Electronic Transactions on Numerical Analysis.
Volume 38, pp. 218-232, 2011.

ISSN 1068-9613.

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A LINEAR CONSTRUCTIVE APPROXIMATION FOR INTEGRABLE FUNCTIONS AND A PARAMETRIC QUADRATURE MODEL BASED ON A GENERALIZATION OF OSTROWSKI-GRÜSS TYPE INEQUALITIES*

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Abstract. A new generalization of Ostrowski-Grüss type inequalities, depending on a parameter λ, is introduced in order to construct a specific linear approximation for integrable functions. Some important subclasses of this inequality such as $\lambda = 1/2, 1$, and $\sqrt{2}/2$ are studied separately. The generalized inequality is employed to establish a parametric quadrature model and obtain its error bounds.

Key words. Ostrowski-Grüss type inequalities, linear constructive approximation, numerical quadrature rules, Chebyshev functional, kernel function.

AMS subject classifications. 26D15, 65D30, 26D20, 65D32.

1. Introduction.

Let $L^p[a,b]$ $(1 \leq p < \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |\, f(t)\,|^p \, dt \right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\ sup}_{x\in[a,b]} |\, f(x)\,| \,.$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$ and a positive weight function $w : [a,b] \to \mathbb{R}^+$ such that $wf, wg, wfg \in L^1[a,b]$, the weighted Chebyshev functional [1] is defined by

$$\mathbf{T}(w,f,g) = \int_a^b w(x)f(x)\,g(x)\,dx - \left(\int_a^b w(x)f(x)\,dx\right)\left(\int_a^b w(x)g(x)\,dx\right). \tag{1.1}$$

If $w(x)$ is uniformly distributed on $[a,b]$ then (1.1) is reduced to the usual Chebyshev functional

$$\begin{aligned}\mathbf{T}(f,g) &= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)\\ &= \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(g(x) - \frac{1}{b-a}\int_a^b g(x)\,dx\right)\,dx\,.\end{aligned}$$

To date, extensive research has been done on the bounds for the usual Chebyshev functional. The first work dates back to 1882 when Chebyshev [1] proved that if $f', g' \in L^\infty[a,b]$, then

$$|\,\mathbf{T}(f,g)\,| \leq \frac{1}{12}(b-a)^2 \,\|\, f' \,\|_\infty \,\|\, g' \,\|_\infty \,.$$

* Received January 3, 2011. Accepted for publication June 17, 2011. Published online August 3, 2011. Recommended by F. Marcellán.

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Later on, in 1934, Grüss [6] showed that

$$|\mathbf{T}(f,g)| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{1.2}$$

where m_1, m_2, M_1 and M_2 are real numbers satisfying the conditions

$$m_1 \le f(x) \le M_1 \quad \text{and} \quad m_2 \le g(x) \le M_2 \quad \text{for all} \quad x \in [a,b].$$

The constant $\frac{1}{4}$ is the best possible in (1.2) in the sense that it cannot be replaced by a smaller quantity.

An inequality related to the usual Chebyshev functional is due to Ostrowski [16] in 1938. If $f : [a,b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left(\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right)(b-a)\,\| f' \|_\infty \tag{1.3}$$

for all $x \in [a,b]$. Today this inequality plays an important role in numerical quadrature rules; see, e.g., [4, 9]. In 1997, Dragomir and Wang [3] introduced a mixture of the inequalities (1.2) and (1.3), called the Ostrowski-Grüss inequality, and showed the following theorem.

THEOREM 1.1. *If $f : [a,b] \to \mathbb{R}$ is a differentiable function with bounded derivative and $\alpha_0 \le f'(t) \le \beta_0$, for all $t \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \le \frac{1}{4}(b-a)(\beta_0 - \alpha_0). \tag{1.4}$$

Due to the importance of the inequality (1.4), many refinements and generalizations have been presented in the literature. For instance, Cheng [2] gave a sharp version of the Ostrowski-Grüss inequality and proved that instead of $\frac{1}{4}$ on the right-hand side of (1.4), the constant $\frac{1}{8}$ should be used; see also [13]. In 2007, Liu [7] applied this inequality for (l,L)-Lipschitzian functions as follows.

THEOREM 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a (l,L)-Lipschitzian function on [a,b], i.e.,*

$$l\,(x_2 - x_1) \le f(x_2) - f(x_1) \le L\,(x_2 - x_1) \quad \text{for} \quad a \le x_1 \le x_2 \le b,$$

where $l, L \in \mathbb{R}$ with $l < L$. Then for all $x \in [a,b]$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \le \frac{b-a}{8}\,(L-l)\,.$$

The following theorem, due to Niezgoda [15], is probably the most recent work about the Ostrowski-Grüss inequality in $L^p[a,b]$ spaces.

THEOREM 1.3. *Let $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I^0}$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. Suppose that $f', \alpha, \beta \in L^p[a,b]$ $(1 \le p < \infty)$ are functions such that $\alpha(t) + \beta(t)$ is a constant function and $\alpha(t) \le f'(t) \le \beta(t)$ for all $t \in [a,b]$. Then for any $x \in [a,b]$ we have the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \le \begin{cases} \frac{1}{4}\,\|\beta-\alpha\|_p\, \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}, \\ \frac{1}{4}\,\|\beta-\alpha\|_1\,, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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As we observe, the general shape of the left-hand side of all these inequalities is the same and variations are in fact employed on the right-hand side of them. In this paper, we first generalize the left-hand side of inequality (1.4) (by adding a free parameter, say λ) and obtain an upper bound for its absolute value in $L^1[a,b]$ and $L^\infty[a,b]$ spaces to arrive at a specific linear approximation for integrable functions. Then we study three special examples of the introduced inequality for $\lambda = \frac{1}{2}, 1$ and $\frac{\sqrt{2}}{2}$. Finally, we establish a parametric quadrature model based on the presented inequality and obtain its error bounds. A section with numerical examples is also given in this sense.

