frac{1}{6}\, q^9 + \frac{1}{10}\, q^{10}, \\ B_3(q) &= \int_0^1 \left|G_3(x;1,q^{-1})\right| x \, dx \overset{q\geq 1}{=} \frac{1}{60q^{10}} \left(6 - 10\, q - 10\, q^2 + 20\, q^3 + 10\, q^4 - 26\, q^5\right) \\ &\quad + \frac{1}{60q^{10}} \left(10\, q^6 + 10\, q^7 - 5\, q^8 - 5\, q^9 + 3\, q^{10}\right). \end{aligned}$$

Similarly, $A_3(q)$ and $B_3(q)$ can be locally minimized as follows:

$$\begin{aligned} &\tfrac{d}{dq} A_3(q) = 0 \Rightarrow q = 0.61464042624976502113, \\ &\tfrac{d}{dq} B_3(q) = 0 \Rightarrow q = 1.6269675037509498717. \end{aligned}$$

The eligible points are given by, respectively, $q_1^l = 0.61464042624976502113$ and $q_2^l = 1.6269675037509498717$, and we have

$$A_3(q_1^l) = 0.0070930130786776837\,92 \quad \text{and} \quad B_3(q_2^l) = 0.00709301307867768379\,15.$$

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In this way, numerical results are compared for different values of q in Table 5.2. Again, this table shows that the values q_1^l and q_2^l may be suitable to choose. As before, the numerical

TABLE 5.2
Numerical results obtained by the formulas (5.2) and (5.4) for Example 5.3.

	The backward q-quadrature rule (5.2)			The forward q-quadrature rule (5.4)	
	Upper Error Bound	Absolute Error		Upper Error Bound	Absolute Error
q			q		
0.11	1.7951172624e-02	1.0672961418e-02	1.12	1.2200481547e-02	9.8452895797e-03
0.21	1.3637132319e-02	8.3917878202e-03	1.22	7.7814438226e-03	5.8556325991e-03
0.31	9.6327446299e-03	6.0931283831e-03	1.32	5.2908405581e-03	3.6982593832e-03
0.41	6.3311138685e-03	4.0288325694e-03	1.42	3.9726805860e-03	2.5752216768e-03
0.51	4.0678839784e-03	2.5023139117e-03	1.52	3.3719884174e-03	2.0549100311e-03
q_1^l	3.2134680935e-03	1.8897764835e-03	q_2^l	3.2134680935e-03	1.8929853463e-03
0.71	4.2502465628e-03	2.6680669716e-03	1.72	3.3285679875e-03	1.9460679524e-03
0.81	7.7627578335e-03	5.4520833925e-03	1.82	3.6124907268e-03	2.1276189732e-03
0.91	1.4320641816e-02	1.1045944166e-02	1.92	3.9990116378e-03	2.3842663387e-03

results of Table 5.2 can be compared with the 3-point Newton-Cotes (classical Simpson) rule as:

$$\int_0^1 xe^x\,dx \approx \frac{1}{6}\left(0(e^0)+4\left(\frac{1}{2}e^{\frac{1}{2}}\right)+1(e^1)\right)\approx 1.0026207280\text{e+}00,$$

whose absolute error is 2.6207283000e-03.

EXAMPLE 5.4. Let $w(x)=\sqrt{x}$ and $f(x)=\dfrac{1}{1+x^2}$ be defined on $[0,1]$. Due to the formula [10]

$$\int_0^1 \frac{x^{\mu-1}}{1+x^p}\,dx=\frac{1}{p}\beta\left(\frac{\mu}{p}\right),$$

in which $p,\mu>0$ and $\beta(x)=\displaystyle\int_0^1\frac{t^{x-1}}{1+t}\,dt$, we have

$$\int_0^1\frac{\sqrt{x}}{1+x^2}\,dx=\frac{1}{2}\beta\left(\frac{3}{4}\right)=0.48749549439936104840.$$

Thanks to relations (5.3) and (5.5) for, e.g., $n=4$ and $a=1$ we obtain

$$\begin{aligned}A_4(q)=\int_0^1|G_4(x;1,q)|\sqrt{x}\,dx &\overset{0<q<1}{=} \frac{4}{99}-\frac{4}{63}q-\frac{4}{63}q^2+\frac{16}{315}q^3+\frac{4}{35}q^4+\frac{8}{63}q^{9/2}\\
&+\frac{4}{35}q^5-\frac{1072}{3465}q^{11/2}-\frac{4}{15}q^6-\frac{32}{315}q^{13/2}+\frac{208}{315}q^{15/2}+\frac{8}{35}q^8-\frac{8}{35}q^{17/2}\\
&-\frac{208}{315}q^9+\frac{32}{315}q^{10}+\frac{8}{15}q^{21/2}+\frac{1072}{3465}q^{11}-\frac{8}{35}q^{23/2}-\frac{8}{63}q^{12}-\frac{8}{35}q^{\frac{25}{2}}\\
&-\frac{32}{315}q^{\frac{27}{2}}+\frac{8}{63}q^{\frac{29}{2}}+\frac{8}{63}q^{\frac{31}{2}}-\frac{8}{99}q^{\frac{33}{2}},\end{aligned}$$

