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# CONVERGENCE ANALYSIS OF GALERKIN POD FOR LINEAR SECOND ORDER EVOLUTION EQUATIONS*

SABRINA HERKT†, MICHAEL HINZE‡, AND RENE PINNAU§

**Abstract.** In this paper, we investigate the proper orthogonal decomposition (POD) discretization method for linear second order evolution equations. We present error estimates for two different choices of snapshot sets, one consisting of solution snapshots only and one consisting of solution snapshots and their derivatives up to second order. We show that the results of [Numer. Math., 90 (2001), pp. 117–148] for parabolic equations can be extended to linear second order evolution equations, and that the derivative snapshot POD method behaves better than the classical one for small time steps. Numerical comparisons of the different approaches are presented, illustrating the theoretical results.

**Key words.** Wave equation, POD, error estimates

**AMS subject classifications.** 34A05

**1. Introduction.** Simulation of industrial problems like flow or heat transfer often requires the solution of large linear or nonlinear systems consisting of tens of thousands of degrees of freedom [9]. Problems of such high dimensions can be handled by using powerful computers with large storage capabilities. Additionally, in some applications, these simulations need to be repeated several times with slightly different input, like in general controller design problems or in the durability simulation of wind turbines [8]. Often even real-time simulation is required, like in multibody dynamics with hardware-in-the-loop or human-in-the-loop systems. In these cases, simulation time becomes an important issue.

Over the years, various methods of model reduction for both linear and nonlinear systems have been developed [11]. These methods allow the construction of low-dimensional reduced models that conserve the essential properties and features of the large model. Whereas the most popular reduction methods such as balanced truncation, moment matching, or analysis of eigenforms only seem to be suitable for linear problems, the method of *proper orthogonal decomposition (POD)* can also be applied to nonlinear systems. Its flexibility in application is based on analyzing a given data set to provide the reduced model, as described in Section 2 of this paper. Originating from fluid dynamic applications including turbulence and coherent structures [2], the method has also proved useful for certain problems in optimal control [5] and in circuit simulations [14].

To justify the method mathematically, Kunisch and Volkwein [6, 7] proved error bounds for POD-Galerkin approximations of linear and nonlinear parabolic equations, respectively. Guided by their numerical analysis, they proposed to also include derivative information in the snapshot set, and proved the superiority of this modified POD approach over the classical one both theoretically and numerically. In [5], the second author and Volkwein extended this analysis to optimal control problems using POD-surrogate models.

In Section 3 of this work, we derive error estimates for POD-Galerkin approximations to linear second order evolution equations based on time discretization with Newmark's scheme. Similar to [6] and [7], we show that convergence can be guaranteed for the derivative approach and for the classical method if the time step size and the dimension of the POD subspace are coupled appropriately. Our numerical experiments with the wave equation, described in

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†Volkswagen AG, Wolfsburg (sabrina.herkt@volkswagen.de)
‡Department of Mathematics, University of Hamburg (michael.hinze@uni-hamburg.de)
§Department of Mathematics, University of Kaiserslautern (pinnau@mathematik.uni-kl.de)

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Section 4, show that the error behavior of both methods strongly depends on the eigenvalues of the correlation matrix.

**2. POD for linear second order evolution equations.** The linear wave equation is a simple example of a partial differential equation of second order. In this section, we outline the mathematical framework required to handle such problems. Furthermore, we describe the discretization by Newmark's method and the POD scheme.

**2.1. Problem description.** Let $V$ and $H$ be real, separable Hilbert spaces for which we require [4, 6]

$$V \hookrightarrow H = H' \hookrightarrow V',$$

where $V'$ denotes the dual of $V$. Each embedding is assumed to be dense and continuous. Further, let $a : V \times V \to \mathbb{R}$ be a continuous, coercive and symmetric bilinear form, i.e., there exist constants $\beta, \kappa \geq 0$ such that

$$\|a(\phi,\psi)\| \leq \beta \|\phi\|_V \|\psi\|_V , \tag{2.1}$$

$$\kappa \|\phi\|_V^2 \leq a(\phi,\phi), \tag{2.2}$$

for all $\phi, \psi \in V$.

As a simple example of a second order evolution equation, we chose the linear wave equation expressed in weak form:

$$\langle \ddot{x}(t), \phi\rangle_H + D\langle \dot{x}(t), \phi\rangle_H + a\,(x(t), \phi) = \langle f(t), \phi\rangle_H \quad \text{for all } \phi \in V \text{ and } t \in [0,T], \tag{2.3a}$$

$$\langle x(0), \psi\rangle = \langle x_0, \psi\rangle_H \quad \text{for all } \psi \in H, \tag{2.3b}$$

$$\langle \dot{x}(0), \psi\rangle = \langle \dot{x}_0, \psi\rangle_H \quad \text{for all } \psi \in H, \tag{2.3c}$$

where $f(t) \in H$ is a given external force and $x(t) \in V$ denotes the unknown deformation over time $t \in [0,T]$. Note that (2.3) incorporates a damping term that corresponds to a Rayleigh-type damping matrix $C$ which is $D$ times the mass matrix [3, 4].

Concerning the existence of a unique solution, the following result is available [3].

PROPOSITION 2.1. *For $f \in L^2((0,T);H)$ and $x_0, \dot{x}_0 \in H$, the weak form problem (2.3) admits a unique solution.*

**2.2. POD-Newmark scheme.** We discretize (2.3) in time using Newmark's scheme [4]; that is, we partition the time interval $[0,T]$ into $m$ subintervals of equal size $\Delta t = T/m$, and we seek a sequence $(X_k) \subset V$, $k = 0,\dots,m$, that satisfies the following equations for each time $t_k = k \cdot \Delta t$:

$$\langle \partial\partial X_k, \phi\rangle_H + D\langle \partial X_k, \phi\rangle_H + a\,(X_k, \phi) = \langle f(t_k), \phi\rangle_H \quad \text{for all } \phi \in V \text{ and } k = 1,\dots m, \tag{2.4a}$$

$$\langle X_0, \psi\rangle = \langle x_0, \psi\rangle_H \quad \text{for all } \psi \in V, \tag{2.4b}$$

$$\langle \partial X_0, \psi\rangle = \langle \partial x_0, \psi\rangle_H \quad \text{for all } \psi \in V. \tag{2.4c}$$

Here we use the derivative approximations

$$\partial X_{k+1} = \frac{2}{\Delta t} X_{k+1} - \frac{2}{\Delta t} X_k - \partial X_k, \tag{2.5a}$$

$$\partial\partial X_{k+1} = \frac{4}{\Delta t^2} X_{k+1} - \frac{4}{\Delta t^2} X_k - \frac{4}{\Delta t}\partial X_k - \partial\partial X_k, \tag{2.5b}$$

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for $k = 1, \dots, m$.