2. A linear constructive approximation for integrable functions. Let us define the following specific kernel on the interval [a,b]:

$$K(x;t,p,q) = \begin{cases} t-p-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right) & t\in[a,x],\\ t-q-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right) & t\in(x,b],\end{cases} \tag{2.1}$$

where $p, q,$ are two free parameters.

THEOREM 2.1. *Let $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I^0}$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If α_0, β_0 are two real constants such that $\alpha_0 \le f'(t) \le \beta_0$ for all $t \in [a,b]$, then for any $x \in [\frac{a+(2\lambda-1)b}{2\lambda}, \frac{b+(2\lambda-1)\,a}{2\lambda}] \subseteq [a,b]$ where $\lambda \in [\frac{1}{2},1]$ we have*

$$\left| f(x) - \frac{1}{\lambda\,(b-a)}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x + \frac{(2\lambda-1)\,a+b}{2\lambda\,(b-a)}f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda\,(b-a)}f(a)\right| \le \frac{\beta_0-\alpha_0}{4\,(b-a)}\,\frac{\lambda^2+(1-\lambda)^2}{\lambda}\,\left((x-a)^2+(b-x)^2\right). \tag{2.2}$$

In other words, the linear approximation

$$f(x) \cong \frac{f(b)-f(a)}{b-a}\,x + \frac{1}{\lambda\,(b-a)}\left(\int_a^b f(t)\,dt + \frac{a+(2\lambda-1)\,b}{2}f(a) - \frac{(2\lambda-1)\,a+b}{2}f(b)\right),$$

has an absolute error less than

$$\frac{\beta_0-\alpha_0}{4\,(b-a)}\,\frac{\lambda^2+(1-\lambda)^2}{\lambda}\,\left((x-a)^2+(b-x)^2\right).$$

Proof. Take $\lambda \in [\frac{1}{2},1]$. Then there exist two real parameters p and q satisfying $q > p$ and $(b-a)/2 \le q-p \le b-a$ so that one has $\lambda = \frac{q-p}{b-a}$. By referring to the kernel (2.1), it can now be verified that

$$\begin{aligned}\int_a^b K\,(x;t,p,q)\;dt &= \int_a^x \left(t-p-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right)\,dt\\ &\quad+\int_x^b \left(t-q-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right)\,dt = 0,\end{aligned}$$

and, moreover,

$$\begin{aligned}\frac{1}{q-p}\int_a^b K\,(x;t,p,q)\;f'(t)\,dt =& f(x) - \frac{1}{\lambda\,(b-a)}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x\\ &+\frac{(2\lambda-1)\,a+b}{2\lambda\,(b-a)}f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda\,(b-a)}f(a)\,.\end{aligned}$$

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Hence, we have

$$
\begin{aligned}
f(x) - \frac{1}{\lambda\,(b-a)}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x& \\
+\frac{(2\lambda-1)\,a+b}{2\lambda\,(b-a)}f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda\,(b-a)}f(a)& \\
= \frac{1}{q-p}\int_a^b K\,(x;t,p,q)\,\left(f'(t)-\frac{\alpha_0+\beta_0}{2}\right)\,dt\,.&
\end{aligned}
$$

On the other side, the assumption $\alpha_0 \le f'(t) \le \beta_0$ results in

$$
\left| f'(t) - \frac{\alpha_0+\beta_0}{2}\right| \le \frac{\beta_0-\alpha_0}{2}.
$$

Therefore,

$$
\begin{aligned}
&\Bigg| f(x) - \frac{1}{\lambda\,(b-a)}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x \\
&\qquad\qquad +\frac{(2\lambda-1)\,a+b}{2\lambda\,(b-a)}f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda\,(b-a)}f(a)\Bigg| \\
&= \frac{1}{q-p}\left|\int_a^b K\,(x;t,p,q)\,\left(f'(t)-\frac{\alpha_0+\beta_0}{2}\right)\,dt\right| \\
&\le \frac{\beta_0-\alpha_0}{2\,(q-p)}\int_a^b |\,K\,(x;t,p,q)\,|\,dt \\
&= \frac{\beta_0-\alpha_0}{2\,\lambda\,(b-a)}\left(\int_a^x \left|t-p-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right|\,dt\right. \\
&\qquad\left.+\int_x^b \left|t-q-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right|\,dt\right)\,.
\end{aligned}
\tag{2.3}
$$

Now, it remains to compute the two integrals of the right-hand side of (2.3). For this purpose, it is sufficient to use a suitable change of variables to get

$$
\begin{aligned}
&\int_a^x \left|t-p-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right|\,dt \\
&\qquad+\int_x^b \left|t-q-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2)\right)\right|\,dt \\
&= \int_{-\lambda\,x+b\lambda-\frac{1}{2}(b-a)}^{(1-\lambda)\,x+b\lambda-\frac{1}{2}(b+a)} |\,z\,|\,dz + \int_{(1-\lambda)\,x+a\lambda-\frac{1}{2}(b+a)}^{-\lambda\,x+a\lambda+\frac{1}{2}(b-a)} |\,w\,|\,dw \\
&= \int_{-\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-x)}^{\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-x)} |\,z\,|\,dz + \int_{-(\lambda-\frac{1}{2})\,(x-a)-\frac{1}{2}(b-x)}^{-(\lambda-\frac{1}{2})\,(x-a)+\frac{1}{2}(b-x)} |\,w\,|\,dw\,.
\end{aligned}
\tag{2.4}
$$