$$
\begin{aligned}
B_4(q) &= \int_0^1 \left|G_4(x;1,q^{-1})\right| \sqrt{x}\,dx \\
&\overset{q\geq 1}{=} \frac{4}{3465q^{\frac{57}{2}}}\Big(-70q^{12}+110q^{13}+110q^{14}-88q^{15}-198q^{16}\Big) \\
&- \frac{4}{3465q^{\frac{57}{2}}}\Big(110q^{\frac{33}{2}}-198q^{17}+268q^{\frac{35}{2}}+462q^{18}+88q^{\frac{37}{2}}-572q^{\frac{39}{2}}-198q^{20}+198q^{\frac{41}{2}}\Big) \\
&+ \frac{4}{3465q^{\frac{57}{2}}}\Big(572q^{21}-88q^{22}-231q^{\frac{45}{2}}-268q^{23}+99q^{\frac{47}{2}}+110q^{24}+99q^{\frac{49}{2}}+44q^{\frac{51}{2}}\Big) \\
&- \frac{4}{3465q^{\frac{57}{2}}}\Big(55q^{\frac{53}{2}}-55q^{\frac{55}{2}}+35q^{\frac{57}{2}}\Big).
\end{aligned}
$$

TABLE 5.3
Numerical results obtained by the formulas (5.2) and (5.4) for Example 5.4.

	The backward q-quadrature rule (5.2)			The forward q-quadrature rule (5.4)	
	Upper Error Bound	Absolute Error		Upper Error Bound	Absolute Error
q			q		
0.13	3.0512629873e-02	1.2017365646e-03	1.16	2.0360940533e-02	9.1332281838e-03
0.23	2.2763742811e-02	1.2078903158e-03	1.26	9.9939866259e-03	4.7374751043e-03
0.33	1.5412690886e-02	2.4908708384e-03	1.36	5.4876306307e-03	2.2275670912e-03
0.43	9.2747352200e-03	2.5294704642e-03	1.46	3.7149380779e-03	8.4944337477e-04
0.53	4.9926976886e-03	1.7136601540e-03	q_2^l	3.2917961400e-03	4.0155929723e-05
q_1^l	3.2917961400e-03	5.9676007359e-04	1.66	3.5694690560e-03	5.0741587699e-04
0.73	5.8285300691e-03	9.7140058693e-04	1.76	4.2246648760e-03	9.3300843598e-04
0.83	1.6632153376e-02	4.2725449384e-03	1.86	5.0858541462e-03	1.2913271919e-03
0.93	4.3835513303e-02	1.0355944272e-02	1.96	6.0564821619e-03	1.6004366887e-03

In a similar manner, it is verified that for the eligible values $q_1^l = 0.6394448209663673339\,2$ and $q_2^l = 1.5638565943637483407$, $A_4(q)$ and $B_4(q)$ are locally minimized, respectively. In this way, numerical results are shown in Table 5.3. Once again, the numerical results of Table 5.3 can be compared with the 4-point Newton-Cotes rule as:

$$
\begin{aligned}
\int_0^1 \frac{\sqrt{x}}{1+x^2}\,dx &\approx \frac{1}{8}\left(\frac{\sqrt{0}}{1+0^2}+3\frac{\sqrt{\frac{1}{3}}}{1+\left(\frac{1}{3}\right)^2}+3\frac{\sqrt{\frac{2}{3}}}{1+\left(\frac{2}{3}\right)^2}+\frac{\sqrt{1}}{1+(1)^2}\right) \\
&\approx 2.8750000000\text{e-}01,
\end{aligned}
$$

whose absolute error is 3.5284062700e-02.

5.2. Application to numerically solving ordinary differential equations. Let us consider the first order differential equation

$$
y'(x) = F(x, y(x)),\ y(0) = y_0,\ x \in [0, T]. \tag{5.6}
$$

We wish to numerically solve (5.6) using the set of basisfunction $\{G_n(x;a,q) = [x-a]_{n,q}\}_{n=0}^{\infty}$.