REMARK 2.2. The initial deformation $X_0$ and velocity $\partial X_0$ in the Newmark problem (2.4) are determined by the initial conditions $x_0$ and $\partial x_0$ from (2.3). The equilibrium equations then yield the acceleration $\partial\partial X_0$.

Like any Galerkin-type method, *proper orthogonal decomposition* is a spatial discretization scheme approximating the solution $X_k$ by a linear combination of basis vectors $\varphi_i \in V$,

$$X_k = \sum_{i=1}^{l} \varphi_i \cdot p_i(t_k) \quad \text{for} \quad k = 1 \dots m, \tag{2.6}$$

where $p_i$ denotes the time-dependent participation factor of the basis vector $i$ in the solution. Setting $V^l = \operatorname{span}\{\varphi_1, \dots, \varphi_l\} \subset V$, the POD-Newmark scheme for the wave equation involves finding a sequence $\{X_k\}_{k=0,\dots,m} \subset V^l$ that satisfies

$$\langle \partial\partial X_k, \phi\rangle_H + D\langle \partial X_k, \phi\rangle_H + a\left(X_k, \phi\right) = \langle f(t_k), \phi\rangle_H \quad \text{for all } \phi \in V^l \text{ and } k = 1, \dots, m, \tag{2.7a}$$

$$\langle X_0, \psi\rangle = \langle x_0, \psi\rangle_H \quad \text{for all } \psi \in V^l, \tag{2.7b}$$

$$\langle \partial X_0, \psi\rangle = \langle \partial x_0, \psi\rangle_H \quad \text{for all } \psi \in V^l. \tag{2.7c}$$

The unique solvability of these equations follows from the following result [3].

PROPOSITION 2.3. *Under the above assumptions there exists a unique solution $X_k \in V^l$ to problem (2.7) for each time level $k = 1, \dots, m$.*

The essential step of the *snapshot POD method* [12] is the construction of the subspace $V^l$. In the usual snapshot POD approach, we take snapshots $X_k$, $k = 1, \dots, m$, of the previously computed solution of problem (2.4). The subspace $V^l$ is chosen to be the best approximation of the snapshot set $\{X_k\}$ in a least squares sense [10]. In this paper we consider POD subspaces built from two different snapshot sets: set $I$ consisting of deformation snapshots $\{x(t_k)\}$ at all time instances, and set $II$ consisting of deformations and derivative approximations $\{x(t_k), \partial x(t_k), \partial\partial x(t_k)\}$. These sets yield the snapshot matrices $Y_I$ and $Y_{II}$ defined by

$$Y_I = [x(t_0), \dots, x(t_m)] \quad \text{and} \tag{2.8}$$

$$Y_{II} = [x(t_0), \dots, x(t_m), \partial x(t_1), \dots, \partial x(t_m), \partial\partial x(t_1), \dots, \partial\partial x(t_{m-1})]. \tag{2.9}$$

Note that the derivative approximations $\partial x(t_k)$ and $\partial\partial x(t_k)$ are elements of the space $V$. Furthermore, their inclusion does not change the dimension of the snapshot set, since they can be expressed as linear combinations of the deformation snapshots

$$\partial X_{k+1} + \partial X_k = \frac{2}{\Delta t}\left(X_{k+1} - X_k\right),$$

$$\partial\partial X_{k+1} + 2\partial\partial X_k + \partial\partial X_{k-1} = \frac{4}{\Delta t^2}\left(X_{k+1} - 2X_k + X_{k-1}\right).$$

We write $Y_{I,II} = [y_0, \dots, y_d]$ with either $d = m$ or $d = 3m - 1$. In both cases, we follow the regular POD recipe [13]:

1. Construct the correlation matrix $C$ from scalar products of the snapshots $y_i$ in the space $X = V$ or the space $X = H$:

$$C_{ij} = \langle y_i, y_j\rangle_X, \quad i, j = 1, \dots d.$$

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2. Solve the eigenvalue problem

$$Cv^k = \lambda_k v^k.$$

3. Define the POD basis vectors by

$$\phi_k = \frac{1}{\sqrt{\lambda_k}} Y \cdot v^k.$$

Each of the orthonormal eigenvectors $v^k$ of the correlation matrix defines a basis vector $\varphi_k$ of the POD subspace. Depending on the number of basis vectors used for the subspace $V^l = \operatorname{span}\{\varphi_1, \dots, \varphi_l\}$, the projection error for

$$P^l y := \sum_{j=1}^{l} \langle y, \varphi_j \rangle_X \cdot \varphi_j$$

can be expressed as

$$\frac{1}{m} \sum_{k=1}^{m} \left\| y_k - \sum_{j=1}^{l} \langle y_k, \varphi_j \rangle_X \cdot \varphi_j \right\|_X^2 = \sum_{j=l+1}^{d} \lambda_j. \tag{2.10}$$

The integer $d < n$ denotes the dimension of the snapshot set $Y$ and $l < d$ is the number of POD basis vectors used for the projection.

The "optimal basis" consists of the eigenvectors corresponding to the $l$ largest eigenvalues and spans the subspace $V^l$ with the smallest projection error of all possible $l$-dimensional subspaces $\hat{V}^l \subset V$. This set of basis vectors is often called the *Karhunen-Loève basis* [11].

**3. Error estimates.** The error of the POD-Newmark scheme is defined as the difference between the numerical solution $X(t)$ of (2.7) and the analytical solution $x(t)$ of (2.3). Our goal is to prove a bound for the $H$-norm of the solution difference.

