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A GRADIENT RECOVERY OPERATOR BASED ON AN OBLIQUE PROJECTION*

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Abstract. We present a construction of a gradient recovery operator based on an oblique projection, where the basis functions of two involved spaces satisfy a condition of biorthogonality. The biorthogonality condition guarantees that the recovery operator is local.

Key words. gradient recovery, a posteriori error estimate, biorthogonal system

AMS subject classifications. 65N30, 65N15, 65N50

1. Introduction.

One reason for the success of the finite element method for solving partial differential equations is that a reliable a posteriori error estimator can be applied to measure the approximation of the finite element solution in any local region [1, 2]. The a posteriori error estimator uses the finite element solution itself to assess the accuracy of the numerical solution. Based on this assessment, the finite element mesh can be locally refined resulting in an adaptive process of controling the global error. The adaptive refinement process is much more efficient than the uniform refinement process in finite element computation.

One of the most popular a posteriori estimators is based on recovery of the gradient of the numerical solution. If the recovered gradient approximates the exact gradient better than the gradient computed directly by using the finite element solution, the comparison gives an a posteriori error estimator. The asymptotic exactness of the estimator is based on some superconvergence results [1, 5, 9, 15, 18, 19].

One can compute the orthogonal projection of the computed gradient of the finite element solution onto the actual finite element space to reconstruct the gradient [3, 7, 11]. As the mass matrix is not diagonal, the recovery process is not local. Although one can use a mass lumping procedure to diagonalize the computed mass matrix, the projection property is not valid and the superconvergence property is, in general, lost after doing the mass lumping procedure. Therefore, in this paper we focus on an oblique projection of the directly computed gradient of the numerical solution. The oblique projection is obtained by using two different finite element spaces, where these two spaces satisfy a biorthogonality property. The trial and test spaces for projecting the finite element gradient are chosen such that arising Gram matrix is diagonal. The biorthogonality property allows the local computation of the recovery operator. We show that the error estimator obtained by using the oblique projection is equivalent to the one obtained by using the orthogonal projection. We introduce our oblique projection in the next section and prove some properties of the recovered gradient.

2. Construction of the gradient recovery operator.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded region with polygonal or polyhedral boundary. We consider a locally quasi-uniform triangulation $\mathcal{T}_h$ consisting of simplices or d-parallelotopes of the domain Ω, where each element $T \in \mathcal{T}_h$ can be transformed affinely to a reference simplex, square or cube. We denote the mesh-size of element T by h_T, and the global mesh-size h is given by $h = \max_{T\in\mathcal{T}_h} h_T$. As the mesh is assumed to be locally quasi-uniform, we also use h_T to measure the mesh-size of elements in the local neighborhood of T.

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Let S_h be the space of finite elements defined on the triangulation $\mathcal{T}_h$,

$$S_h := \{v_h \in C^0(\Omega) : \, v_h|_T \in \mathcal{P}_p(T), \; T \in \mathcal{T}_h\}, \quad p \in \mathbb{N},$$

where $\mathcal{P}_p(T)$ is the space of polynomials of total degree less than or equal to p in T if T is a reference simplex and $\mathcal{P}_p(T)$ is the space of polynomials of degree less than or equal to p in each variable if T is a d-parallelotope.

As mentioned in the introduction, we use oblique projection to compute the projection of the gradient onto the finite element space S_h. The projection process can be thought of as a Petrov–Galerkin formulation for the gradient ∇u, where the trial space is $[S_h]^d$, and the basis functions of the test space are constructed in a special way. Let the space of the standard finite element functions S_h be spanned by the basis $\{\phi_1, \dots, \phi_n\}$. We construct the basis $\{\mu_1, \dots, \mu_n\}$ of the space M_h of test functions so that the basis functions of S_h and M_h satisfy a biorthogonality relation

$$\int_\Omega \mu_i \, \phi_j \, dx = c_j \delta_{ij}, \; c_j \neq 0, \; 1 \leq i, j \leq n, \tag{2.1}$$

