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CONDITION NUMBER ANALYSIS FOR VARIOUS FORMS OF BLOCK MATRIX PRECONDITIONERS*

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Dedicated to Richard S. Varga on the occasion of his 80th birthday

Abstract. Various forms of preconditioners for elliptic finite element matrices are studied, based on suitable block matrix partitionings. Bounds for the resulting condition numbers are given, including a study of sensitivity to jumps in the coefficients and to the constant in the strengthened Cauchy-Schwarz-Bunyakowski inequality.

Key words. preconditioning, Schur complement, domain decomposition, Poincaré–Steklov operator, approximate block factorization, strengthened Cauchy-Schwarz-Bunyakowski inequality

AMS subject classifications. 65F10, 65N22

1. Introduction.

Preconditioning is an essential part of an efficient iterative solution method when solving large-scale linear and nonlinear systems of equations. This paper deals with systems arising from the finite element discretization of elliptic partial differential equations.

The efficiency of a preconditioner is mostly judged by the condition number of the resulting preconditioned operator, and in applications it is important to know whether the condition number depends critically on certain problem parameters such as jumps in the material coefficients. The most efficient preconditioners are based on some block partitioning of the matrix. Common structures are block tridiagonal and two-by-two partitionings. Elementwise constructed preconditioners can be efficient as they can be constructed locally and relatively cheaply but still can provide a significant reduction of the condition number of the unpreconditioned operator.

Block tridiagonal matrices arise in many applications. For instance, such a structure arises when decomposing the domain of definition of an elliptic operator using unidirectional stripes, or more generally, for a decomposition such that (in addition to a corresponding portion of the original boundary) each subdomain has a common boundary only with its previous and next neighbours in the sequence of subdomains. This subdivision can often be done according to different values of the coefficients in the differential operator, i.e., different materials in the underlying physical domain. Each diagonal block in the matrix corresponds to the restriction of the operator to one of the subdomains, and ordering the nodes in each domain in groups and then the domains consecutively, results in a block tridiagonal matrix.

An interesting example of matrices of two-by-two block structure arises by ordering the interior domain nodes separately from the interface nodes and ordering all interface nodes last. This in turn results in a block diagonal submatrix with uncoupled blocks, which are only coupled to the interface nodes ordered last. The part of the system which corresponds to the different interior node sets can then be solved in parallel.

In both cases there arise Schur complement matrices when solving systems with these matrices. For the block tridiagonal case, they arise at each step of the consecutive elimination of the pivot blocks, and in the latter case by elimination of the interior nodes. Schur complement matrices are in general full matrices and must be approximated by some sparse matrix

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in the construction of the preconditioner. The construction of such approximations and the analysis of condition numbers of the Schur complements, both on continuous and discrete level, are the main topic of this paper.

We give here first a general framework for the analysis of approximations of Schur complement matrices. Consider a symmetric positive definite bilinear form $a(u, v)$, and let U_1, U_2 be two subspaces of a linear space V, where the intersection of U_1, U_2 contains only the trivial element. Here the spaces can be more general function spaces as well, but in our applications V is a finite element space, i.e., spanned by a set of finite element basis functions. As has been shown in early publications [3, 4, 11, 13, 14], the strengthened Cauchy-Schwarz-Bunyakowski inequality plays a fundamental role in the analysis of matrices partitioned in two-by-two block form. The inequality takes the form

$$a(u, v) \le \gamma \Big(a(u, u)\, a(v, v) \Big)^{1/2}, \qquad \forall u \in U_1,\ v \in U_2,$$

where $\gamma < 1$ is the smallest such constant and is referred to as the CBS constant. In fact, γ is the cosine of the angle between the two subspaces, measured by the inner product $a(u, v)$. For matrices in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

the CBS inequality can be written as

$$x_1^T A_{12}\, x_2 \le \gamma \Big((x_1^T A_{11}\, x_1)\, (x_2^T A_{22}\, x_2) \Big)^{1/2}, \qquad \forall x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}.$$

Alternatively, we can define γ by $\gamma = \varrho\,(A_{22}^{-1/2} A_{21} A_{11}^{-1} A_{12} A_{22}^{-1/2})^{1/2}$, where $\varrho(.)$ denotes the spectral radius. Hence γ measures the size of the off-diagonal blocks in relation to the diagonal blocks. It is readily seen that

$$(1-\gamma^2) x_2^T A_{22}\, x_2 \le x_2^T S_2\, x_2 \le x_2^T A_{22}\, x_2, \qquad \forall x_2 \in \mathbb{R}^{n_2},$$

where $S_2 := A_{22} - A_{21} A_{11}^{-1} A_{12}$ is the Schur complement matrix. Hence the condition number satisfies

$$\kappa(A_{22}^{-1} S_2) \le \frac{1}{1-\gamma^2}. \tag{1.1}$$

The remainder of the paper is organized as follows. In Section 2 we discuss briefly the factorization of block tridiagonal matrices. We are in particular interested in approximating the arising Schur complement matrices in such a way that their quality is insensitive to jumps in the coefficients in the differential operator. This will be discussed in Section 3. Section 4 is devoted to an algebraic derivation of condition numbers in the approximations of matrices partitioned in two-by-two block form, where the pivot block is block diagonal, such that the condition number depends only on the CBS constant. A continuous analogue of the method of Section 3 is presented on some model problems in Section 5. In Section 6 we analyze the case of using elementwise approximations of Schur complements, and how to define them so that they also become insensitive to coefficient jumps.

Except when it is otherwise stated, the inequalities

$$A \le B, \quad A < B$$

between two symmetric matrices (of the same order) mean that $B - A$ is positive semidefinite or positive definite, respectively. The notation $\varrho(A)$ for a symmetric positive semidefinite matrix A stands for its maximal eigenvalue. The spectral condition number of A is defined by $\kappa(A) = \lambda_{max}(A)/\lambda_{min}(A)$.

2. Recursive approximation of Schur complements. Let us consider a symmetric, positive definite matrix A with tridiagonal block structure

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{m,m-1} & A_{mm} \end{bmatrix}. \tag{2.1}$$

Here $A_{ij} = A_{ji}^T$ for all i, j. The exact block factorization of A takes the form

$$A = (S + L_A)S^{-1}(S + L_A^T),$$

where $S = blockdiag(S_1, \dots, S_m)$ and L_A is the strictly lower block triangular part of A. Here the Schur complements S_i are determined recursively as

$$\begin{aligned} &S_1 := A_{11}, \\ &S_2 := A_{22} - A_{21}S_1^{-1}A_{12}, \\ &\ldots.. \\ &S_i := A_{ii} - A_{i,i-1}S_{i-1}^{-1}A_{i-1,i}, \\ &\ldots.. \end{aligned} \tag{2.2}$$

for $i \leq m$.