THEOREM 3.1. *Let $x(t)$ be the regular solution of (2.3) and $X_k, k = 1, \dots, m$, be the solution of (2.7) at each time step $t_k$. Let the POD subspace $V^l$ be constructed from snapshot set $Y_I$ or $Y_{II}$, respectively. Then there exist constants $C_I$ and $C_{II}$ depending on $T$, $D$, $x^{(3)}$ and $x^{(4)}$, but not on $\Delta t$, $m$ or $l$, such that for $\Delta t \le 1$,*

$$\begin{aligned} &\frac{1}{m} \sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \le \\ &\quad C_I \Bigg( \left\|X_0 - P^l x(t_0)\right\|_H^2 + \left\|X_1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X_0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\qquad + \Delta t \left\|\partial X_1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \left( \frac{1}{\Delta t^4} + \frac{1}{\Delta t} + 1 \right) \sum_{j=l+1}^{d} \lambda_{Ij} \Bigg) \end{aligned} \tag{3.1}$$

*for snapshots constructed via $Y_I$, and*

$$\begin{aligned} &\frac{1}{m} \sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \le \\ &\quad C_{II} \Bigg( \left\|X_0 - P^l x(t_0)\right\|_H^2 + \left\|X_1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X_0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\qquad + \Delta t \left\|\partial X_1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \sum_{j=l+1}^{d} \lambda_{IIj} \Bigg) \end{aligned} \tag{3.2}$$

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*for snapshots constructed via* $Y_{II}$.

REMARK 3.2. These estimates are constructed in a similar way to the estimates given in [6] and [7]. In analogy to [6, Lemma 2] we have: For all $x \in V$,

$$\|x\|_H \leq \sqrt{\|M\|_2 \cdot \|K^{-1}\|_2} \cdot \|x\|_V \text{ for all } x \in V^l, \tag{3.3}$$

$$\|x\|_V \leq \sqrt{\|K\|_2 \cdot \|M^{-1}\|_2} \cdot \|x\|_H \text{ for all } x \in V^l, \tag{3.4}$$

$$\text{with} \quad M_{ij} = \langle \Phi_i, \Phi_j \rangle_H, \quad K_{ij} = \langle \Phi_i, \Phi_j \rangle_V, \tag{3.5}$$

where $\|\cdot\|_2$ denotes the spectral norm for symmetric matrices. $M$ and $K$ are called the system's mass matrix and stiffness matrix, respectively.

These inequalities allow us to set up an error estimate in the $H$-norm and also to control the error in the $V$-norm as long as $\|M\|_2$, $\|M^{-1}\|_2$, $\|K\|_2$ and $\|K^{-1}\|_2$ are bounded. Hence, we restrict ourselves to the $H$-norm in the following.

REMARK 3.3. Note that the eigenvalues $\lambda_{Ij}$ and $\lambda_{IIj}$ are not identical. The weighting of snapshots is changed by inclusion of the derivative approximations, which leads to different choices of basis vectors for the subspaces $V_I^l$ and $V_{II}^l$. In both cases, the snapshot correlation matrix $C$ is generally not invertible, so the sum of the eigenvalues remains finite.

*Proof.* Recall that $X_k$ is the solution of the POD system (2.7) at times $t_k = k \cdot \Delta t$, $k = 0, \ldots, m$, and $x(t_k)$ is the corresponding solution of the original system (2.4). In order to estimate

$$\frac{1}{m} \sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \tag{3.6}$$

we decompose the local error into a projection part $\rho$ and a part $\vartheta$ arising from the numerical discretization procedure:

$$X_k - x(t_k) = \underbrace{X_k - P^l x(t_k)}_{=:\vartheta_k} + \underbrace{P^l x(t_k) - x(t_k)}_{=:\rho_k}, \tag{3.7}$$

which yields

$$\frac{1}{m} \sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \leq \frac{2}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 + \frac{2}{m} \sum_{k=1}^{m} \|\rho_k\|_H^2 . \tag{3.8}$$

For an estimate of $\|\rho_k\|_H^2$ we use the error bound (2.10). Case $I$ is constructed in the "classical" way and simply yields the POD projection error [13]

$$\frac{1}{m+1} \sum_{k=0}^{m} \left\| x_k - \sum_{j=1}^{l} \langle x_k, \phi_j \rangle_X \cdot \phi_j \right\|_X^2 = \sum_{j=l+1}^{d} \lambda_{Ij}. \tag{3.9}$$

Here, $x_k$ denotes the snapshot $x(t_k)$.

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For later use we derive

$$\sum_{k=1}^{m-1}\left\|\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2$$

$$=\sum_{k=1}^{m-1}\frac{1}{\Delta t^4}\left\|x_{k+1}-2x_k+x_{k-1}-P^l x_{k+1}+2P^l x_k-P^l x_{k-1}\right\|_X^2$$

$$\leq\frac{4}{\Delta t^2}\sum_{k=1}^{m-1}2\Big(\left\|x_{k+1}-P^l x_{k+1}\right\|_X^2+\left\|x_{k-1}-P^l x_{k-1}\right\|_X^2\Big)$$

$$\leq\frac{16}{\Delta t^2}\sum_{k=0}^{m}\left\|x_k-P^l x_k\right\|_X^2$$

$$\leq\frac{16}{\Delta t^2}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij} \tag{3.10}$$

and

$$\sum_{k=1}^{m-1}\left\|\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2$$

$$=\sum_{k=1}^{m-1}\frac{16}{\Delta t^4}\left\|x_{k+1}-2x_k+x_{k-1}-P^l x_{k+1}+2P^l x_k-P^l x_{k-1}\right\|_X^2$$

$$\leq\frac{16}{\Delta t^4}\sum_{k=1}^{m-1}4\Big(\left\|x_{k+1}-P^l x_{k+1}\right\|_X^2+\left\|2x_k-2P^l x_k\right\|_X^2+\left\|x_{k-1}-P^l x_{k-1}\right\|_X^2\Big)$$

$$\leq\frac{384}{\Delta t^4}\sum_{k=0}^{m}\left\|x_k-P^l x_k\right\|_X^2$$

$$\leq\frac{384}{\Delta t^4}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij}. \tag{3.11}$$

For the second case, the POD error is analogously defined for the sum over all snapshots (solution $x$ and derivatives $\partial x$ and $\partial\partial x$):

$$\frac{1}{3m}\sum_{k=0}^{m}\left\|x_k-\sum_{j=1}^{l}\langle x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2+$$

$$+\frac{1}{3m}\sum_{k=1}^{m}\left\|\partial x_k-\sum_{j=1}^{l}\langle\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2+$$

$$+\frac{1}{3m}\sum_{k=1}^{m-1}\left\|\partial\partial x_k-\sum_{j=1}^{l}\langle\partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2=\sum_{j=l+1}^{d}\lambda_{IIj}, \tag{3.12}$$

which yields

$$\frac{1}{m}\sum_{k=1}^{m}\|\rho_k\|_X^2 = \frac{1}{m}\sum_{k=1}^{m}\left\|x_k - \sum_{j=1}^{l}\langle x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 \le 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj},$$
$$\frac{1}{m}\sum_{k=1}^{m}\left\|\partial x_k - \sum_{j=1}^{l}\langle \partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 \le 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj},$$
$$\frac{1}{m}\sum_{k=1}^{m-1}\left\|\partial\partial x_k - \sum_{j=1}^{l}\langle \partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 \le 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj}.$$