But since in Theorem 2.1 we assumed

$$
x \in [\frac{a+(2\lambda-1)b}{2\lambda}, \frac{b+(2\lambda-1)\,a}{2\lambda}] \subseteq [a,b] \quad \text{for any} \quad \lambda \in [\frac{1}{2},1],
\tag{2.5}
$$

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the following results can directly be deduced

$$
\begin{aligned}
&-\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)\le 0\,,\\
&\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)\ge 0,\\
&-(\lambda-\frac{1}{2})\,(\,x-a)-\frac{1}{2}(b-x)\le 0\,,\\
&-(\lambda-\frac{1}{2})\,(\,x-a)+\frac{1}{2}(b-x)\ge 0\,.
\end{aligned} \tag{2.6}
$$

Thus, (2.4) is computed as

$$
\begin{aligned}
&\int_{-\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)}^{\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)}|\,z\,|\;dz+\int_{-(\lambda-\frac{1}{2})\,(\,x-a)-\frac{1}{2}(b-x)}^{-(\lambda-\frac{1}{2})\,(\,x-a)+\frac{1}{2}(b-x)}|\,w\,|\;dw\\
&\quad=\int_{-\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)}^{0}(-z)\,dz+\int_{0}^{\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)}z\,dz\\
&\qquad\qquad+\int_{-(\lambda-\frac{1}{2})\,(\,x-a)-\frac{1}{2}(b-x)}^{0}(-w)\,dw+\int_{0}^{-(\lambda-\frac{1}{2})\,(\,x-a)+\frac{1}{2}(b-x)}w\,dw\\
&\quad=\frac{\lambda^2+(1-\lambda)^2}{2}\,\left((x-a)^2+(b-x)^2\right)\,,
\end{aligned}
$$

which completes the proof. □

Note that the general condition (2.5) is in fact a solution of the first and fourth inequality in (2.6), because

$$
\begin{aligned}
&-\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-\,x)\le 0\;\;\Rightarrow\;\;\frac{a+(2\lambda-1)\,b}{2\lambda}\le x\,,\\
&-(\lambda-\frac{1}{2})\,(\,x-a)+\frac{1}{2}(b-x)\ge 0\;\;\Rightarrow\;\;x\;\le\frac{b+(2\lambda-1)\,a}{2\lambda}\,,
\end{aligned}
$$

and

$$
a\le\frac{a+(2\lambda-1)b}{2\lambda}\le x\le\frac{b+(2\lambda-1)\,a}{2\lambda}\le b\;\Leftrightarrow\;\lambda\in[\frac{1}{2},1]. \tag{2.7}
$$

This means that, by accepting the condition (2.7) as the region of solutions, the second and third inequalities in (2.6) are automatically satisfied.

Although there are various instances for $\lambda\in[\frac{1}{2},1]$, three special cases $\lambda=\frac{1}{2}$, 1 and $\lambda=\frac{\sqrt{2}}{2}$ may be of more interest as the following examples show.

EXAMPLE 2.2. If $\lambda=\frac{1}{2}$, the eligible region in (2.5) is the same as $x\in[a,b]$. So, according to (2.2) we have

$$
\begin{aligned}
&\left|f(x)-\frac{2}{b-a}\int_a^b f(t)\,dt+\frac{b-x}{b-a}f(b)+\frac{x-a}{b-a}f(a)\right|\\
&\quad\le\frac{\beta_0-\alpha_0}{4\,(b-a)}\,\left((x-a)^2+(b-x)^2\right)\\
&\quad\le\frac{\beta_0-\alpha_0}{4\,(b-a)}\,\max_{x\in[a,b]}\left((x-a)^2+(b-x)^2\right)=\frac{1}{4}(b-a)(\beta_0-\alpha_0)\,,
\end{aligned} \tag{2.8}
$$

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which is valid for all $x \in [a,b]$ if and only if $\alpha_0 \le f'(x) \le \beta_0$.

REMARK 2.3. If we set $\lambda = \frac{q-p}{b-a} = \frac{1}{2}$ in (2.1), then the following particular kernel is derived

$$K(x;t,a,b) = \begin{cases} t - \frac{x+a}{2} & t \in [a,x] \\ t - \frac{x+b}{2} & t \in (x,b]\,. \end{cases}$$

Cheng in [2, Eq. (3.3)] used this kernel but obtained the "incorrect version" of inequality (2.8) as

$$\left| f(x) - \frac{2}{b-a}\int_a^b f(t)\,dt + \frac{b-x}{b-a} f(b) + \frac{x-a}{b-a} f(a) \right| \le \frac{\beta_0 - \alpha_0}{8\,(b-a)} \left((x-a)^2 + (b-x)^2 \right).$$

For instance, substituting $f(x) = e^x$ and $[a,b] = [1,2]$ in the corrected inequality (2.8) implies that the linear approximation

$$e^{x-1} \cong (e-1)\,x - 1,$$

has an absolute error less than

$$\frac{e-1}{4} \left((x-1)^2 + (2-x)^2 \right),$$

for all $x \in [1,2]$.