where $n := \dim M_h = \dim S_h$, δ_{ij} is the Kronecker symbol, and c_j a scaling factor, and is always positive. This scaling factor c_j can be chosen proportionally to the area $|\operatorname{supp} \phi_j|$. It is easy to show that a local basis on the reference element $\hat{T}$ can be easily constructed so that equation (2.1) holds. In the following, we give these basis functions for linear finite elements in two and three dimensions. Since we do not require an approximation property of these basis functions, the construction is only based on a reference element. Therefore, it is easy to construct these basis functions for any finite element space. For the reference triangle $\hat{T} := \{(x, y) : \, 0 \leq x, 0 \leq y, x + y \leq 1\}$, we have

$$\hat{\mu}_1 := 3 - 4x - 4y, \; \hat{\mu}_2 := 4x - 1, \text{ and } \hat{\mu}_3 := 4y - 1,$$

where the basis functions $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$ are associated with three vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ of the reference triangle. For the reference tetrahedron $\hat{T} := \{(x, y, z) : \, 0 \leq x, 0 \leq y, 0 \leq z, x + y + z \leq 1\}$, we have

$$\hat{\mu}_1 := 4 - 5x - 5y - 5z, \; \hat{\mu}_2 := 5x - 1, \text{ and } \hat{\mu}_3 := 5y - 1, \; \hat{\mu}_4 := 5z - 1,$$

where the basis functions $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ associated with four vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of the reference tetrahedron.

If we start by constructing these biorthogonal basis functions locally, and they span the same polynomial space as the finite element basis functions locally, these basis functions are unique up to a scaling factor. The global basis functions for the test space are constructed by glueing the local basis functions together following exactly the same procedure of constructing global finite element basis functions from the local ones. These global basis functions then satisfy the condition of biorthogonality (2.1) with global finite element basis functions, and $\operatorname{supp} \phi_i = \operatorname{supp} \mu_i, \; 1 \leq i \leq n$. The stability requirement for the biorthogonal system is that

$$\beta = \inf_{\phi_h \in S_h} \sup_{\mu_h \in M_h} \frac{\int_\Omega \phi_h \mu_h \, dx}{\|\phi_h\|_{L^2(\Omega)} \|\mu_h\|_{L^2(\Omega)}} > 0. \tag{2.2}$$

We will see that this constant enters into the error estimate.

REMARK 2.1. Biorthogonal basis functions are very popular in the context of mortar finite elements [12, 13, 17], where these basis functions should also satisfy an appropriate approximation property [13, 14], and therefore, difficult to construct for higher order simplicial

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meshes in three dimensions. However, here these basis functions are used only as test functions to compute the projection, and so we do not need the approximation property of these basis functions. This allows an easy construction of local functions in a reference element.

Our gradient recovery technique is based on the oblique projection operator $Q_h : L^2(\Omega) \to S_h$, which is defined as

$$\int_\Omega Q_h v \, \mu_h \, dx = \int_\Omega v \mu_h \, dx, \; v \in L^2(\Omega), \; \mu_h \in M_h. \tag{2.3}$$

It is easy to verify that Q_h is well-defined and is the identity if restricted to S_h. Hence, Q_h is a projection onto the space S_h. We note that Q_h is not the orthogonal projection onto S_h but an oblique projection onto it. Oblique projectors are studied extensively in [10], and different proofs on an identity on the norm of oblique projections are provided in [16].

REMARK 2.2. If we use trial and test functions from the same space S_h, we obtain an orthogonal projection. Then, the locality of the operator Q_h can be obtained only by mass lumping. After mass lumping, the projection property of the operator Q_h in terms of L^2-inner product is lost.