Hence, we get the estimate

$$\begin{aligned}\sum_{k=1}^{m-1}&\left\|\partial\left(x_{k+1}+2x_k+x_{k-1}\right) - \sum_{j=1}^{l}\langle\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2\\ &\le 24\sum_{k=1}^{m-1}\left\|\partial x_k - \sum_{j=1}^{l}\langle\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2\\ &\le 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj}, \end{aligned}\tag{3.13}$$

and an analogous estimate holds for the second derivatives

$$\begin{aligned}\sum_{k=1}^{m-1}&\left\|\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right) - \sum_{j=1}^{l}\langle\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2\\ &\le 24\sum_{k=1}^{m-1}\left\|\partial\partial x_k - \sum_{j=1}^{l}\langle\partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2\\ &\le 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj}. \end{aligned}\tag{3.14}$$

For an estimate of $\|\vartheta_k\|_H^2 = \left\|X_k - P^l x(t_k)\right\|_H^2$, we state the following identity:

$$\begin{aligned}&\langle\partial\partial\vartheta_k,\psi\rangle_H + D\langle\partial\vartheta_k,\psi\rangle_H + a\left(\vartheta_k,\psi\right)\\ &= \langle\partial\partial X_k,\psi\rangle_H - \langle\partial\partial P^l x(t_k),\psi\rangle_H + D\cdot\langle\partial X_k,\psi\rangle_H - D\cdot\langle\partial P^l x(t_k),\psi\rangle_H\\ &\quad + a\left(X_k,\psi\right) - a\left(P^l x(t_k),\psi\right)\\ &= \langle f(t_k),\psi\rangle_H - a\left(P^l x(t_k),\psi\right) - \langle\partial\partial P^l x(t_k),\psi\rangle_H - D\langle\partial P^l x(t_k),\psi\rangle_H\\ &= \langle f(t_k),\psi\rangle_H - a\left(x(t_k),\psi\right) - \langle\partial\partial P^l x(t_k),\psi\rangle_H - D\langle\partial P^l x(t_k),\psi\rangle_H\\ &= \langle\ddot{x}(t_k),\psi\rangle_H + D\langle\dot{x}(t_k),\psi\rangle_H - \langle\partial\partial P^l x(t_k),\psi\rangle_H\\ &= \langle\left(\ddot{x}(t_k) - \partial\partial P^l x(t_k)\right) + D\left(\dot{x}(t_k) - \partial P^l x(t_k)\right),\psi\rangle_H\\ &=: \langle v_k,\psi\rangle_H, \end{aligned}\tag{3.15}$$

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which holds for all $\psi \in V^l$.

Hence, the sequence $\vartheta_k$ can be regarded as the solution of a linear, damped wave equation with the "force term" $v_k$. In analogy to the centered scheme described in [4], the Newmark scheme for this equation can be written as

$$\frac{1}{\Delta t^2}\langle \vartheta_{k+1} - 2\vartheta_k + \vartheta_{k-1}, \psi\rangle_H + \frac{2D}{\Delta t}\langle \vartheta_{k+1} - \vartheta_{k-1}, \psi\rangle_H + \\ + \frac{1}{4} a\left(\vartheta_{k+1} + 2\vartheta_k + \vartheta_{k-1}, \psi\right) = \frac{1}{4}\langle v_{k+1} + 2v_k + v_{k-1}, \psi\rangle_H .$$

For convenience, we define $\gamma_k = v_{k+1} + 2v_k + v_{k-1}$. Choosing $\psi = \vartheta_{k+1} - \vartheta_{k-1} \in V^l$ as a test function in (3.15), we get

$$\underbrace{\frac{1}{\Delta t^2}\langle \vartheta_{k+1} - 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1}\rangle_H}_{=:T_1} + \underbrace{\frac{2D}{\Delta t}\left\|\vartheta_{k+1} - \vartheta_{k-1}\right\|_H^2}_{=:d^k \le 0} + \\ + \underbrace{\frac{1}{4} a\left(\vartheta_{k+1} + 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1}\right)}_{=:T_2} = \underbrace{\frac{1}{4}\langle \gamma_k, \vartheta_{k+1} - \vartheta_{k-1}\rangle_H}_{=:W^k} .$$

Further, it holds that

$$\begin{aligned} T_1 &= \frac{1}{\Delta t^2}\langle \vartheta_{k+1} - 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1}\rangle_H = \\ &= \frac{1}{\Delta t^2}\langle (\vartheta_{k+1} - \vartheta_k) - (\vartheta_k - \vartheta_{k-1}), (\vartheta_{k+1} - \vartheta_k) + (\vartheta_k - \vartheta_{k-1})\rangle_H = \\ &= \frac{1}{\Delta t^2}\left(\left\|\vartheta_{k+1} - \vartheta_k\right\|_H^2 - \left\|\vartheta_k - \vartheta_{k-1}\right\|_H^2\right) \end{aligned}$$

and

$$\begin{aligned} T_2 &= \frac{1}{4} a\left(\vartheta_{k+1} + 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1}\right) = \\ &= \frac{1}{4} a\left((\vartheta_{k+1} + \vartheta_k) + (\vartheta_k + \vartheta_{k-1}), (\vartheta_{k+1} + \vartheta_k) - (\vartheta_k + \vartheta_{k-1})\right) \\ &= \frac{1}{4}\left[a\left(\vartheta_{k+1} + \vartheta_k, \vartheta_{k+1} + \vartheta_k\right) - a\left(\vartheta_k + \vartheta_{k-1}, \vartheta_k + \vartheta_{k-1}\right)\right] . \end{aligned}$$

This yields

$$\begin{aligned} &E^{k+1} + d^k \;=\; E^k + W^k \\ \text{with}\quad &E^{k+1} := \left\|\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t}\right\|_H^2 + \frac{1}{4} a\left(\vartheta_{k+1} + \vartheta_k, \vartheta_{k+1} + \vartheta_k\right) . \end{aligned}$$