The application of this factorization to solve a linear system involves the solution of the block triangular factors using a forward and a backward sweep. At each of them, systems with matrices $S_i, i = 1, \dots, m$, appear that must be solved. In addition, matrix-vector multiplications with L_A and L_A^T, respectively, appear. In general, the S_i are full matrices and their construction and the computation of actions of S_i^{-1} can be expensive.

Our goal is to approximate S_i by some matrix X_i which is sparse and the computation of X_i and X_i^{-1} applied to vectors are cheap. We define $X = blockdiag(X_1, X_2, ..., X_m)$, and let the preconditioner C be defined by $C = (X + L_A)X^{-1}(X + L_A^T)$. At the same time, the approximation must be sufficiently accurate. For instance, it has been shown in [1] that the following lower bound holds for the condition number: $\kappa(C^{-1}A) \geq \min_i \kappa(X_i^{-1}S_i)$.

Since systems with the matrices A_{ii} are generally inexpensive to solve, we could try A_{ii} as an approximation of S_i. At each step of the method we then deal with a two-by-two block matrix in the form

$$\begin{bmatrix} A'_{ii} & A_{i,i+1} \\ A_{i+1,i} & A_{i+1,i+1} \end{bmatrix},$$

where

$$A'_{ii} = \begin{bmatrix} A_{11} & A_{12} & 0 & \dots & \dots & 0 \\ A_{21} & A_{22} & A_{23} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & A_{i,i-1} & A_{ii} \end{bmatrix}, \qquad i = 1, 2, ..., m-1.$$

As pointed out in the introduction, the accuracy of the approximation $A_{i+1,i+1}$ of $S_{i+1} := A_{i+1,i+1} - A_{i+1,i}(A'_{ii})^{-1}A_{i,i+1}$ is given by $\kappa(A_{i+1,i+1}^{-1}S_{i+1}) = 1/(1-\gamma_{i+1}^2)$. Here γ_{i+1} depends on the stage of the elimination. For a model elliptic problem with constant coefficients on a unit square and constant mesh size h, it can be seen (see, e.g., [1]) that the

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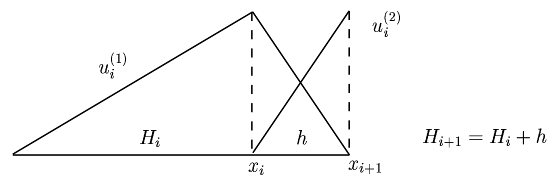

FIG. 2.1. *The functions $u_i^{(1)}$ and $u_i^{(2)}$ for which the CBS constant is taken.*

basis functions which give rise to the γ-constant at stage i are as shown in Figure 2.1, where $0 < x_i < 1$, $x_{i+1} - x_i = h$.

Since $a(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v$, one finds $\gamma_{i+1}^2 = 1 - (h/H_{i+1})$. In the limit as $i \to m$ and $H_i \to 1$, one finds $\gamma_m^2 = 1 - h$. For more general problems, such as with variable coefficients, one gets $\gamma_{i+1}^2 = 1 - O(h/H_{i+1})$ and $\gamma_m^2 = 1 - O(h)$. It follows that the quality of this approximation deteriorates with increasing stage numbers i.

As discussed in several publications (see, e.g., [1, 9, 18]), the approximation method can be improved in various ways. A simple method is to use a diagonal compensation in some form, where

$$X_i := A_{ii} - D_i\,, \tag{2.3}$$

where D_i is a diagonal matrix, such that

$$D_i v_i = A_{i,i-1} X_{i-1}^{-1} A_{i-1,i}\, v_i \tag{2.4}$$

for some given positive vector v_i.

First, let v_i be the eigenvector to $A_{i,i-1} X_{i-1}^{-1} A_{i-1,i}$ corresponding to the smallest eigenvalue ξ_i of this matrix. Then

$$D_i v_i = \xi_i v_i\,,$$

i.e., $D_i = \xi_i I_i$ is a multiple of the identity matrix for the ith block. Since ξ_i is the smallest eigenvalue, it follows that $A_{i,i-1} X_{i-1}^{-1} A_{i-1,i} \geq D_i$ and then $A_{ii} - A_{i,i-1} X_{i-1}^{-1} A_{i-1,i} \leq A_{ii} - D_i = X_i$. Here

$$C - A = X + L_A X^{-1} L_A^T - D_A,$$

where $D_A = blockdiag(A_{11}, A_{22}, \ldots, A_{mm})$, and

$$(C - A)_{ii} = X_i + A_{i,i-1} X_{i-1}^{-1} A_{i-1,i} - A_{ii} \geq 0.$$

Hence $C \geq A$, which yields

$$\varrho(C^{-1}A) \leq 1.$$

In this method we must estimate the smallest eigenvalue of $C^{-1}A$, which we will not do here as the choice of X should rather be such that the smallest eigenvalue of $C^{-1}A$ is bounded below by unity or some positive constant $\alpha \leq 1$.

Consider now the choice $v_i := \mathbf{e}_i = (1, \ldots, 1)$, i.e., $\mathbf{e}_i$ has all components equal to unity. Then X_i is obtained from

$$X_i := A_{ii} - D_i, \tag{2.5}$$

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where

$$D_i \mathbf{e}_i = A_{i,i-1} X_{i-1}^{-1} A_{i-1,i} \mathbf{e}_i \,. \tag{2.6}$$

Assume here for simplicity that A is an M-matrix. Then we have componentwise

$$A_{ii}^{-1} \geq 0, \quad A_{i,i-1} \leq 0, \quad A_{i-1,i} \leq 0.$$

It follows by induction that $X_{i-1}^{-1} \geq 0$ componentwise. Hence $A_{ii} - A_{i,i-1} X_{i-1}^{-1} A_{i-1,i}$ is a Z-matrix, i.e., all its off-diagonal components are non-positive. Since $(A_{ii} - D_i)\mathbf{e}_i = (A_{ii} - A_{i,i-1} X_{i-1}^{-1} A_{i-1,i})\mathbf{e}_i$, it holds that if this vector is nonzero then $X_i = A_{ii} - D_i$ is positive definite, and also an M-matrix. Should the matrix lose positive definiteness (by having $(A_{ii} - D_i)\mathbf{e}_i = 0$), we must perturb the matrices A_{ii} with some (small) positive number. This has been discussed, e.g., in [1]; see also [5].

Assuming that no perturbation is required, we have

$$X_i = A_{ii} - D_i \leq A_{ii} - A_{i,i-1} X_{i-1}^{-1} A_{i-1,i},$$

with the inequality in the positive semidefinite sense. Therefore

$$(C - A)_{ii} = X_i + A_{i,i-1} X_{i-1}^{-1} A_{i-1,i} - A_{ii} \leq 0,$$

that is, $C \leq A$ and

$$\lambda_i(C^{-1}A) \geq 1.$$

Hence we have a lower bound. The upper bound follows from a theorem in [18], there stated in a somewhat more general form.