And/or, substituting $[a,b] = [0,1]$ and $x = 1/3,\ 1/5$ into (2.8) respectively gives the following error bounds for two three-point quadratures

$$\left| \frac{1}{6} f(0) + \frac{1}{2} f(\frac{1}{3}) + \frac{1}{3} f(1) - \int_0^1 f(x)\,dx \right| \le \frac{5}{72}\,(\beta_0 - \alpha_0)\,,$$

and

$$\left| \frac{1}{10} f(0) + \frac{1}{2} f(\frac{1}{5}) + \frac{2}{5} f(1) - \int_0^1 f(x)\,dx \right| \le \frac{17}{200}\,(\beta_0 - \alpha_0)\,.$$

EXAMPLE 2.4 (The sharp version of Ostrowski-Grüss inequality for $\lambda = 1$; see [2]). For $\lambda = 1$ we can directly obtain the sharp version of inequality (1.4) as follows. Since $\lambda = \frac{q-p}{b-a} = 1$ and without loss of generality taking $q = b$ and $p = a$, the general kernel (2.1) is simplified to

$$K(x;t,a,b) = \begin{cases} t - x + \frac{b-a}{2} & t \in [a,x], \\ t - x - \frac{b-a}{2} & t \in (x,b]\,. \end{cases}$$

Now, according to Theorem 2.1 and relation (2.3) we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \\ &\quad \le \frac{\beta_0 - \alpha_0}{2\,(b-a)} \left(\int_a^x \left| t - (x - \frac{b-a}{2}) \right| dt + \int_x^b \left| t - (x + \frac{b-a}{2}) \right| dt \right) \\ &\quad = \frac{1}{8}(b-a)(\beta_0 - \alpha_0)\,, \end{aligned}$$

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for all $x \in [a, b]$ and $\alpha_0 \le f'(x) \le \beta_0$, because

$$\int_a^x \left| t - (x - \frac{b-a}{2}) \right| dt + \int_x^b \left| t - (x + \frac{b-a}{2}) \right| dt$$
$$= \int_{\frac{b+a}{2}-x}^{\frac{b-a}{2}} |\, z\,|\, dz + \int_{-\frac{b-a}{2}}^{\frac{b+a}{2}-x} |\, z\,|\, dz = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |\, z\,|\, dz = \frac{(b-a)^2}{4}\,.$$

EXAMPLE 2.5 (An optimized value for λ). Since the right-hand side of inequality (2.2) is in terms of the variable λ, by minimizing the function

$$g(\lambda) = \frac{\lambda^2 + (1-\lambda)^2}{\lambda} \quad \text{on} \quad [1/2, 1], \tag{2.9}$$

one obtains the optimized value $\lambda = \frac{\sqrt{2}}{2} \in [\frac{1}{2}, 1]$. Hence, substituting this value into (2.2) gives

$$\begin{aligned}
&\left| f(x) - \frac{\sqrt{2}}{b-a} \int_a^b f(t)\, dt - \frac{f(b)-f(a)}{b-a}\, x \right. \\
&\qquad\qquad \left. + \frac{(2-\sqrt{2})\, a + \sqrt{2}\, b}{2\,(b-a)} f(b) - \frac{(2-\sqrt{2})\, b + \sqrt{2}\, a}{2\,(b-a)} f(a) \right| \\
&\le \frac{(\sqrt{2}-1)(\beta_0 - \alpha_0)}{2(b-a)} \;\big((x-a)^2 + (b-x)^2\big) \\
&\quad \Leftrightarrow x \in [\frac{(2-\sqrt{2})\, b + \sqrt{2}\, a}{2}, \frac{(2-\sqrt{2})\, a + \sqrt{2}\, b}{2}]\,.
\end{aligned}$$

REMARK 2.6. Although $\alpha_0 \le f'(x) \le \beta_0$ is a general condition in Theorem 2.1, sometimes one might not be able to easily obtain both bounds of α_0 and β_0 for f'. In this case, one can consider two further theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

THEOREM 2.7. *Let $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a, b] \subset \mathbf{I}^0$. If α_0 is a real constant such that $\alpha_0 \le f'(t)$ for all $t \in [a, b]$, then for any $\lambda \in [\frac{1}{2}, 1]$ and all $x \in [a, b]$ we have*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\lambda\,(b-a)} \int_a^b f(t)\, dt - \frac{f(b)-f(a)}{b-a}\, x \right. \\
&\qquad\qquad \left. + \frac{(2\lambda-1)\, a + b}{2\lambda\,(b-a)} f(b) - \frac{a + (2\lambda-1)\, b}{2\lambda\,(b-a)} f(a) \right| \\
&\le \frac{b-a}{2\lambda} \left(\mathbf{S} - \alpha_0\right)\,,
\end{aligned} \tag{2.10}$$

where $\mathbf{S} = (f(b) - f(a))/(b - a)$.

Proof. Again take $\lambda \in [\frac{1}{2}, 1]$. By noting that there exist two parameters *p* and *q* satisfying