Due to the coercivity of the bilinear form $a$ we have

$$\left\|\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t}\right\|_H^2 \le E^{k+1} \tag{3.16}$$

and

$$
\begin{aligned}
E^{k+1} + d^k &= E^1 + \sum_{i=1}^{k} W^i = E^1 + \frac{1}{4}\sum_{i=1}^{k} \langle \gamma_i, \vartheta_{i+1} - \vartheta_{i-1} \rangle_H \\
&= E^1 + \frac{1}{4}\sum_{i=1}^{k} \langle \gamma_i, (\vartheta_{i+1} - \vartheta_i) + (\vartheta_i - \vartheta_{i-1}) \rangle_H \\
&= E^1 + \frac{1}{4}\sum_{i=1}^{k} \langle \gamma_i, \vartheta_{i+1} - \vartheta_i \rangle_H + \frac{1}{4}\sum_{i=1}^{k} \langle \gamma_i, \vartheta_i - \vartheta_{i-1} \rangle_H \\
&= E^1 + \frac{1}{4}\left( \sum_{i=1}^{k-1} \langle \gamma_i, \vartheta_{i+1} - \vartheta_i \rangle_H + \langle \gamma_k, \vartheta_{k+1} - \vartheta_k \rangle_H \right) + \\
&\qquad + \frac{1}{4}\left( \sum_{p=1}^{k-1} \langle \gamma_{p+1}, \vartheta_{p+1} - \vartheta_p \rangle_H + \langle \gamma_1, \vartheta_1 - \vartheta_0 \rangle_H \right) \\
&= E^1 + \frac{1}{4}\langle \gamma_k, \vartheta_{k+1} - \vartheta_k \rangle_H + \frac{1}{4}\langle \gamma_1, \vartheta_1 - \vartheta_0 \rangle_H \\
&\qquad + \frac{1}{4}\sum_{i=1}^{k-1} \langle \gamma_{i+1} + \gamma_i, \vartheta_{i+1} - \vartheta_i \rangle_H .
\end{aligned}
$$

Using Young's inequality and $\Delta t \le 1$, we get

$$
\begin{aligned}
\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 &\le E^1 + \frac{\Delta t}{32} \|\gamma_1\|_H^2 + \frac{\Delta t}{2} \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{32} \|\gamma_k\|_H^2 + \frac{\Delta t}{2} \left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 + \\
&\qquad + \sum_{i=1}^{k-1} \frac{\Delta t}{32} \|\gamma_{i+1} + \gamma_i\|_H^2 + \sum_{i=1}^{k-1} \frac{\Delta t}{2} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \\
&\le E^1 + \frac{\Delta t}{32} \|\gamma_1\|_H^2 + \frac{\Delta t}{2} \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{32} \|\gamma_k\|_H^2 + \frac{1}{2} \left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 + \\
&\qquad + \sum_{i=1}^{k-1} \frac{\Delta t}{32} \|\gamma_{i+1} + \gamma_i\|_H^2 + \sum_{i=1}^{k-1} \frac{\Delta t}{2} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 .
\end{aligned}
$$

This yields

$$
\begin{aligned}
\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 &\le 2 \cdot E^1 + \frac{\Delta t}{16} \|\gamma_1\|_H^2 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{16} \|\gamma_k\|_H^2 + \\
&\qquad + \sum_{i=1}^{k-1} \frac{\Delta t}{16} \|\gamma_{i+1} + \gamma_i\|_H^2 + \sum_{i=1}^{k-1} \Delta t \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \\
&\le 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \sum_{i=1}^{k} \frac{\Delta t}{4} \|\gamma_i\|_H^2 + \sum_{i=1}^{k-1} \Delta t \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 .
\end{aligned}
$$

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We use the discrete Gronwall lemma [1], which yields

$$\begin{aligned}
&\sum_{i=1}^{k} \Delta t \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \leq \\
&\leq (1+\Delta t)^k \sum_{i=1}^{k} (1+\Delta t)^{-i} \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \sum_{j=1}^{i-1} \frac{\Delta t}{4} \|\gamma_j\|_H^2 \right) \\
&\leq e^T \cdot \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 \right) + \sum_{i=1}^{k} \frac{\Delta t}{4} \|\gamma_i\|_H^2 + \sum_{i=2}^{k} \sum_{j=1}^{i-1} \frac{\Delta t}{4} \|\gamma_j\|_H^2 \\
&\leq e^T \cdot \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 \right) + \sum_{i=1}^{k} \frac{\Delta t}{4} \|\gamma_i\|_H^2 + \frac{\Delta t}{4} \sum_{i=2}^{k} (k-i) \|\gamma_i\|_H^2 \\
&\leq e^T \cdot \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{k} \|\gamma_i\|_H^2 \right).
\end{aligned}$$

Therefore,

$$\Delta t \sum_{i=1}^{m} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \leq e^T \cdot \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \|\gamma_i\|_H^2 \right),$$

which depends only on the initial conditions $\vartheta_0$ and $\dot{\vartheta}_0$ and on the sequence $(\gamma_k)$.

Further, we have

$$\begin{aligned}
\|\vartheta_{k+1}\|_H^2 &\leq \left\| \vartheta_1 + \sum_{i=1}^{k-1} (\vartheta_{i+1} - \vartheta_i) \right\|_H^2 \\
&\leq 2 \|\vartheta_1\|_H^2 + 2k \sum_{i=1}^{k} \|\vartheta_{i+1} - \vartheta_i\|_H^2 \\
&\leq 2 \|\vartheta_1\|_H^2 + 2m\Delta t \sum_{i=1}^{m} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \\
&\leq 2 \|\vartheta_1\|_H^2 + 2Te^T \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \|\gamma_i\|_H^2 \right),
\end{aligned}$$

which yields for the averaged sum

$$\frac{1}{m} \sum_{k=0}^{m} \|\vartheta_k\|_H^2 \leq 2 \|\vartheta_1\|_H^2 + 2Te^T \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \|\gamma_i\|_H^2 \right).$$

In the following, we construct a bound for the right hand side terms which are dominated by the sum over $\|\gamma_k\|_H^2$. Again, this sequence $\|\gamma_k\|_H^2 = \|v_{k+1} + 2v_k + v_{k-1}\|_H^2$ is separated into two terms, a "projection" and a "discretization" part:

$$\begin{aligned}
v_k &= \ddot{x}(t_k) - \partial\partial P^l x(t_k) + D\left(\dot{x}(t_k) - \partial P^l x(t_k)\right) \\
&= \underbrace{\ddot{x}(t_k) - \partial\partial x(t_k)}_{=:w_k} + \underbrace{\partial\partial x(t_k) - \partial\partial P^l x(t_k)}_{=:z_k} + D \left( \underbrace{\dot{x}(t_k) - \partial x(t_k)}_{=:\widetilde{w}_k} + \underbrace{\partial x(t_k) - \partial P^l x(t_k)}_{=:\widetilde{z}_k} \right),
\end{aligned}$$