Now we analyze the approximation property of the operator Q_h in the L^2- and H^1-norms. For an arbitrary element $T' \in \mathcal{T}_h$, $S(T')$ denotes the patch of T'. The closure of $S(T')$ is defined as

$$\bar{S}(T') = \bigcup \{\bar{T} \in \mathcal{T}_h : \partial T \cap \partial T' \neq \emptyset\}. \tag{2.4}$$

Let $\mathbf{Q}_h$ be the vector version of the operator Q_h so that $\mathbf{Q}_h : [L^2(\Omega)]^d \longrightarrow [S_h]^d$ and

$$\mathbf{Q}_h \mathbf{u} = (Q_h u_1, \cdots, Q_h u_d) \quad \text{for} \quad \mathbf{u} \in [L^2(\Omega)]^d.$$

Then $\mathbf{Q}_h$ is our gradient recovery operator. We show that the operator $\mathbf{Q}_h$ satisfies the properties (**R**2)–(**R**3) stated for a gradient recovery operator $\mathbf{G}_X$ in [1, pp. 72–73]:

- (**R**2) If $x_0 \in T$, then the value of the recovered gradient depends only on values of ∇v sampled on the patch $S(T)$.
- (**R**3) $\mathbf{G}_X : S_h \to S_h \times S_h$ is a linear operator, and there exists a constant C independent of h such that

$$\|\mathbf{G}_X v\|_{L^\infty(T)} \leq C \|v\|_{W^{1,\infty}(S(T))}, \;\; T \in \mathcal{T}_h, \; v \in S_h.$$

Since S_h and M_h form a biorthogonal system, we can write Q_h as

$$Q_h v = \sum_{i=1}^{n} \frac{\int_\Omega \mu_i \, v \, dx}{c_i} \phi_i, \tag{2.5}$$

which shows that the operator Q_h is local, and hence $\mathbf{Q}_h$ satisfies the property (**R**2) of operator $\mathbf{G}_X$ stated in [1, p. 73].

By using the above representation, we can show that Q_h is stable in the L^2-norm.

LEMMA 2.3. *For $v \in L^2(\Omega)$, we have*

$$\|Q_h v\|_{L^2(T')} \leq C \|v\|_{L^2(S(T'))},$$

and hence

$$\|Q_h v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega)}. \tag{2.6}$$

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Proof. Using the definition of Q_h as given in (2.5), we have

$$\|Q_h v\|_{L^2(T')} = \left\| \sum_{\substack{1\le i\le n\\ T'\subset \operatorname{supp}\phi_i}} \frac{\int_\Omega \mu_i\, v\, dx}{c_i}\phi_i \right\|_{L^2(T')}.$$

Since $\operatorname{supp}\phi_i = \operatorname{supp}\mu_i$ for $1 \le i \le n$, we have

$$\int_\Omega \mu_i\, v\, dx = \int_{\operatorname{supp}\phi_i} \mu_i\, v\, dx.$$

Denoting the support of ϕ_i by S_i and applying the Cauchy–Schwarz inequality yields to

$$\left| \int_{S_i} \mu_i\, v\, dx \right| \le \|\mu_i\|_{L^2(S_i)} \|v\|_{L^2(S_i)},$$

so that

$$\|Q_h v\|_{L^2(T')} \le \sum_{\substack{1\le i\le n\\ T'\subset S_i}} \frac{\|\mu_i\|_{L^2(S_i)}\|v\|_{L^2(S_i)}}{c_i} \|\phi_i\|_{L^2(T')}.$$

Since c_i is proportional to the area $|S_i|$, we estimate the L^2-norm by the L^∞-norm and use the local quasi-uniformity to obtain

$$\|\mu_i\|_{L^2(S_i)} \|\phi_i\|_{L^2(T')} \le C\, c_i,$$

where C is independent of the mesh-size. Thus

$$\|Q_h v\|_{L^2(T')} \le C \sum_{\substack{1\le i\le n\\ T'\subset S_i}} \|v\|_{L^2(S_i)}.$$

Noting that the element T' is fixed and summation is restricted to those $i's$ for which $T' \subset S_i$, we have

$$\|Q_h v\|_{L^2(T')} \le C \|v\|_{L^2(S(T'))},$$

where $S(T')$ is as defined in (2.4). The L^2-stability (2.6) then follows by summing this estimate over all elements $T' \in \mathcal{T}_h$. □

In the following, $P_h : L^2(\Omega) \to S_h$ denotes the L^2-orthogonal projection onto S_h. It is well-known that the operator P_h is stable in the L^2- and H^1-norms. Using the stability of the operator Q_h in the L^2-norm, and of operator P_h in the H^1-norm, we can show that Q_h is also stable in the H^1-norm. We refer to [13] for the proof of this result.