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For this purpose, the unknown solution may be approximated as

$$y(x) \approx y_n(x) = \sum_{k=0}^{n} a_k G_k(x;a,q).$$

We need to compute the first derivative of $G_k(x;a,q)$ for $k = 0,1,\ldots,n$. It is easy to see that

$$\frac{d}{dx} G_k(x;a,q) = G_k(x;a,q) \left(\sum_{j=0}^{k-1} \frac{1}{x - aq^j} \right) = \sum_{j=0}^{k-1} \prod_{j \neq r=0}^{k-1} (x - aq^r).$$

Therefore we have

$$\frac{d}{dx} y(x) \approx \frac{d}{dx} y_n(x) = \sum_{k=0}^{n} a_k \frac{d}{dx} G_k(x;a,q) = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k-1} \prod_{j \neq r=0}^{k-1} (x - aq^r) \right). \tag{5.7}$$

Now substituting (5.7) into (5.6) yields

$$\begin{aligned} &\sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k-1} \prod_{j \neq r=0}^{k-1} (x - aq^r) \right) \approx F\left(x, \sum_{k=0}^{n} a_k G_k(x;a,q) \right), \\ &y_n(0) = \sum_{k=0}^{n} a_k G_k(0;a,q) = y_0. \end{aligned} \tag{5.8}$$

To evaluate the unknown coefficients a_k, $k = 0,1,\ldots,n$, we can use different methods such as the well-known collocation method. In this sense, an appropriate choice of the collocation points plays a key role to obtain efficient and stable numerical approximations. To reach this goal, we can choose a set of Chebyshev-Gauss collocating points on the interval $(0,T]$, as

$$\tau_i = \frac{T}{2} - \frac{T}{2} \cos\left(\frac{2i+1}{n+1} \frac{\pi}{2} \right), \ i = 0,1,\ldots,n-1,$$

which is used in [23]. Thus from (5.8) we get the following nonlinear system of algebraic equations

$$\begin{aligned} &\sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k-1} \prod_{j \neq r=0}^{k-1} (\tau_i - aq^r) \right) \approx F\left(\tau_i, \sum_{k=0}^{n} a_k G_k(\tau_i;a,q) \right), \ i = 0,1,\ldots,n-1, \\ &\sum_{k=0}^{n} a_k G_k(0;a,q) = y_0. \end{aligned} \tag{5.9}$$

The unknown coefficients a_k in (5.9) can be estimated by using the Newton method. Note that all computations and figures are carried out by the MAPLE software.

EXAMPLE 5.5. Consider the equation

$$y' + (\sin x) y = \mathrm{e}^{-x} \left((x-1)\sin x + \cos x - \cos^2 x - x + 2 \right), \quad y(0) = 0,$$

with the exact solution $y(x) = \mathrm{e}^{-x}(x + \sin x)$. After evaluating the unknown coefficients corresponding to the nonlinear system (5.9) by the Maple, the exact and numerical solutions of the equation for $n = 10,\ a = 1,\ q = 0.5$ and error functions $E_n(x) = y(x) - y_n(x)$ for

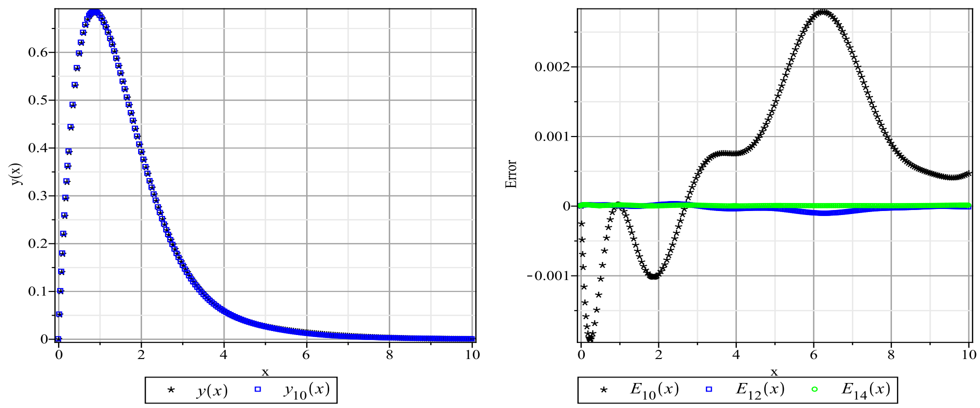

FIG. 5.1. *Numerical solution versus the exact one (left) and comparison of the error functions $E_n(x)$ for some values of n (right) for Example 5.5.*