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$(b-a)/2 \leq q-p \leq b-a$ such that $\lambda = \frac{q-p}{b-a}$, we have

$$\begin{aligned}
&\Bigg| f(x) - \frac{1}{\lambda(b-a)} \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x \\
&\qquad\qquad + \frac{(2\lambda-1)\,a+b}{2\lambda(b-a)} f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda(b-a)} f(a) \Bigg| \\
&= \frac{1}{q-p} \left| \int_a^b K(x;t,p,q)\,\left(f'(t)-\alpha_0\right)\,dt \right| \\
&\leq \frac{1}{q-p} \int_a^b \left| K(x;t,p,q) \right| \left(f'(t)-\alpha_0\right)\,dt \\
&\leq \frac{1}{q-p} \max_{t\in[a,b]} \left| K(x;t,p,q) \right| \int_a^b \left(f'(t)-\alpha_0\right)\,dt \\
&\leq \frac{(f(b)-f(a)-\alpha_0(b-a))}{\lambda(b-a)} \times \\
&\qquad \max_{x\in[a,b]} \left\{ \max_{t\in[a,x]} \left| t-p-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2) \right) \right|, \right. \\
&\qquad\qquad \left. \max_{t\in(x,b]} \left| t-q-\frac{1}{b-a}\left((q-p)\,x+ap-bq+\frac{1}{2}(b^2-a^2) \right) \right| \right\} \\
&= \frac{(f(b)-f(a)-\alpha_0(b-a))}{\lambda(b-a)} \times \\
&\qquad \max_{x\in[a,b]} \left\{ \left| -\frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-x) \right|, \left| \frac{1}{2}(x-a)+(\lambda-\frac{1}{2})\,(b-x) \right|, \right. \\
&\qquad\qquad \left. \left| -(\lambda-\frac{1}{2})\,(x-a)-\frac{1}{2}(b-x) \right|, \left| -(\lambda-\frac{1}{2})\,(x-a)+\frac{1}{2}(b-x) \right| \right\} \\
&= \frac{(f(b)-f(a)-\alpha_0(b-a))}{\lambda(b-a)} \times \\
&\qquad \max_{\lambda\in[\frac{1}{2},1]} \left\{ \left| (\lambda-\frac{1}{2})\,(b-a) \right|, \left| -\frac{1}{2}\,(b-a) \right|, \left| \frac{1}{2}\,(b-a) \right|, \left| -(\lambda-\frac{1}{2})(b-a) \right| \right\} \\
&= \frac{(f(b)-f(a)-\alpha_0(b-a))}{2\lambda} = \frac{b-a}{2\lambda}\left(\mathbf{S}-\alpha_0\right). \qquad \square
\end{aligned}$$

THEOREM 2.8. *Let $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If β_0 is a real constant such that $f'(t) \leq \beta_0$ for all $t \in [a,b]$, then for any $\lambda \in [\frac{1}{2},1]$ and all $x \in [a,b]$ we have*

$$\begin{aligned}
&\Bigg| f(x) - \frac{1}{\lambda(b-a)} \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\,x \\
&\qquad\qquad + \frac{(2\lambda-1)\,a+b}{2\lambda(b-a)} f(b) - \frac{a+(2\lambda-1)\,b}{2\lambda(b-a)} f(a) \Bigg| \\
&\quad \leq \frac{b-a}{2\lambda}\left(\beta_0-\mathbf{S}\right),
\end{aligned} \tag{2.11}$$

where $\mathbf{S} = (f(b)-f(a))/(b-a)$.
The proof is similar to that of Theorem 2.7.

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3. A parametric quadrature model and its error bound. One of the advantages of Theorems 2.1, 2.7, and 2.8 is to establish some new models of quadrature rules and to obtain their error bounds using inequalities (2.2), (2.10) and (2.11). For this purpose, we should first state a short summary of numerical quadrature rules. Consider a general $(n+1)$-point weighted quadrature formula of the form

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=0}^{n} w_k\, f(x_k) + R_{n+1}(f), \tag{3.1}$$

where $w(x)$ is a positive weight function on [a,b], $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are nodes and weight coefficients, respectively, and $R_{n+1}(f)$ is the corresponding error [5, 12].
Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most *d*. The quadrature formula (3.1) has degree of exactness *d* if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}(p) = 0$. In addition, if $R_{n+1}(p) \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (3.1) has precise degree of exactness *d*. The convergence order of quadrature formula (3.1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k),$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)\, L(x\,;\, x_k) + r_{n+1}(f; x),$$

where

$$L(x\,;\, x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \qquad (k = 0, 1, \ldots, n)\,,$$

we obtain (3.1), with

$$w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)\, w(x)}{x - x_k}\, dx \quad (k = 0, 1, \ldots, n)\,, \tag{3.2}$$

and

$$R_{n+1}(f) = \int_a^b r_{n+1}(f; x)\, w(x)\, dx.$$

Note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f; x) = 0$ and therefore $R_{n+1}(f) = 0$.

Quadrature formulae obtained in this way are known as interpolatory. It is well known that any interpolatory quadrature (3.1) with nonnegative coefficients (3.2) is convergent for all continuous functions on $[a, b]$, [12]. Usually the simplest interpolatory quadrature formula of type (3.1) with predetermined nodes $\{x_k\}_{k=0}^n \in [a, b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a + kh\}_{k=0}^n$ with $h = (b - a)/n$, the classical Newton-Cotes formula is derived. On the other hand, the interpolatory formulae of the maximal degree of exactness are known as Gaussian quadrature formulae whose construction is closely connected with orthogonal polynomials. For details on quadrature

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formulae see, for example, [5, pp. 152-185] and [12, pp. 319-361]. In addition, several other types of standard and nonstandard quadratures have been recently developed, e.g., in [8, 10, 11, 14].