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yielding finally

$$\|\gamma_k\|_H^2 \le 4\|w_{k+1}+2w_k+w_{k-1}\|_H^2 + 4\|z_{k+1}+2z_k+z_{k-1}\|_H^2 + \\ + 4\|\widetilde{w}_{k+1}+2\widetilde{w}_k+\widetilde{w}_{k-1}\|_H^2 + 4\|\widetilde{z}_{k+1}+2\widetilde{z}_k+\widetilde{z}_{k-1}\|_H^2 .$$

Due to Taylor's theorem, we have

$$\begin{aligned} &\|w_{k+1}+2w_k+w_{k-1}\|_H^2 \\ &= \|\ddot{x}(t_{k+1})+2\ddot{x}(t_k)+\ddot{x}(t_{k-1}) - (\partial\partial x(t_{k+1})+2\partial\partial x(t_k)+\partial\partial x(t_{k-1}))\|_H^2 \\ &= \left\|\ddot{x}(t_{k+1})+2\ddot{x}(t_k)+\ddot{x}(t_{k-1}) - \frac{4}{\Delta t^2}\left(x(t_{k+1})-2x(t_k)+x(t_{k-1})\right)\right\|_H^2 \\ &\le K\Delta t^4, \end{aligned}$$

where $K$ is independent of $\Delta t$, $m$, and $l$, which leads to

$$\sum_{k=1}^{m-1}\|w_{k+1}+2w_k+w_{k-1}\|_H^2 \le K\Delta t^3.$$

Accordingly, we find for $\widetilde{w}$

$$\sum_{k=1}^{m-1}\|\widetilde{w}_{k+1}+2\widetilde{w}_k+\widetilde{w}_{k-1}\|_H^2 \le K\Delta t^3,$$

where $K>0$ is independent of $\Delta t$, $m$ and $l$. The estimates for $z_k = \partial\partial x(t_k) - \partial\partial P^l x(t_k)$ and $\widetilde{z}_k = \partial x(t_k) - \partial P^l x(t_k)$ depend on the particular choice of the POD subspace; see equations (3.10)–(3.11) and (3.13)–(3.14) for case $I$ and $II$, respectively. For case $I$,

$$\sum_{k=1}^{m-1}\|z_{k+1}+2z_k+z_{k-1}\|_X^2 \le \frac{24}{\Delta t^4}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij}$$

and

$$\sum_{k=1}^{m-1}\|\widetilde{z}_{k+1}+2\widetilde{z}_k+\widetilde{z}_{k-1}\|_X^2 \le \frac{16}{\Delta t^2}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij}.$$

For case $II$,

$$\sum_{k=1}^{m-1}\|z_{k+1}+2z_k+z_{k-1}\|_X^2 \le 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj}$$

and

$$\sum_{k=1}^{m-1}\|\widetilde{z}_{k+1}+2\widetilde{z}_k+\widetilde{z}_{k-1}\|_X^2 \le 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj}.$$

Combining the estimates for $z_k$ , $\widetilde{z}_k$, $w_k$, and $\widetilde{w}_k$, we have

$$\begin{aligned} \|\gamma_k\|_H^2 &= \|v_{k+1}+2v_k+v_{k-1}\|_H^2 \\ &\le 4\|w_{k+1}+2w_k+w_{k-1}\|_H^2 + 4\|z_{k+1}+2z_k+z_{k-1}\|_H^2 + \\ &\quad + 4D\|\widetilde{w}_{k+1}+2\widetilde{w}_k+\widetilde{w}_{k-1}\|_H^2 + 4D\|\widetilde{z}_{k+1}+2\widetilde{z}_k+\widetilde{z}_{k-1}\|_H^2 . \end{aligned}$$

We get for case $I$

$$\sum_{i=1}^{m} \|\gamma_i\|_H^2 \leq K_1 \Delta t^2 + K_2 \frac{m+1}{\Delta t^4} \sum_{j=l+1}^{d} \lambda_{Ij},$$

and for case $II$

$$\sum_{i=1}^{m} \|\gamma_i\|_H^2 \leq K_3 \Delta t^3 + K_4\, m \sum_{j=l+1}^{d} \lambda_{IIj}.$$

In conclusion, the error estimate for case $I$ can be written as

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \leq \\ &\leq 4 \sum_{j=l+1}^{d} \lambda_{Ij} + 4\,\|\vartheta_1\|_H^2 + \\ &\quad + 4Te^T \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + K\Delta t^4 + \left( \frac{12}{\Delta t^3} + \frac{4D}{\Delta t} \right)(m+1) \sum_{j=l+1}^{d} \lambda_{Ij} \right). \end{aligned}$$

The term $E^1$ contains the expression $(\vartheta_1 - \vartheta_0)/\Delta t$, which can be regarded as an extended initial condition for the velocities $(\partial\vartheta_1 + \partial\vartheta_0)/2$ due to Newmark's scheme (2.5a). Hence, we get

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \leq \\ &\leq C_I \Bigg( \left\|X_0 - P^l x(t_0)\right\|_H^2 + \left\|X_1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X_0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\qquad + \Delta t \left\|\partial X_1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \left( \frac{1}{\Delta t^4} + \frac{1}{\Delta t} + 1 \right) \sum_{j=l+1}^{d} \lambda_{Ij} \Bigg), \end{aligned}$$

with $C_I$ independent of $\Delta t$ and $m$.

Case $II$ yields the estimate

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \|X_k - x(t_k)\|_H^2 \leq \\ &\leq 6 \sum_{j=l+1}^{d} \lambda_{IIj} + 4\,\|\vartheta_1\|_H^2 + \\ &\quad + 4Te^T \left( 2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + K\Delta t^4 + 36T(1+D) \cdot \sum_{j=l+1}^{d} \lambda_{IIj} \right), \end{aligned}$$

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which can similarly be interpreted as

$$\frac{1}{m}\sum_{k=1}^{m}\|X_k - x(t_k)\|_H^2 \leq$$
$$\leq C_{II}\Bigg( \left\|X_0 - P^l x(t_0)\right\|_H^2 + \left\|X_1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X_0 - P^l \dot{x}(t_0)\right\|_H^2$$
$$+\Delta t \left\|\partial X_1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \sum_{j=l+1}^{d} \lambda_{IIj}\Bigg), \tag{3.17}$$

where $C_{II}$ is independent of $\Delta t$ and $m$. □

In both cases, we find terms that are independent of the time step $\Delta t$. Both cases also contain terms that depend on $\Delta t$ in the numerator. These terms vanish as $\Delta t$ goes to zero.