THEOREM 2.1. *Let A be a symmetric positive definite matrix partitioned in $m \times m$ block form. Let $C = (X + L)X^{-1}(X + L^T)$, where X is symmetric positive definite block diagonal and L is strictly lower block triangular, both with consistent partitioning to A. Let $\mu_i := \lambda_i(X^{-1}K)$, where $K = A - L - L^T$, let $\sigma := \max \mu_i$ and assume that $\sigma < 2$. Then*

$$\kappa(C^{-1}A) \leq min\Big\{\frac{1}{2-\sigma},\ \sum_{i=1}^{m} \mu_i - \alpha(m-1)\Big\}.$$

In particular, if $L = L_A$, then $\mu_i = \lambda_i(X^{-1}D_A)$ and if

$$D_i \leq \varrho A_{ii} \quad \text{for some} \quad \varrho < 1/2, \tag{2.7}$$

then by (2.5),

$$X_i \geq (1 - \varrho)A_{ii}.$$

Hence

$$\sigma \leq \lambda_{max}(X_i^{-1}A_{ii}) \leq \frac{1}{1-\varrho} < 2. \tag{2.8}$$

REMARK 2.2. The above two choices have somewhat opposite properties. In particular, if $\mathbf{e}_i$ is also an eigenvector of $A_{i,i-1} X_{i-1}^{-1} A_{i-1,i}$ for the smallest eigenvalue, then it can be seen that $A_{i,i-1} X_{i-1}^{-1} A_{i-1,i}$ is a multiple of the identity matrix. In the following we assume that this does not hold.

In the following section it is analyzed how the Schur complements depend on jumps in the coefficients in the differential operator.

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3. Schur complements for elliptic problems with jumps in their coefficients. Let us consider a domain decomposition (DD) method for an elliptic problem discretized with FEM, such that (in addition to a corresponding portion of the outer boundary) each subdomain has a common boundary only with its previous and next neighbours in the sequence of subdomains. Elliptic operators with different constant diffusion coefficient in each subdomain often arise in the context of various (DD) procedures [2, 15, 16, 17, 19]. Our goal in this section is to study the sensitivity of Schur complements to coefficient jumps.

In the classical DD approach, the interior domain nodes are ordered separately from the interface nodes and all interface nodes are ordered last. Like in multigrid methods, to avoid large condition numbers of the corresponding Schur complements, an efficient method has proved to be to introduce one or more proper auxiliary coarse spaces that have a global balancing effect; see, e.g., the BDD method [19] and the approach of so-called exotic coarse spaces [16] in a Schwarz method framework.

An alternative to the above approach is to take the interface nodes into account together with the previous subdomain in the mentioned sequence of subdomains. This approach, considered in the present paper, leads to a tridiagonal block structure as in (2.1). It will be verified for a model problem that the condition numbers of the Schur complements are sensitive to the jump in the first approach (namely, proportional to the magnitude of the jump) but are not in the second approach. That is, one can have independence of jumps without introducing auxiliary problems.

For simplicity, the detailed study is given for a decomposition of the domain Ω in three subdomains Ω_1, Ω_2 and Ω_3. According to the above, we have common boundaries $\Gamma_1 := \overline{\Omega}_1 \cap \overline{\Omega}_2$ and $\Gamma_2 := \overline{\Omega}_2 \cap \overline{\Omega}_3$, but Ω_1 and Ω_3 have no common boundary. We will first formulate the block forms of the stiffness matrix under the two mentioned approaches for an isotropic Poisson equation. Then we rewrite the stiffness matrices under different diffusion coefficient in each Ω_i, and study the variation of the corresponding condition numbers.

3.1. Basic block forms for the isotropic Poisson equation. Let us consider the Poisson equation with homogeneous Dirichlet boundary conditions. The FEM subspace is chosen with piecewise linear basis functions, assumed either to have node points on one of Γ_i or to have its support entirely in one of Ω_i.

In the classical DD approach, the stiffness matrix is written in the block form

$$A = \begin{bmatrix} A_{11} & 0 & 0 & A_{1,\Gamma_1} & 0 \\ 0 & A_{22} & 0 & A_{2,\Gamma_1} & A_{2,\Gamma_2} \\ 0 & 0 & A_{33} & 0 & A_{3,\Gamma_2} \\ A_{\Gamma_1,1} & A_{\Gamma_1,2} & 0 & A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & A_{\Gamma_2,2} & A_{\Gamma_2,3} & 0 & A_{\Gamma_2,\Gamma_2} \end{bmatrix}. \tag{3.1}$$

Here $A_{i,\Gamma_j} = A_{j,\Gamma_i}^T$ for all i, j. Then one lets

$$A_{\Gamma 1} := A_{1\Gamma}^T := \begin{bmatrix} A_{\Gamma_1,1} \\ 0 \end{bmatrix}, \quad A_{\Gamma 2} := A_{2\Gamma}^T := \begin{bmatrix} A_{\Gamma_1,2} \\ A_{\Gamma_2,2} \end{bmatrix}, \quad A_{\Gamma 3} := A_{3\Gamma}^T := \begin{bmatrix} 0 \\ A_{\Gamma_2,3} \end{bmatrix}, \tag{3.2}$$

$$A_{\Gamma\Gamma} := \begin{bmatrix} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & A_{\Gamma_2,\Gamma_2} \end{bmatrix}, \tag{3.3}$$

and thus obtains the more concise form

$$A = \begin{bmatrix} A_{11} & 0 & 0 & A_{1\Gamma} \\ 0 & A_{22} & 0 & A_{2\Gamma} \\ 0 & 0 & A_{33} & A_{3\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma 3} & A_{\Gamma\Gamma} \end{bmatrix}. \tag{3.4}$$

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The solution of the corresponding linear system can be reduced to solving systems with $\Sigma_i := A_{ii}$, $i = 1, 2, 3$, and an additional system with the Schur complement matrix

$$\Sigma := A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} - A_{\Gamma 3} A_{33}^{-1} A_{3\Gamma} \,. \tag{3.5}$$

In the other approach, the interface nodes are taken into account together with the previous subdomain. Under this reordering, the stiffness matrix in (3.1) can be rewritten as