In this section, we introduce a parametric type of quadrature formulae having equidistant nodes and parametric coefficients. In other words, the coefficients of this quadrature are all in terms of a parameter λ that respectively generates the midpoint rule for $\lambda = 1$, the trapezoidal rule for $\lambda = 0$ and the Simpson rule for $\lambda = \frac{2}{3}$. Then we extend the established quadrature to a general *m*-point formula. For this purpose, we first write inequality (2.2) as a three-point quadrature model to arrive at

$$\begin{aligned}\int_a^b f(t)\,dt \cong \left(-\frac{1}{2}a + \lambda x - (\lambda - \frac{1}{2})\,b\right)\,f(a) + \lambda(b-a)\,f(x)\\ + \left((\lambda - \frac{1}{2})\,a - \lambda x + \frac{1}{2}\,b\right)\,f(b),\end{aligned} \tag{3.3}$$

with an error less than

$$\frac{1}{4}\,(\beta_0 - \alpha_0)\,(\lambda^2 + (1-\lambda)^2)\,((x-a)^2 + (b-x)^2), \tag{3.4}$$

for all $\lambda \in [\frac{1}{2}, 1]$ and $x \in [\frac{a+(2\lambda-1)b}{2\lambda}, \frac{b+(2\lambda-1)\,a}{2\lambda}]$ if and only if $\alpha_0 \le f'(t) \le \beta_0$.

The important point is that the function $(x-a)^2 + (b-x)^2$ in the error bound (3.4) can still be minimized on the aforesaid interval with the solution $x = (a+b)/2$. Hence, since $\frac{a+b}{2} \in [\frac{a+(2\lambda-1)b}{2\lambda}, \frac{b+(2\lambda-1)\,a}{2\lambda}]$ for any $\lambda \in [\frac{1}{2}, 1]$, substituting this point in (3.3) yields our optimized quadrature rule

$$\int_a^b f(t)\,dt \cong \frac{b-a}{2}\left((1-\lambda)f(a) + 2\lambda\,f(\frac{a+b}{2}) + (1-\lambda)f(b)\right) \tag{3.5}$$

with an error less than

$$\frac{(b-a)^2(\lambda^2 + (1-\lambda)^2)\,(\beta_0 - \alpha_0)}{8} \tag{3.6}$$

for all $\lambda \in [\frac{1}{2}, 1]$ if and only if $\alpha_0 \le f'(t) \le \beta_0, \forall t \in [a, b]$.

Fortunately, the eligible region for λ in the optimized quadrature (3.5) is extendable to $\lambda \in [0, 1]$, because this region satisfies (2.6). In other words, by defining the particular kernel

$$K(\frac{a+b}{2}; t, p, q) = K^*(t;\lambda) = \begin{cases} t - \frac{a+b}{2} + \lambda\,\frac{b-a}{2} & t \in [a, \frac{a+b}{2}], \\ t - \frac{a+b}{2} - \lambda\,\frac{b-a}{2} & t \in (\frac{a+b}{2}, b], \end{cases}$$

for $\lambda \in [0, 1]$ we have

$$\int_a^b K^*\,(t;\lambda)\;dt = 0,$$

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and also if $\alpha_0 \leq f'(t) \leq \beta_0$ then

$$\begin{aligned}
&\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left((1-\lambda)f(a) + 2\lambda\, f(\frac{a+b}{2}) + (1-\lambda)f(b) \right) \right| \\
&\quad = \left| \int_a^b K^*(t;\lambda) \left(f'(t) - \frac{\alpha_0+\beta_0}{2} \right) dt \right| \leq \frac{\beta_0-\alpha_0}{2} \int_a^b |K^*(t;\lambda)|\,dt \\
&\quad = \frac{\beta_0-\alpha_0}{2}\left(\int_a^{\frac{a+b}{2}} \left| t - \frac{a+b}{2} + \lambda\frac{b-a}{2} \right| dt + \int_{\frac{a+b}{2}}^b \left| t - \frac{a+b}{2} - \lambda\frac{b-a}{2} \right| dt \right) \\
&\quad = \frac{\beta_0-\alpha_0}{2}\left(\int_{\frac{b-a}{2}(\lambda-1)}^{\frac{b-a}{2}\lambda} |z|\,dz + \int_{\frac{b-a}{2}(-\lambda)}^{\frac{b-a}{2}(1-\lambda)} |z|\,dz \right) \\
&\quad = \frac{(\beta_0-\alpha_0)(b-a)^2(\lambda^2+(1-\lambda)^2)}{8}\,.
\end{aligned}$$

This means that the quadrature model (3.5) is valid for all $\lambda \in [0,1]$.
Based on the above-mentioned result and according to (2.10), the three-point quadrature (3.5) should also have an error less than

$$\frac{1}{2}(b-a)^2(\mathbf{S}-\alpha_0),$$

if $\mathbf{S} = (f(b)-f(a))/(b-a)$, $\alpha_0 \leq f'(t)$ and $t \in [a,b]$.

The precision degree of the quadrature model (3.5) is in general $d=1$ for any $\lambda \in [0,1]$ and only for $\lambda = \frac{2}{3}$ it increases to $d=2$. However, as we pointed out, substituting $\lambda = 1$ into (3.5) and (3.6) yields the well-known midpoint rule together with its error bound

$$\left| \int_a^b f(t)\,dt - (b-a)f(\frac{a+b}{2}) \right| \leq \frac{1}{8}(b-a)^2(\beta_0-\alpha_0)\,,$$

and $\lambda = 0$ gives the trapezoidal rule with its error bound

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\,(f(a)+f(b)) \right| \leq \frac{1}{8}(b-a)^2(\beta_0-\alpha_0)\,,$$

and finally $\lambda = \frac{2}{3}$ generates the Simpson rule with its error bound

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b) \right) \right| \leq \frac{5}{72}(b-a)^2(\beta_0-\alpha_0)\,.$$