LEMMA 2.4. *For* $w \in H^1(\Omega)$, *we have*

$$|Q_h w|_{H^1(\Omega)} \le C |w|_{H^1(\Omega)}.$$

The following lemma establishes the approximation property of the operator Q_h for a function $v \in H^s(\Omega)$. We refer to [6, 8] for the interpolation theory of functions in $H^s(\Omega)$.

LEMMA 2.5. *For a function $v \in H^{s+1}(\Omega)$, $s > 0$, there exists a constant C independent of the mesh-size h so that*

$$\begin{aligned} \|v - Q_h v\|_{L^2(\Omega)} &\leq Ch^{r+1}|v|_{H^{r+1}(\Omega)}, \\ \|v - Q_h v\|_{H^1(\Omega)} &\leq Ch^{r}|v|_{H^{r+1}(\Omega)}, \end{aligned} \tag{2.7}$$

where $r := \min\{s, p\}$.

Proof. We start with a triangle inequality

$$\|v - Q_h v\|_{L^2(\Omega)} \leq \|v - P_h v\|_{L^2(\Omega)} + \|P_h v - Q_h v\|_{L^2(\Omega)}.$$

Since Q_h acts as an identity on S_h, we have

$$\|v - Q_h v\|_{L^2(\Omega)} \leq \|v - P_h v\|_{L^2(\Omega)} + \|Q_h(P_h v - v)\|_{L^2(\Omega)}.$$

Now we use the equation (2.6) to get

$$\|v - Q_h v\|_{L^2(\Omega)} \leq C\|v - P_h v\|_{L^2(\Omega)}.$$

The first inequality of (2.7) now follows by using the approximation property of the orthogonal projection P_h onto S_h; see [4]. The second inequality of (2.7) is proved similarly using the stability of Q_h in the H^1-norm and the approximation property of the orthogonal projection P_h onto S_h. ☐

LEMMA 2.6. *Let $v_h \in S_h$ and $u \in H^s(\Omega)$ with $s > \frac{d}{2}$. Then for all $T \in \mathcal{T}_h$,*

$$\|\mathbf{Q}_h \nabla v_h\|_{L^\infty(T)} \leq C\|\nabla v_h\|_{L^\infty(S(T))} \tag{2.8}$$

and

$$\|\mathbf{Q}_h \nabla I_h u\|_{L^\infty(T)} \leq C\|\nabla u\|_{L^\infty(S(T))}, \tag{2.9}$$

where I_h is the Lagrange interpolation operator.

Proof. The formula (2.5) for $Q_h v$ yields

$$\|Q_h v\|_{L^\infty(T)} = \left\| \sum_{\substack{1\leq i\leq n \\ T'\subset \operatorname{supp}\phi_i}} \frac{\int_\Omega \mu_i\, v\, dx}{c_i} \phi_i \right\|_{L^\infty(T)}.$$

As $\nabla v_h \in L^\infty(S(T))$, we can follow the arguments of the proof of Lemma 2.3, and obtain the estimate (2.8). To obtain the estimate (2.9), we start with the mean value theorem as in [5],

$$\|\nabla I_h u\|_{L^\infty(S(T))} \leq \|\nabla u\|_{L^\infty(S(T))},$$

and apply the estimate (2.8). ☐

We note that Lemma 2.6 corresponds to property (**R**3) of the operator $\mathbf{G}_X$ stated in [1, pp. 72–73]. We show that the operator $\mathbf{Q}_h$ has the same approximation property as the L^2-projection operator. We note that the L^2-projection operator is not local, whereas our new projection operator $\mathbf{Q}_h$ is local. Hence, it is ideal to use this operator as a gradient recovery operator for a posteriori error estimation.