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{1,\Gamma_1} & 0 & 0 & 0 \\ A_{\Gamma_1,1} & A_{\Gamma_1,\Gamma_1} & A_{2,\Gamma_1} & 0 & 0 \\ 0 & A_{\Gamma_1,2} & A_{22} & A_{2,\Gamma_2} & 0 \\ 0 & 0 & A_{\Gamma_2,2} & A_{\Gamma_2,\Gamma_2} & A_{3,\Gamma_2} \\ 0 & 0 & 0 & A_{\Gamma_2,3} & A_{33} \end{bmatrix}, \tag{3.6}$$

where we introduce the notation

$$\tilde{A}_{11} := \begin{bmatrix} A_{11} & A_{1,\Gamma_1} \\ A_{\Gamma_1,1} & A_{\Gamma_1,\Gamma_1} \end{bmatrix}, \qquad \tilde{A}_{12} := \begin{bmatrix} 0 & 0 \\ A_{2,\Gamma_1} & 0 \end{bmatrix}, \qquad \tilde{A}_{21} := \begin{bmatrix} 0 & A_{\Gamma_1,2} \\ 0 & 0 \end{bmatrix}, \tag{3.7}$$

$$\tilde{A}_{22} := \begin{bmatrix} A_{22} & A_{2,\Gamma_2} \\ A_{\Gamma_2,2} & A_{\Gamma_2,\Gamma_2} \end{bmatrix}, \qquad \tilde{A}_{23} := \begin{bmatrix} 0 \\ A_{3,\Gamma_2} \end{bmatrix}, \qquad \tilde{A}_{32} := \begin{bmatrix} 0 \\ A_{\Gamma_2,3} \end{bmatrix}, \tag{3.8}$$

to obtain the concise form

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & A_{33} \end{bmatrix}. \tag{3.9}$$

In the Schur complement approach, here only the first block remains unchanged: $S_1 := \tilde{A}_{11}$, and the solution of the original system can now be reduced to solving two additional systems corresponding to Schur complements, determined recursively as

$$S_2 := \tilde{A}_{22} - \tilde{A}_{21} S_1^{-1} \tilde{A}_{12}, \qquad S_3 := A_{33} - \tilde{A}_{32} S_2^{-1} \tilde{A}_{23} \,. \tag{3.10}$$

Using notation (3.7)-(3.8) and letting

$$S_{\Gamma_1} := A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}, \tag{3.11}$$

we obtain

$$S_2 = \begin{bmatrix} A_{22} - A_{\Gamma_1,2} S_{\Gamma_1}^{-1} A_{2,\Gamma_1} & A_{2,\Gamma_2} \\ A_{\Gamma_2,2} & A_{\Gamma_2,\Gamma_2} \end{bmatrix}. \tag{3.12}$$

The similar formula for S_3 will not be needed here.

3.2. Conditioning properties for problems with jumps in their coefficients. Now we can turn to the case of our interest. Instead of the above Poisson equation, we consider the FEM solution of an elliptic problem with a different constant diffusion coefficient in each Ω_i. That is, in weak form, one seeks $u \in V_h \subset H_0^1(\Omega)$, such that

$$\int_\Omega w \, \nabla u \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in V_h, \tag{3.13}$$

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where w is a weight function on Ω, such that

$$w_{|\Omega_i} \equiv w_i, \qquad i = 1, 2, 3.$$

In our model problem we assume

$$w_1 \geq w_2 \geq w_3 \tag{3.14}$$

and are interested in the case

$$w_1 >> w_2\,. \tag{3.15}$$

When varying these coefficients, in order to avoid the loss of ellipticity in the limit, we also assume that there exists a constant $\alpha > 0$ such that

$$w_3 \geq \alpha w_2\,. \tag{3.16}$$

Below, we will find that if we vary the ratio w_1/w_2 unboundedly, then the condition numbers also grow to infinity for the Schur complement in (3.5) but remain bounded for the Schur complements in (3.10).

Let us first consider the classical DD approach again. The stiffness matrix (3.1) is then modified as follows. The entries corresponding to basis functions with support in Ω_i are multiplied by the weight w_i. For simplicity, assume that for the node points on one of Γ_i, the support of the basis function is symmetric w.r.t the node point, and thus its parts intersecting with the two domains have equal measure. (An opposite case will be mentioned in Remark 3.3.) Then the entries corresponding to such basis functions are multiplied by $(w_i + w_j)/2$. Therefore, the stiffness matrix has the form

$$A = \begin{bmatrix} w_1 A_{11} & 0 & 0 & w_1 A_{1,\Gamma_1} & 0 \\ 0 & w_2 A_{22} & 0 & w_2 A_{2,\Gamma_1} & w_2 A_{2,\Gamma_2} \\ 0 & 0 & w_3 A_{33} & 0 & w_3 A_{3,\Gamma_2} \\ w_1 A_{\Gamma_1,1} & w_2 A_{\Gamma_1,2} & 0 & \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & w_2 A_{\Gamma_2,2} & w_3 A_{\Gamma_2,3} & 0 & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} \end{bmatrix}. \tag{3.17}$$

With these modifications, one readily sees that the Schur complement (3.5) becomes

$$\Sigma(w) := W A_{\Gamma\Gamma} - w_1 A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - w_2 A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} - w_3 A_{\Gamma 3} A_{33}^{-1} A_{3\Gamma}, \tag{3.18}$$

where W is the two-by-two block diagonal matrix, blockdiag$\left(\frac{w_1+w_2}{2} I_{\Gamma 1},\ \frac{w_2+w_3}{2} I_{\Gamma 2}\right)$.

PROPOSITION 3.1. *There exist constants $c_1, c_2 > 0$ independent of w, such that*

$$\kappa(\Sigma(w)) \geq c_1 \frac{w_1}{w_2} + c_2\,. \tag{3.19}$$

Proof. Using (3.2)-(3.3), a simple calculation yields

$$\begin{aligned} \tilde{\Sigma}(w) &:= \frac{1}{w_2}\Sigma(w) \\ &= \begin{bmatrix} \frac{w_1}{w_2}\Sigma_1 + \frac{1}{2}A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & \frac{1}{2}(1+\frac{w_3}{w_2})A_{\Gamma_2,\Gamma_2} \end{bmatrix} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} - \frac{w_3}{w_2} A_{\Gamma 3} A_{33}^{-1} A_{3\Gamma}, \end{aligned} \tag{3.20}$$

where $\Sigma_1 := \frac{1}{2}A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1}A_{11}^{-1}A_{1,\Gamma_1}$. Here $\Sigma_1 \geq 0$ (i.e., it is positive semidefinite) and is not a zero matrix since it is a Schur complement, corresponding to the positive definite matrix $\tilde{A}_{11}$ modified by setting a zero diffusion coefficient outside Ω_1. Further, $A_{\Gamma_i,\Gamma_i} > 0$, $i = 1, 2$, and $\frac{1}{2}(1 + \frac{w_3}{w_2}) \leq 1$, owing to (3.14). Hence the matrix

$$G(w) := \begin{bmatrix} \frac{w_1}{w_2}\,\Sigma_1 + \frac{1}{2}A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & \frac{1}{2}(1 + \frac{w_3}{w_2})A_{\Gamma_2,\Gamma_2} \end{bmatrix}$$

satisfies $\lambda_{max}(G(w)) \geq \frac{w_1}{w_2}\lambda_{max}(\Sigma_1)$ and $\lambda_{min}(G(w)) \leq \lambda_{min}(A_{\Gamma_2,\Gamma_2})$, which yields for the condition number of $G(w)$ that

$$\kappa(G(w)) \geq \frac{w_1}{w_2}\frac{\lambda_{max}(\Sigma_1)}{\lambda_{min}(A_{\Gamma_2,\Gamma_2})}\,.$$

The condition numbers of the other two terms in (3.20) are bounded. Since $\kappa(\Sigma(w)) = \kappa(\tilde{\Sigma}(w))$, we obtain (3.19). ∎

COROLLARY 3.2. *If we vary $\frac{w_1}{w_2}$ unboundedly, then*

$$\kappa(\Sigma(w)) = O\big(\frac{w_1}{w_2}\big) \to \infty \quad as\ \frac{w_1}{w_2} \to \infty.$$

REMARK 3.3. The above sensitivity to $\frac{w_1}{w_2}$ may be reduced if the supports of the basis functions on Γ_1 are not assumed to be symmetric with respect to the node point, but their parts intersecting with Ω_2 have small measure. However, this would in turn lead to inpractically small element widths and very large gradients of the basis functions near Γ_1.