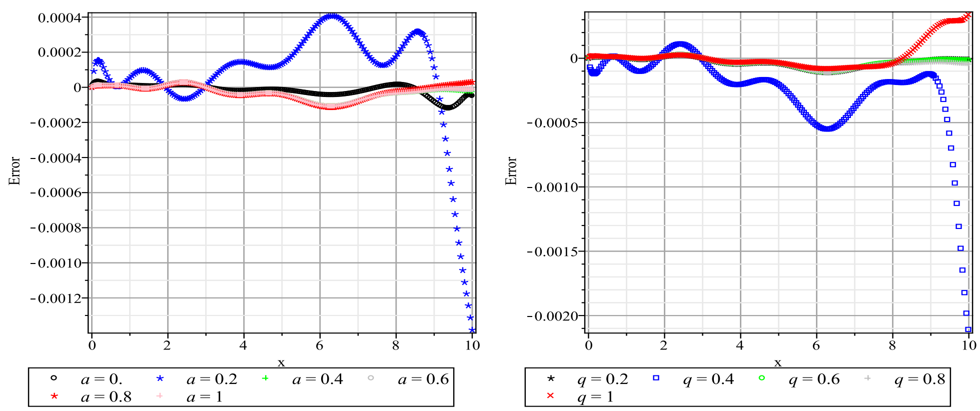

FIG. 5.2. *Behavior of $E_n(x)$ for $n = 10,\ q = 0.5$ and some values of a (left) and for some values of q and $a = 1$ (right) for Example 5.5.*

$n = 10, 12, 14$ and $a = 1,\ q = 0.5$ are shown in Figure 5.1. Moreover, the behavior of the error functions $E_n(x)$ are displayed in Figure 5.2 for some values of a and $q = 0.5$ in the left figure and also different values of q and for $a = 1$ in the right figure.

EXAMPLE 5.6. Consider the nonlinear equation

$$y' + \sin y = \sin x + x \cos x + \sin\left(1 + x \sin x\right),\ y(0) = 1,$$

with the exact solution $y(x) = 1 + x \sin x$. Comparing the exact solution versus numerical solution for $n = 10,\ a = 1$ and $q = 0.5$ and error functions $E_n(x) = y(x) - y_n(x)$ for some values of n, $a = 1$ and $q = 0.5$ are shown in Figure 5.3. Also, the behavior of the error functions $E_n(x)$ are displayed in Figure 5.4 for some values of a and $q = 0.5$ in the left figure and also different values of q and for $a = 1$ in the right figure.

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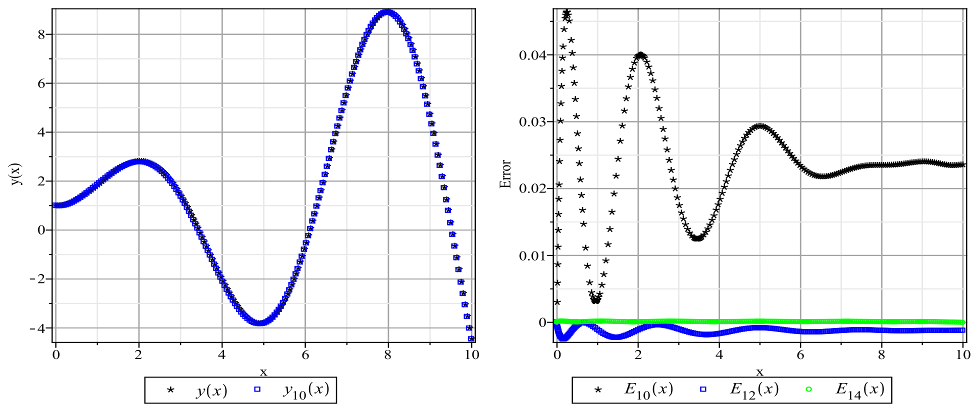

FIG. 5.3. *Numerical solution versus the exact one (left) and comparison of the error functions $E_n(x)$ for some values of n (right) for Example* 5.6.

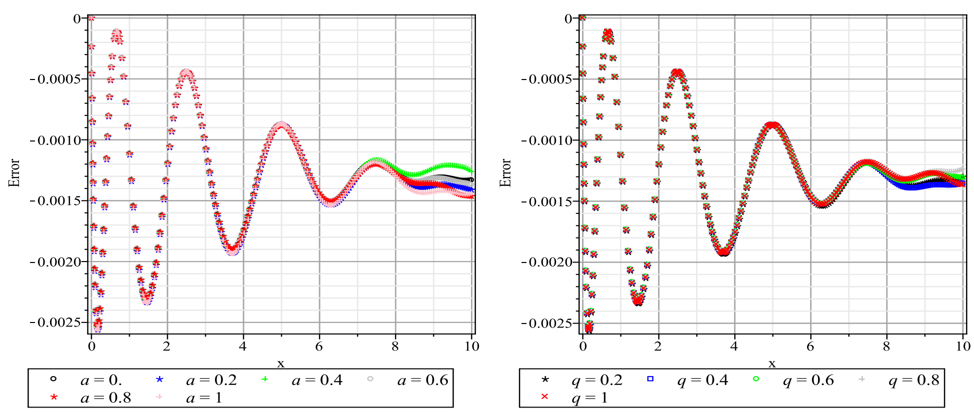

FIG. 5.4. *Behavior of $E_n(x)$ for $n = 10,\ q = 0.5$ and some values of a (left) and for some values of q and $a = 1$ (right) for Example* 5.6.

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