In case $I$, which only uses the deformation snapshots, the error estimate additionally contains a term that carries $\Delta t$ in the denominator. For this particular choice of the POD subspace the error bound tends to infinity with $\Delta t \to 0$. This means that convergence cannot be assured formally if a snapshot set consisting of deformations only is used. If velocities and accelerations are added into the set, convergence can be deduced from (3.17).

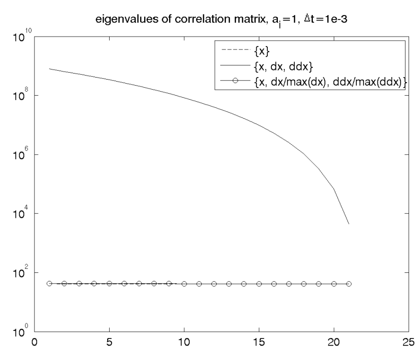


FIGURE 4.1. *Decay of eigenvalues of the snapshot correlation matrix, for* $a_i = 1$.

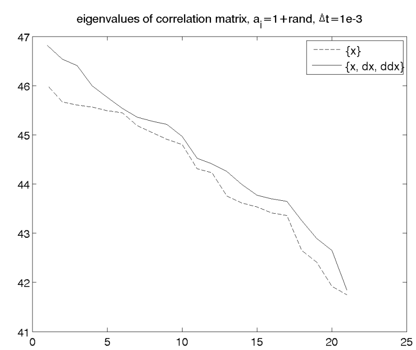


FIGURE 4.2. *Decay of eigenvalues of the snapshot correlation matrix, for* $a_i = 1 + rand$.

**4. Numerical results.** For a numerical comparison of the different POD techniques discussed above, a simple test model was set up in MATLAB. The example shows a one-dimensional linear wave equation on the interval $\Omega = (0, L)$ with homogeneous Dirichlet boundary conditions, which can be regarded as a vibrating string fixed at both ends.

Mathematically, our model problem is described by the initial-boundary value problem

$$\mu \cdot \ddot{x}(s,t) - S \cdot x''(s,t) = f(s,t) \quad \text{in } (0,L)\times(0,T), \tag{4.1a}$$
$$x(s,0) = x_0 \quad \text{in } (0,L), \tag{4.1b}$$
$$\dot{x}(s,0) = \dot{x}_0 \quad \text{in } (0,L), \tag{4.1c}$$
$$x(s,t) = 0 \quad \text{on } \partial\Omega = \{0,L\} \quad \text{for all } t \in (0,T). \tag{4.1d}$$

We chose $L = 1$, $S = 1$, $\mu = 1$, and $T = 2$, and the initial deformation $x_0$ is a weighted sum of sinusoidal shapes

$$x_0 = x(t_0) = \sum_{i=1}^{n} a_i \cdot \sin\left(i \cdot \pi \frac{s}{L}\right),$$

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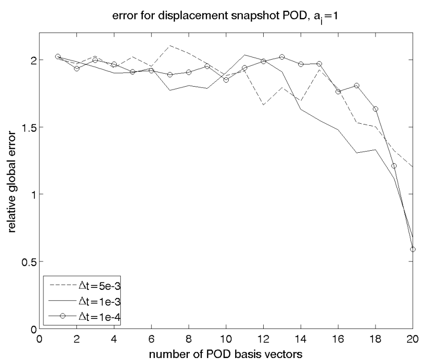


FIGURE 4.3. *Error norms for deformation snapshot set, $a_i = 1$.*

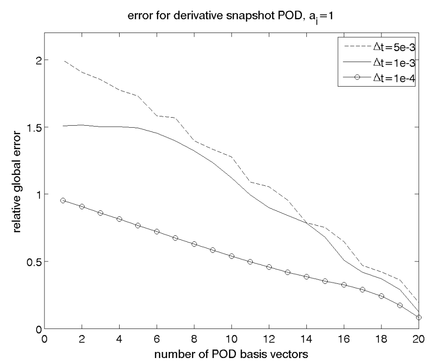


FIGURE 4.4. *Error norms for derivative snapshot set, $a_i = 1$.*

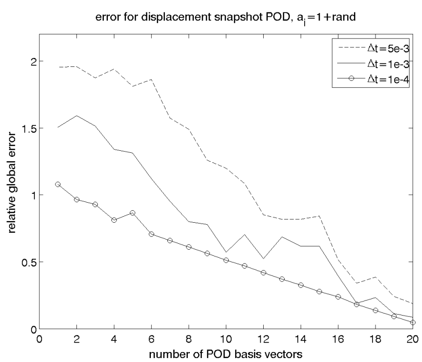


FIGURE 4.5. *Error norms for deformation snapshot set, $a_i = 1 + rand$.*

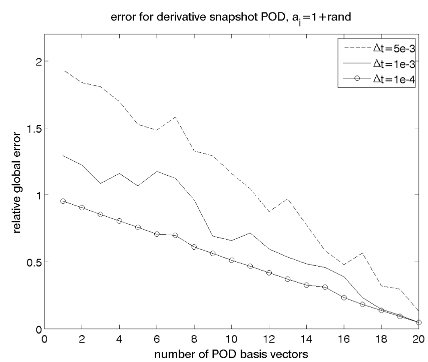


FIGURE 4.6. *Error norms for derivative snapshot set, $a_i = 1 + rand$.*

with weights $a_i \in \mathbb{R}$. Furthermore, we set the external force $f(t)$ to zero, yielding the analytical solution and its derivatives:

$$x(s,t) = \sum_{i=1}^{n} a_i \cdot \sin\left(i\pi \frac{s}{L}\right) \cdot \cos\left(i\pi \frac{c}{L} t\right), \tag{4.2}$$

$$\dot{x}(s,t) = \sum_{i=1}^{n} -a_i \cdot \sin\left(i\pi \frac{s}{L}\right) \cdot \sin\left(i\pi \frac{c}{L} t\right) \cdot i\pi \frac{c}{L}, \tag{4.3}$$

$$\ddot{x}(s,t) = \sum_{i=1}^{n} -a_i \cdot \sin\left(i\pi \frac{s}{L}\right) \cdot \cos\left(i\pi \frac{c}{L} t\right) \cdot i^2\pi^2 \frac{c^2}{L^2}, \quad \text{with} \quad c = \sqrt{\frac{S}{\mu}}. \tag{4.4}$$

The POD method was realized using snapshots at $m+1$ uniformly distributed points in time. To observe the error behavior with decreasing time step, we investigate three different step sizes dividing the interval into $m = 400$, 2000, and 20000 subintervals. We use Newmark's method for the time integration, and discretize in space using 500 linear finite elements.