Let us now consider the second approach. We study the Schur complements (3.10) modified with respect to the diffusion coefficient w. The corresponding modification of the matrix $\tilde{A}$ in (3.6) comes by first replacing the considered blocks of (3.1) by the corresponding blocks of (3.17), and then using the same reassembling as for (3.6). Then the Schur complement S_2 in (3.12) becomes modified as

$$S_2(w) := \begin{bmatrix} w_2 A_{22} - \, w_2^2\, A_{\Gamma_1,2}\, S_{\Gamma_1}(w)^{-1} A_{2,\Gamma_1} & w_2 A_{2,\Gamma_2} \\ w_2 A_{\Gamma_2,2} & \frac{1}{2}(w_2 + w_3)A_{\Gamma_2,\Gamma_2} \end{bmatrix}, \tag{3.21}$$

where S_{Γ_1} in (3.11) has been replaced by

$$S_{\Gamma_1}(w) := \frac{w_1 + w_2}{2}\, A_{\Gamma_1,\Gamma_1} - w_1 A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}\,. \tag{3.22}$$

Introducing the notation

$$S_2^{11}(w) := A_{22} - \, w_2\, A_{\Gamma_1,2} S_{\Gamma_1}(w)^{-1} A_{2,\Gamma_1}\,, \tag{3.23}$$

we have

$$S_2(w) := \begin{bmatrix} w_2 S_2^{11}(w) & w_2 A_{2,\Gamma_2} \\ w_2 A_{\Gamma_2,2} & \frac{1}{2}(w_2 + w_3)A_{\Gamma_2,\Gamma_2} \end{bmatrix}. \tag{3.24}$$

LEMMA 3.4. *There holds* $S_{\Gamma_1}(w) \geq w_2\, S_{\Gamma_1}$.

Proof. We have

$$S_{\Gamma_1}(w) = w_2\Big[\frac{1}{2}\big(\frac{w_1}{w_2} + 1\big)A_{\Gamma_1,\Gamma_1} - \frac{w_1}{w_2}A_{\Gamma_1,1}A_{11}^{-1}A_{1,\Gamma_1}\Big]$$

$$= w_2\Big[\frac{w_1}{w_2}\big(\frac{1}{2}A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1}A_{11}^{-1}A_{1,\Gamma_1}\big) + \frac{1}{2}A_{\Gamma_1,\Gamma_1}\Big]\,.$$

Since, by assumption, $w_1 \geq w_2$, we obtain

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$$S_{\Gamma_1}(w) \geq w_2 \Big[\big(\frac{1}{2} A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}\big) + \frac{1}{2} A_{\Gamma_1,\Gamma_1} \Big] = w_2\, S_{\Gamma_1}. \qquad \square$$

Similarly to (3.23), let us denote the top left block of (3.12) by

$$S_2^{11} := A_{22} - A_{\Gamma_1,2} S_{\Gamma_1}^{-1} A_{2,\Gamma_1}\,, \tag{3.25}$$

and then let

$$\tilde{S}_2 := \begin{bmatrix} S_2^{11} & A_{2,\Gamma_2} \\ A_{\Gamma_2,2} & \frac{1}{2}(1+\alpha) A_{\Gamma_2,\Gamma_2} \end{bmatrix} \tag{3.26}$$

with α from (3.16). Now we can prove the following required boundedness.

PROPOSITION 3.5. *The condition number of $S_2(w)$ satisfies*

$$\kappa(S_2(w)) \leq \frac{\lambda_{max}(\tilde{A}_{22})}{\lambda_{min}(\tilde{S}_2)}.$$

Hence it is bounded independently of w.

Proof. Clearly $S_2^{11}(w) \leq A_{22}$, and $\frac{1}{2}(w_2 + w_3) \leq w_2$ owing to (3.14). Hence

$$S_2(w) \leq w_2 \begin{bmatrix} A_{22} & A_{2,\Gamma_2} \\ A_{\Gamma_2,2} & A_{\Gamma_2,\Gamma_2} \end{bmatrix} = w_2\, \tilde{A}_{22}\,. \tag{3.27}$$

To find a lower bound for $S_2(w)$, note that Lemma 3.4 and the definitions (3.23) and (3.25) yield

$$S_2^{11}(w) \geq S_2^{11}\,. \tag{3.28}$$

Substituting (3.28) into (3.24), and using (3.16) and (3.26), respectively, we then obtain

$$S_2(w) \geq \begin{bmatrix} w_2 S_2^{11} & w_2 A_{2,\Gamma_2} \\ w_2 A_{\Gamma_2,2} & \frac{1}{2}(1+\alpha) w_2\, A_{\Gamma_2,\Gamma_2} \end{bmatrix} = w_2\, \tilde{S}_2\,.$$

Here $\tilde{S}_2 > 0$, since by the above, $w_2 \tilde{S}_2$ is the Schur complement $S_2(w)$ in the case $w_3 = \alpha w_2$. Together with (3.27), we obtain the required statement. $\square$

Finally, we consider the second Schur complement S_3 from (3.10). When replacing its considered blocks from (3.1) by the corresponding blocks of (3.17), we observe that each of the blocks A_{33}, $\tilde{A}_{32}$ and $\tilde{A}_{23}$ is multiplied by w_3. Hence the matrix S_3 becomes modified as

$$S_3(w) := w_3\, A_{33} - w_3^2\, \tilde{A}_{32}\, S_2(w)^{-1} \tilde{A}_{23}\,. \tag{3.29}$$

We can easily prove again the following required boundedness.

PROPOSITION 3.6. *The condition number of $S_3(w)$ satisfies*

$$\kappa(S_3(w)) \leq \frac{\lambda_{max}(A_{33})}{\lambda_{min}(S_3)}.$$

Hence it is bounded independently of w.