In this sense, there is still a fourth case that should be highlighted in the parametric quadrature (3.5). Since $\lambda^2 + (1-\lambda)^2$ is minimized at $\lambda = \frac{1}{2} \in [0,1]$, the quadrature rule

$$\int_a^b f(t)\,dt \cong \frac{b-a}{4}\left(f(a) + 2\,f(\frac{a+b}{2}) + f(b) \right), \tag{3.7}$$

has a minimum error bound, i.e., $(b-a)^2\,(\beta_0-\alpha_0)/16$, among all cases depending on λ. Note that in the case $\lambda = \frac{\sqrt{2}}{2}$ (Example 2.5) we minimized the function $g(\lambda)$ in (2.9) to obtain an optimized bound for the right-hand side of inequality (2.2), while in the case $\lambda = \frac{1}{2}$

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we minimized the function $\lambda\, g(\lambda) = \lambda^2 + (1-\lambda)^2$ to obtain an optimized bound for the error absolute value of the quadrature rule (3.5). This means that by replacing $\lambda = \frac{\sqrt{2}}{2}$ and $x = (a+b)/2$ in (3.3), the quadrature

$$\int_a^b f(t)\,dt \cong \frac{b-a}{4}\left((2-\sqrt{2})f(a) + 2\sqrt{2}\, f(\frac{a+b}{2}) + (2-\sqrt{2})f(b)\right),$$

has an error less than $(2-\sqrt{2})\,(b-a)^2\,(\beta_0-\alpha_0)/8$, which is, however, larger than the error bound of (3.7), i.e., $(b-a)^2\,(\beta_0-\alpha_0)/16$.

By using inequality (2.2) and the well known triangle inequality, we can now generalize the quadrature formula (3.3) to a composite form as follows.

THEOREM 3.1. *Let $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I^0}$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If α_0, β_0 are real constants such that $\alpha_0 \le f'(t) \le \beta_0$ for $t \in [a,b]$, then for every partition $I_n = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of the interval $[a,b]$ and for any intermediate point vector $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})$ that satisfies $p_i \in \left[\frac{x_i+(2\lambda-1)\,x_{i+1}}{2\lambda}, \frac{x_{i+1}+(2\lambda-1)\,x_i}{2\lambda}\right] \subseteq [x_i, x_{i+1}]$ for $\lambda \in [\frac{1}{2}, 1]$ and $i = 0, 1, \ldots, n-1$, we have*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - A_\lambda(f; I_n, \mathbf{p})\right| \\ &\qquad \le \frac{(\beta_0-\alpha_0)(\lambda^2+(1-\lambda)^2)}{4}\left(\sum_{i=0}^{n-1}(p_i-x_i)^2 + (x_{i+1}-p_i)^2\right), \end{aligned} \tag{3.8}$$

where A_λ denotes the generalized quadrature rules defined by

$$\begin{aligned} A_\lambda(f; I_n, \mathbf{p}) = &\sum_{i=0}^{n-1}(\lambda\, p_i - \frac{1}{2}x_i - (\lambda - \frac{1}{2})\,x_{i+1})\,f(x_i) \\ &+ \sum_{i=0}^{n-1}(-\lambda\, p_i + (\lambda-\frac{1}{2})\,x_i + \frac{1}{2}x_{i+1})\,f(x_{i+1}) \\ &+ \lambda\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,f(p_i)\,. \end{aligned} \tag{3.9}$$

Proof. By substituting $x = p_i \in \left[\frac{x_i+(2\lambda-1)\,x_{i+1}}{2\lambda}, \frac{x_{i+1}+(2\lambda-1)\,x_i}{2\lambda}\right] \subseteq [x_i, x_{i+1}]$ and $[a,b] = [x_i, x_{i+1}]$ in inequality (2.2) we have

$$\begin{aligned} &\Big|\lambda\,(x_{i+1}-x_i) f(p_i) + (\lambda\, p_i - \frac{1}{2}x_i - (\lambda-\frac{1}{2})\,x_{i+1})\,f(x_i) \\ &\qquad\qquad + (-\lambda\, p_i + (\lambda-\frac{1}{2})\,x_i + \frac{1}{2}x_{i+1})\,f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\,dt\Big| \\ &\quad \le \frac{(\beta_0-\alpha_0)\,(\lambda^2+(1-\lambda)^2)}{4}\;\left((p_i-x_i)^2 + (x_{i+1}-p_i)^2\right). \end{aligned}$$

Now, summing the above inequalities over i from 0 to $n-1$ and then using the triangle inequality proves the theorem straightforwardly. ∎

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3.1. On the convergence of quadrature rules given in Theorem 3.1. Let us reconsider formula (3.1) and set for convenience

$$I(f)=\int_a^b w(t)\,f(t)\,dt \qquad \text{and} \qquad I_{n+1}(f)=\sum_{k=0}^{n} w_k\,f(x_k).$$

As we know, the main problem in numerical integration is to find a system of nodes $\{x_k\}_{k=0}^n$ and weights $\{w_k\}_{k=0}^n$ such that the quadrature error

$$|\,R_{n+1}(f)\,| = |\,I(f)-I_{n+1}(f)\,|,$$

tends to zero as $n\to\infty$. In such a case, we say that the quadrature rule $I_{n+1}(f)$ converges to $I(f)$.
On the other hand, if we have

$$\lim_{n\to\infty}|\,R_{n+1}(f)\,| = \left|I(f)-\lim_{n\to\infty} I_{n+1}(f)\right| \le c,$$