THEOREM 2.7. *We have*

$$\|\nabla I_h u - \mathbf{P}_h \nabla I_h u\|_{L^2(\Omega)} \leq \|\nabla I_h u - \mathbf{Q}_h \nabla I_h u\|_{L^2(\Omega)} \leq \frac{1}{\beta}\|\nabla I_h u - \mathbf{P}_h \nabla I_h u\|_{L^2(\Omega)}, \tag{2.10}$$

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where $\beta > 0$ is given in (2.2), and $\mathbf{P}_h$ is the vector version of the L^2-projection operator P_h.

Proof. Due to the property of orthogonal projection, it is well-known that

$$\|\nabla I_h u - \mathbf{P}_h \nabla I_h u\|_{L^2(\Omega)} \leq \|\nabla I_h u - \mathbf{Q}_h \nabla I_h u\|_{L^2(\Omega)}.$$

The second part of the inequality is obtained by using

$$\|\mathbf{Q}_h\| = \|\mathbf{I} - \mathbf{Q}_h\|, \quad \text{see [16]},$$

where the norm of the operator is taken with respect to the L^2-norm, and $\mathbf{I}$ is the identity operator. Let $\boldsymbol{\phi}_h$ be an arbitrary element in S_h^d. Applying

$$\begin{aligned}\|\nabla I_h u - \mathbf{Q}_h \nabla I_h u\|_{L^2(\Omega)} &= \|(\mathbf{I} - \mathbf{Q}_h)(\nabla I_h u - \boldsymbol{\phi}_h)\|_{L^2(\Omega)} \\ &\leq \|\mathbf{I} - \mathbf{Q}_h\| \|\nabla I_h u - \boldsymbol{\phi}_h\|_{L^2(\Omega)} \leq \|\mathbf{Q}_h\| \|\nabla I_h u - \boldsymbol{\phi}_h\|_{L^2(\Omega)}.\end{aligned}$$

Furthermore, for $\mathbf{u} \in L^2(\Omega)^d$, we have

$$\|\mathbf{Q}_h \mathbf{u}\| \leq \frac{1}{\beta} \sup_{\boldsymbol{\mu}_h \in M_h^d} \frac{\int_\Omega \boldsymbol{\mu}_h \mathbf{Q}_h \mathbf{u}\, dx}{\|\boldsymbol{\mu}_h\|_{L^2(\Omega)}} \leq \frac{1}{\beta} \frac{\int_\Omega \boldsymbol{\mu}_h \mathbf{u}\, dx}{\|\boldsymbol{\mu}_h\|_{L^2(\Omega)}} \leq \frac{1}{\beta} \|\mathbf{u}\|_{L^2(\Omega)}. \qquad \square$$

Since the error estimator based on L^2-projection is asymptotically exact [3, 7, 11] even for mildy unstructured meshes, the error estimator based on this oblique projection is also asymptotically exact for such meshes. However, the L^2-projection is not local and hence expensive to compute. Our new oblique projection gives a local gradient recovery operator, which is easy and cheap to compute.

The error estimator on the element T is defined as

$$\eta_T = \|\mathbf{Q}_h \nabla u_h - \nabla u_h\|_{L^2(T)},$$

where u_h is the finite element solution of some boundary value problem. If the finite element solution u_h and the Lagrange interpolant $I_h u$ of the true solution u satisfies

$$|u_h - I_h u|_{H^1(\Omega)} \leq C(u) h^{p+\tau}$$

for some $\tau \in (0, 1]$ and $C(u) > 0$ independent of h, then the error estimator can be proved to be asymptotically exact as in [1, 3, 5, 9, 11].

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