Proof. Obviously $S_3(w) \leq w_3\, A_{33}$. Further, in (3.24) we can estimate each w_2 below by w_3 and $S_2^{11}(w)$ below by S_2^{11} using (3.28), such that we obtain

$$S_2(w) \geq w_3 \begin{bmatrix} S_2^{11} & A_{2,\Gamma_2} \\ A_{\Gamma_2,2} & A_{\Gamma_2,\Gamma_2} \end{bmatrix} = w_3\, S_2\,,$$

and substitution into (3.29) yields $S_3(w) \geq w_3\, A_{33} - w_3\, \tilde{A}_{32}\, S_2^{-1} \tilde{A}_{23} = w_3\, S_3$. The two bounds imply the desired estimate. $\square$

3.3. On the growth of condition number with the number of subdomains. Whereas we have obtained jump independence in the previous subsection, these estimates are inherently unable to compensate for the number of subdomains. This follows if we relate the new estimates to those on the original Schur complements (whose condition number is known to increase with the number of subdomains). Namely, the appearance of the new constants w_i makes each inequality worse (or unchanged if the constants coincide), therefore $S_i(w)$ cannot be better conditioned than the original S_i.

In fact, for $S_2(w)$, definition (3.10) implies $S_2 \le \tilde{A}_{22}$, and (3.26) and the ordering $w_3 \le w_2$ implies $S_2 \ge \tilde{S}_2$. Similarly, (3.10) implies $S_3 \le \tilde{A}_{33}$. Hence the bounds in Propositions 3.5 and 3.6, respectively, satisfy

$$\frac{\lambda_{max}(\tilde{A}_{22})}{\lambda_{min}(\tilde{S}_2)} \ge \frac{\lambda_{max}(S_2)}{\lambda_{min}(S_2)} = \kappa(S_2) \quad \text{and} \quad \frac{\lambda_{max}(A_{33})}{\lambda_{min}(S_3)} \ge \frac{\lambda_{max}(S_3)}{\lambda_{min}(S_3)} = \kappa(S_3).$$

One can see that Proposition 3.5 can be extended to the case of more than the three subdomains considered in our example, if similar conditions are assumed. In particular, we assume a stripe-type decomposition (i.e., each subdomain has a common boundary only with its previous and next neighbours in the sequence of subdomains), and the subdomains are numbered such that the weights w_i are ordered monotonically. Then the proof of Proposition 3.5 can be repeated such that the role of the 1st, 2nd and 3rd subdomains are played by the $(i-1)$th, ith and $(i+1)$th subdomains, respectively. Using the above arguments, however, the bounds obtained for $\kappa(S_i(w))$ cannot be less than $\kappa(S_i)$.

As shown in the introduction, the condition numbers $\kappa(S_i)$ deteriorate even in the preconditioned form (1.1). This shows an important motivation for the efficient preconditioning of the Schur complements. A possible improvement was given in Section 2, and in the sequel we will study other block orderings to avoid the recursive growth of the condition numbers. The next section yields estimates in terms of the constant γ in the strengthened Cauchy-Schwarz-Bunyakowski inequality.

4. Odd-even partitioning of subdomains. We assume now that we have ordered the subdomains in an odd-even fashion so that the finite element matrix takes the form

$$A = \begin{bmatrix} A_1 & 0 & A_{13} \\ 0 & A_2 & A_{23} \\ A_{31} & A_{32} & A_3 \end{bmatrix}. \tag{4.1}$$

Here A_i, $i = 1, 2$, correspond to interior node points and A_3 to edge and vertex node points. Clearly the matrices A_i themselves are block diagonal. This matrix can be factored into the form

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ A_{31} & A_{32} & S \end{bmatrix} \begin{bmatrix} I_1 & 0 & A_1^{-1}A_{13} \\ 0 & I_2 & A_2^{-1}A_{23} \\ 0 & 0 & I_3 \end{bmatrix}, \tag{4.2}$$

where S is the Schur complement matrix

$$S_3 = A_{33} - A_{31}A_1^{-1}A_{13} - A_{32}A_2^{-1}A_{23} \tag{4.3}$$

and some simpler matrix is used in a corresponding approximate block matrix factorization. Although the actions of the matrices A_i^{-1} can be computed readily separately for each subdomain, the major problem remains how to precondition the matrix S_3.

As indicated in Section 2, and shown in papers on domain decomposition methods (see, e.g., [23, 22] and also [1]), if we just use A_{33} as preconditioner then the condition number

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$\kappa(A_{33}^{-1}S_3)$ grows as $O(h/H)$ (as $h \to 0$), where h, H are the characteristic mesh sizes for the fine mesh and for the subdomains, respectively. Furthermore, as mentioned in Section 3, the condition number of S_3 itself deteriorates as the magnitude of coefficient jumps increase, which makes the construction of an efficient preconditioner to S_3 additionally difficult.

Instead, we will use a preconditioner of A that takes contributions from both interior and boundary points into account. This is similar to the second approach in Section 3.

The preconditioner B will be in additive form

$$B = B_1 + B_2. \tag{4.4}$$

The matrices B_1, B_2 are formed from the inverse matrices

$$\begin{bmatrix} B_{11} & 0 & B_{13} \\ 0 & I_2 & 0 \\ B_{31} & 0 & \left(S_3^{(1)}\right)^{-1} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & A_{13} \\ 0 & I_2 & 0 \\ A_{31} & 0 & A_3 \end{bmatrix}^{-1},$$

and

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & \left(S_3^{(2)}\right)^{-1} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & A_2 & A_{23} \\ 0 & A_{32} & A_3 \end{bmatrix}^{-1},$$

where

$$S_3^{(i)} = A_3 - A_{3i}A_i^{-1}A_{i3}, \qquad i = 1, 2,$$

and the matrices B_{ij} need not be given as we only aim at a bound on $\kappa(BA)$ in which B_{ij} will not appear. To form B_i, the sub-block identity matrices are deleted from the above, i.e.,

$$B_1 := \begin{bmatrix} B_{11} & 0 & B_{13} \\ 0 & 0 & 0 \\ B_{31} & 0 & \left(S_3^{(1)}\right)^{-1} \end{bmatrix}, \qquad B_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & \left(S_3^{(2)}\right)^{-1} \end{bmatrix}. \tag{4.5}$$

We will show that by use of perturbations of the subblocks in the position $(3, 3)$ of the inverses, we can derive a condition number κ of the preconditioned matrix which depends only on the CBS constant $\gamma = \varrho\left(A_3^{-1/2}(A_{31}A_1^{-1}A_{13} + A_{32}A_2^{-1}A_{23})A_3^{-1/2}\right)^{1/2}$, since $\kappa \leq 1/(1-\gamma^2)^{1/2}$.