(i.e., a constant number independent of n), then

$$\exists\ c^* < c:\quad \left|I(f)-\lim_{n\to\infty} I_{n+1}(f)\right| = c^* \quad\Leftrightarrow\quad \lim_{n\to\infty}\left(I_{n+1}(f)\pm c^*\right)=I(f).$$

This means that there exists a quadrature $I^*_{n+1}(f)=I_{n+1}(f)\pm c^*$ that converges to $I(f)$.
By noting these initial comments, the right-hand side of relation (3.8) shows that the convergence or divergence of the quadrature (3.9) directly depends on selecting the partition $I_n=\{a=x_0<x_1<\ldots<x_n=b\}$ and also the intermediate points $\{p_i\}_{i=0}^{n-1}$ such that the limit value

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n-1}(p_i-x_i)^2+(x_{i+1}-p_i)^2\right)$$

converges or diverges. In other words, if the series $\sum\limits_{i=0}^{\infty}(p_i-x_i)^2+(x_{i+1}-p_i)^2$ is convergent, then the quadrature $A_\lambda(f;I_n,\mathbf{p})$ in (3.9) does not diverge. For example, take

$$p_i=\frac{x_i+x_{i+1}}{2}\in\left[\frac{x_i+(2\lambda-1)\,x_{i+1}}{2\lambda},\frac{x_{i+1}+(2\lambda-1)\,x_i}{2\lambda}\right]$$

and substitute into inequality (3.8) to get

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{2}\Bigg((1-\lambda)f(x_i)\right.\\ &\qquad\qquad\left.+2\lambda\,f(\frac{x_i+x_{i+1}}{2})+(1-\lambda)\,f(x_{i+1})\Bigg)\right|\\ &\le\frac{(\beta_0-\alpha_0)(\lambda^2+(1-\lambda)^2)}{8}\left(\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\right),\end{aligned}\tag{3.10}$$

which is in fact a generalization of the optimized rule (3.5) and is valid for $\lambda\in[0,1]$. Now, (3.10) is divergent if $\sum\limits_{i=0}^{\infty}(x_{i+1}-x_i)^2$ is divergent and vice versa. For instance, if we choose

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the values x_i in the partition $I_n = \{a = x_0 < x_1 < \ldots < x_n = b\}$ to be of the form $x_{i+1} - x_i = 1/(i+1)$, which is equivalent to $x_i = x_0 + H(i) = a + \sum_{k=1}^{i} \frac{1}{k}$, then the limit case of (3.10) would be

$$\Bigg| \int_a^\infty f(t)\,dt - \sum_{i=0}^{\infty} \frac{1}{2(i+1)} \Big((1-\lambda) f(a+H(i))) + 2\lambda f\left(a + H(i) + \frac{1}{2(i+1)}\right) + (1-\lambda) f\left(a + H(i) + \frac{1}{i+1}\right)\Bigg| \le \frac{(\beta_0 - \alpha_0)(\lambda^2 + (1-\lambda)^2)}{8} \frac{\pi^2}{6} \qquad \forall \lambda \in [0,1].$$

As another example, if $\lambda = \frac{1}{2}$ in (3.9), then the quadrature

$$\int_a^b f(t)\,dt \cong \sum_{i=0}^{n-1} \left(\frac{p_i - x_i}{2} f(x_i) + \frac{x_{i+1} - x_i}{2} f(p_i) + \frac{x_{i+1} - p_i}{2} f(x_{i+1}) \right),$$

having an error less than

$$\frac{\beta_0 - \alpha_0}{8} \left(\sum_{i=0}^{n-1} (p_i - x_i)^2 + (x_{i+1} - p_i)^2 \right)$$

would be convergent for all $p_i \in [x_i, x_{i+1}]$ if the series $\sum_{i=0}^{\infty} (p_i - x_i)^2 + (x_{i+1} - p_i)^2$ is convergent.

4. Numerical experiments. In this section, we present some numerical evidence that illustrates our theoretical results (i.e., the given error bounds). For this purpose we apply the optimized quadrature rule (3.7) having a minimum error bound $E_0 = (b-a)^2 (\beta_0 - \alpha_0)/16$ among all cases depending on the parameter λ. Clearly this bound shows that whenever the values $(b-a)^2$ and $\beta_0 - \alpha_0$ are simultaneously small, the quadrature rule (3.7) is accurate, as Table 4.1 illustrates it. In this table, we have considered the well known special function $\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) = \int_0^x e^{-t^2} dt$ for different values of $x = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ and $\frac{1}{32}$. By noting the value E_0, this table shows whenever x is smaller the bound E_0 is also smaller, and we have therefore a better result for the quadrature (3.7). Note that if $f(t) = e^{-t^2}$ $(t > 0)$, then we have $f'(t) \le 0$ and $f''(t) = 2e^{-t^2}(2t^2 - 1)$, respectively.

TABLE 4.1
Numerical results for $\int_0^x e^{-t^2} dt$ by using the quadrature (3.7).

x	exact value	quadrature (3.7)	|error|	bound E_0
1	0.7468241330	0.7313702519	0.0154538811	0.05361024279
1/2	0.4612810064	0.4572033636	0.0040776428	0.01216876223
1/4	0.2448878872	0.2442753710	0.0006125162	0.00183479114
1/8	0.1243519988	0.1242718492	0.0000801496	0.00024035557
1/16	0.0624187150	0.0624085812	0.0000101338	0.00003039860
1/32	0.0312398305	0.0312385601	0.0000012704	0.00000381097

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Acknowledgments. This research was supported by a grant from "Bonyade Mellie Nokhbegan" No. PM/1184.

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