For the first case, in which only displacements are included in the POD set, the snapshots are simply $\{x(t_k)\}_{k=0,\dots,m}$. In the second case, where deformations, velocities and accelerations were taken into account, the snapshot set was

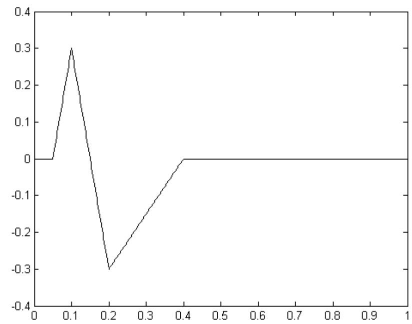

FIGURE 4.7. *Initial condition for the linear wave equation.*

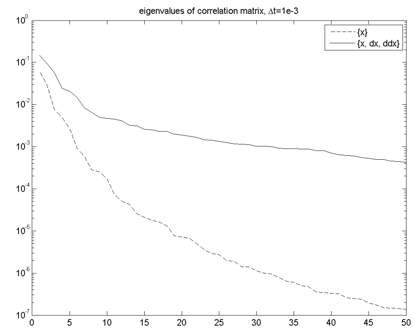


FIGURE 4.8. *Decay of eigenvalues for the POD snapshot sets, $\Delta t = 10^{-3}$.*

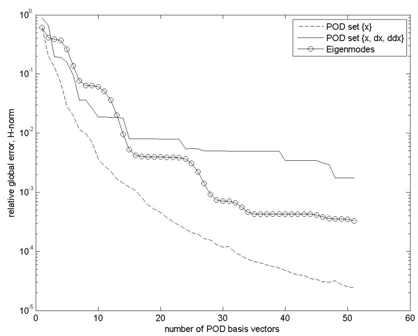


FIGURE 4.9. *Error norms for POD and eigenmode analysis, $H$-norm, $\Delta t = 10^{-3}$.*

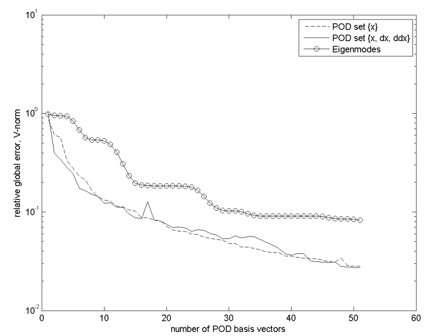


FIGURE 4.10. *Error norms for POD and eigenmode analysis, $V$-norm, $\Delta t = 10^{-3}$.*

$x(t_0), \ldots, x(t_m), \dot{x}(t_0), \ldots, \dot{x}(t_m), \ddot{x}(t_0), \ldots, \ddot{x}(t_m)$.

REMARK 4.1. When using unweighted snapshots, the eigenvalues of the derivative set differ from those from the deformation set by a factor of $10^8$ (Figure 4.1). This is due to the fact that the velocities and accelerations are about 2, respectively 3, orders of magnitude larger than the deformations. This difference leads not only to large eigenvalues but also to an overrating of the derivatives in the correlation matrix. For this reason, the first and second derivative snapshots were divided by their respective maxima over space and time.

Furthermore, to investigate the influence of the eigenvalues of the correlation matrix, we compare two different initial conditions: one with uniformly weighted sinusoidal shapes ($a_i = 1$), and one with randomly varying weights ($a_i = 1 + \mathrm{rand}$, $\max(\mathrm{rand}) = 0.05$). The former yields a nearly constant distribution of eigenvalues up to the dimension of $a$, whereas the eigenvalues for the latter set decay linearly (Figures 4.1 and 4.2). Note that for problems including damping, the eigenvalues usually decay exponentially.

Figures 4.3 and 4.4 compare the norms of the relative global errors for the case $a_i = 1$. In this setup, the classical snapshot POD method shows no improvement with decreasing time step size, whereas the one which uses derivative snapshots performs significantly better.

In the case of a random distribution of sinusoidal weights, both methods show a diminishing error for smaller time steps (Figures 4.5 and 4.6). One possible reason is the influence of the eigenvalue decay on the error norm which dominates the error in this case.

REMARK 4.2. In all cases mentioned above, the absolute values of the error norms are still high ($> 10\%$). The dimension of the model corresponds to the dimension of $a$ (here: $dim(a) = 21$). As soon as a larger number of POD vectors is used, the error drops instantly. This behavior is also seen in the eigenvalue distribution, yielding $\lambda_i = 0$ for $i > dim(a)$. Therefore, a setup with such a weighting of modes actually forces the user to work with all occurring eigenvalues, as every neglected basis vector still has a considerable influence on the solution. In this case, dimension reduction is risky and the example shall only be seen as a constructed model to demonstrate the error behavior.

As a second example, we use a non-smooth initial condition $u_0$ (Figure 4.7) in the same setup as above and compare the POD methods with the classical eigenmode method frequently used for linear systems. Furthermore, we set the damping factor $d = 10$. In the case of high damping, we get an exponential decay of eigenvalues (Figure 4.8). A fast eigenvalue decay leads to a small error in subspace approximation of the snapshot set (see (2.10)). This yields a better condition for the POD method than in the example above. Figure 4.9 shows a comparison of the relative global errors of both POD and the eigenmode methods. The errors are computed in the $H$-norm $\frac{1}{m}\sum_{k=1}^{m}\|X_k - x(t_k)\|_H^2$. In this case, the derivative POD method performs slightly worse than the classical one. The errors for the eigenmode method range between the ones of both POD methods. If we measure the error in the $V$-norm, both POD methods perform better than the eigenmodes (Figure 4.10).

**5. Conclusion.** We study the POD method for the linear wave equation and compare two choices of snapshot sets: set $I$ consists of deformation snapshots, and set $II$ additionally contains velocities and accelerations. As for parabolic problems, there is no convergence guarantee for simple deformation snapshots. Only the incorporation of additional derivative snapshots yields an error bound that diminishes with diminishing time steps.

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