Let first the preconditioner B be defined by (4.4)-(4.5). The matrix A is split as

$$A = \hat{A}_1 + \hat{A}_2 - E,$$

where

$$\hat{A}_1 = \begin{bmatrix} A_1 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & A_3 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2 & A_{23} \\ 0 & A_{32} & A_3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix}.$$

Then

$$B_1\hat{A}_1 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \qquad B_2\hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

A computation shows that

$$
\begin{aligned}
ABA &= A\left(B_1\hat{A}_1 + B_1(\hat{A}_2 - E) + B_2\hat{A}_2 + B_2(\hat{A}_1 - E)\right)\\
&= A\left(I + blockdiag(0,0,I_3) + B_1(\hat{A}_2 - E) + B_2(\hat{A}_1 - E)\right)\\
&= A + \begin{bmatrix} 0 & 0 & A_{13}\\ 0 & 0 & A_{23}\\ 0 & 0 & A_3\end{bmatrix} + \begin{bmatrix} I_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_3\end{bmatrix}\begin{bmatrix} 0 & 0 & 0\\ 0 & A_2 & A_{23}\\ 0 & A_{32} & 0\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{bmatrix}\begin{bmatrix} A_1 & 0 & A_{13}\\ 0 & 0 & 0\\ A_{31} & 0 & 0\end{bmatrix}\\
&\quad + (\hat{A}_2 - E)B_1(\hat{A}_2 - E) + (\hat{A}_1 - E)B_2(\hat{A}_1 - E)\\
&= A + \begin{bmatrix} 0 & 0 & A_{13}\\ 0 & 0 & A_{23}\\ A_{31} & A_{32} & A_3\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0\\ 0 & A_2 & A_{23}\\ 0 & A_{32} & 0\end{bmatrix}\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \left(S_3^{(1)}\right)^{-1}\end{bmatrix}\begin{bmatrix} 0 & 0 & 0\\ 0 & A_2 & A_{23}\\ 0 & A_{32} & 0\end{bmatrix}\\
&\quad + \begin{bmatrix} A_1 & 0 & A_{13}\\ 0 & 0 & 0\\ A_{31} & 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \left(S_3^{(2)}\right)^{-1}\end{bmatrix}\begin{bmatrix} A_1 & 0 & A_{13}\\ 0 & 0 & 0\\ A_{31} & 0 & 0\end{bmatrix}\\
&= A + \begin{bmatrix} A_{13}\left(S_3^{(2)}\right)^{-1}A_{31} & 0 & A_{13}\\ 0 & A_{23}\left(S_3^{(1)}\right)^{-1}A_{32} & A_{23}\\ A_{31} & A_{32} & A_3\end{bmatrix} =: A + F.
\end{aligned}
$$

Hence

$$ABA - A = F \tag{4.6}$$

and

$$A^{1/2}BA^{1/2} = I + A^{-1/2}FA^{-1/2}.$$

Let A_3 be split as $A_3 = A_3^{(1)} + A_3^{(2)}$, where $A_3^{(i)}$ $(i = 1, 2)$ arises from contributions to edge nodes from odd and even numbered subdomains, respectively. Then the matrices

$$\begin{bmatrix} A_1 & 0 & A_{13}\\ 0 & 0 & 0\\ A_{31} & 0 & A_3^{(1)}\end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 0 & 0 & 0\\ 0 & A_2 & A_{23}\\ 0 & A_{32} & A_3^{(2)}\end{bmatrix}$$

are the full contributions from odd and even numbered subdomains, respectively, so they are positive semidefinite. Hence $A_3^{(i)} - A_{3i}A_i^{-1}A_{i3}$ $(i = 1, 2)$ are also positive semidefinite, thus

$$S_3^{(1)} = A_3^{(1)} + A_3^{(2)} - A_{31}A_1^{-1}A_{13} \geq A_3^{(2)} \quad\text{or}\quad \left(A_3^{(2)}\right)^{-1} \geq \left(S_3^{(1)}\right)^{-1}$$

and similarly, $\left(A_3^{(1)}\right)^{-1} \geq \left(S_3^{(2)}\right)^{-1}$. Hence

$$A_{13}\left(S_3^{(2)}\right)^{-1}A_{31} \leq A_{13}\left(A_3^{(1)}\right)^{-1}A_{31} \leq A_1, \quad A_{23}\left(S_3^{(1)}\right)^{-1}A_{32} \leq A_{23}\left(A_3^{(2)}\right)^{-1}A_{32} \leq A_2\,,$$

and by (4.6),

$$ABA - A \leq \begin{bmatrix} A_1 & 0 & A_{13}\\ 0 & A_2 & A_{23}\\ A_{31} & A_{32} & A_3\end{bmatrix} = A.$$

Therefore

$$A^{1/2}BA^{1/2} \leq 2I \quad \text{and} \quad \lambda_{max}(BA) \leq 2. \tag{4.7}$$

To derive a lower bound, we will use perturbations. Let then B be defined as above, and let the preconditioner $\tilde{B}$ to A be defined as

$$\tilde{B} := B + \Delta, \qquad \text{where} \quad \Delta := \delta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} \quad \text{for some } \delta \geq 0. \tag{4.8}$$

The intention is to keep δ sufficiently small so as not to increase the upper bound too much. We have $A\tilde{B}A - A = ABA - A + A\Delta A$, and we wish to find a positive number ξ sufficiently large to make $(\xi+1)A\tilde{B}A \geq A$.

Here

$$(\xi+1)A\tilde{B}A - A = \xi A + (\xi+1)(ABA - A) + (\xi+1)A\Delta A.$$

Further,

$$A\Delta A = \delta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{4.9}$$

and, using (4.6), we have

$$\begin{aligned} (\xi+1)(A\tilde{B}A - A) = \xi &\begin{bmatrix} A_1 & 0 & \alpha A_{13} \\ 0 & A_2 & \alpha A_{23} \\ \alpha A_{31} & \alpha A_{32} & \alpha^2 D_3 \end{bmatrix} \\ &+(\xi+1)\begin{bmatrix} A_{13}\,(S_3^{(2)})^{-1} A_{31} & 0 & \beta A_{13} \\ 0 & A_{23}\,(S_3^{(1)})^{-1} A_{32} & \beta A_{23} \\ \beta A_{31} & \beta A_{32} & \beta^2\,(S_3^{(1)} + S_3^{(2)}) \end{bmatrix} \\ &+\Big((2\xi+1) + (\xi+1)\delta - 2\beta^2(\xi+1)\Big)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix}, \end{aligned} \tag{4.10}$$

where

$$D_3 = A_{31}A_1^{-1}A_{13} + A_{32}A_2^{-1}A_{23}, \qquad S_3^{(1)} + S_3^{(2)} = 2A_3 - D_3,$$

and ξ, α, β are positive numbers satisfying

$$\begin{aligned} &\xi\alpha + (\xi+1)\beta = 2\xi + 1 && \text{(equating the off-diagonal matrices),} \\ &\xi\alpha^2 = (\xi+1)\beta^2 && \text{(equating the } D_3\text{-terms),} \\ &2(\xi+1)\beta^2 = (2\xi+1) + (\xi+1)\delta && \text{(equating the } A_3\text{-terms).} \end{aligned} \tag{4.11}$$

Now we can prove

THEOREM 4.1. *Assume that A_1 in (4.1) is positive definite, and let $\tilde{B}$ be defined in (4.8). Then*

$$\lambda_{min}(\tilde{B}A) \geq \frac{1}{\xi+1} \quad \textit{and} \quad \lambda_{max}(\tilde{B}A) \leq 2 + \frac{\delta}{1-\gamma^2},$$

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where $\xi = \xi(\delta)$ satisfies the equation (4.11) and

$$\gamma = \varrho\,(A_3^{-1/2}(A_{31}A_1^{-1}A_{13} + A_{32}A_2^{-1}A_{23})A_3^{-1/2})^{1/2}.$$

As $\delta \to 0$, $\xi \to \infty$, the condition number is asymptotically bounded by

$$\kappa(\tilde{B}A) \lessapprox \frac{1}{2\sqrt{2}}\Big(2\delta^{-1/2} + \frac{\delta^{1/2}}{1-\gamma^2}\Big),$$

and

$$\min_{\delta>0} \kappa(\tilde{B}A) \lessapprox \frac{1}{\sqrt{1-\gamma^2}}.$$

Proof. We have

$$\begin{bmatrix} A_1 & 0 & \alpha A_{13} \\ 0 & A_2 & \alpha A_{23} \\ \alpha A_{31} & \alpha A_{32} & \alpha^2 D_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ \alpha A_{31}A_1^{-1} & \alpha A_{32}A_1^{-1} & I_3 \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 & 0 & \alpha A_1^{-1}A_{31} \\ 0 & I_2 & \alpha A_1^{-1}A_{32} \\ 0 & 0 & I_3 \end{bmatrix}$$

and

$$\begin{bmatrix} A_{13}\,(S_3^{(2)})^{-1}A_{31} & 0 & \beta A_{13} \\ 0 & A_{23}\,(S_3^{(1)})^{-1}A_{32} & \beta A_{23} \\ \beta A_{31} & \beta A_{32} & \beta^2\,(S_3^{(1)} + S_3^{(2)}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ 0 & 0 & \beta\,S_3^{(2)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (S_3^{(2)})^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & 0 & \beta\,S_3^{(2)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & \beta\,S_3^{(1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (S_3^{(1)})^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{32} & \beta\,S_3^{(1)} \end{bmatrix}.$$

It follows that the first two terms in (4.10) are positive semidefinite. Since by (4.11) the last term in (4.10) is zero, it follows that

$$(\xi + 1)A\tilde{B}A \geq A,$$

whence $\lambda_{min}(\tilde{B}A) \geq \dfrac{1}{\xi+1}$. Further, using (4.7) and (4.9),

$$\lambda_{max}(\tilde{B}A) = \sup_{x\neq 0} \frac{x^T A\tilde{B}Ax}{x^T Ax} \leq 2 + \sup_{x\neq 0} \frac{x^T A\Delta Ax}{x^T Ax} = 2 + \delta \sup_{x\neq 0} \frac{x_3^T A_3 x_3}{x^T Ax}$$

$$= 2 + \delta \sup_{x\neq 0} \frac{x_3^T A_3 x_3}{x_3^T S_3 x_3} = 2 + \frac{\delta}{1-\gamma^2},$$

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where S_3 is from (4.3).

A computation from (4.11) shows that

$$\alpha = \sqrt{\frac{\xi+1}{\xi}}\beta, \qquad \sqrt{\xi+1}(\sqrt{\xi+1}+\sqrt{\xi})\beta = 2\xi+1,$$

and

$$\frac{1}{\xi+1} = \frac{1+\zeta+2\sqrt{\zeta}}{\sqrt{\zeta+1}}\sqrt{\delta} \le 2\sqrt{2\delta},$$

where $\zeta = \dfrac{\xi}{\xi+1}$. Hence $\zeta < 1$. If $\delta \to 0$, then $\xi \to \infty$, $\zeta \to 1$, and hence

$$\lambda_{min}(\tilde{B}A) \ge \frac{1}{\xi+1} \approx 2\sqrt{2\delta}.$$

For the condition number we have

$$\kappa(\tilde{B}A) \lessapprox \frac{2+\delta/(1-\gamma^2)}{2\sqrt{2\delta}}$$

which is minimized for $\delta = 2(1-\gamma^2)$. Hence

$$\min_{\delta>0} \kappa(\tilde{B}A) \lessapprox \frac{1}{\sqrt{1-\gamma^2}}. \qquad \square$$

As follows from Section 2, for partitioning in subdomains, $\gamma^2 = 1 - O(h/H)$, where h, H are the characteristic mesh sizes for the finest elements and for the macroelements, respectively. Hence it follows from Theorem 4.1 for the condition number that

$$\kappa(\tilde{B}A) = O\Big((h/H)^{-1/2}\Big).$$

Therefore, the number of conjugate gradient iterations to solve a system with matrix A, using the preconditioner $\tilde{B} = \tilde{B}(\delta)$ with $\delta = \dfrac{2}{1-\gamma^2} = O(h/H)$, increases at most as $O\Big((h/H)^{-1/4}\Big)$, which is fairly minor. For instance, $(h/H)^{-1/4} = 2$, respectively 4, if $H = 16h$ or $H = 256h$.

We remark that for convenience of the derivation of the condition number, we have formulated the matrix A based on an odd-even ordering. However, since the matrix B is given in additive form, we can actually implement the actions of the local element inverses in any order, or even in parallel. The method can be further improved by use of a perturbation matrix in the form $\Delta = \delta\, blockdiag(0, 0, A_3^{-1}S_3A_3^{-1})$ instead of (4.8). It turns out that in this case the condition number does not depend on γ but only on δ, which can be chosen independently of the coefficient in the differential operator. This shows independence of both the mesh parameter and coefficient jumps, but will not be discussed further in this paper.

The above method can be further extended by use of a splitting of node points in a coarse mesh set and a remaining fine mesh set. For the corresponding two-by-two block matrix the above method can be applied for the pivot block matrix, and the arising Schur complement matrix in the block preconditioner can likewise be preconditioned elementwise. This will be discussed in Section 6; see further analysis in [10, 12], and for related results [21]. In the following section elementwise preconditioners are also analyzed by more analytical means.

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5. Some model analysis on the continuous level. A continuous analogue of the method of Section 3 is presented now on some model problems, including the introduction of a certain modified Poincaré–Steklov operator for the interfaces. This study on the continuous level can help the understanding of the properties of the studied factorization approach.

5.1. Preliminaries: the Poincaré–Steklov operator. As pointed out in Section 3, the analysis of standard domain decomposition methods relies strongly on the Poincaré–Steklov operator; see, e.g., [23, 22]. In this subsection we give a brief description, following [22].

Let us consider a boundary value problem

$$\left\{ \begin{array}{l} -\Delta u = f, \quad \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{array} \right. \tag{5.1}$$

where Ω is a bounded domain with Lipschitz boundary and $f \in L^2(\Omega)$. The domain Ω is decomposed in two nonoverlapping domains Ω_1 and Ω_2, whose common boundary is denoted by Γ, further, we let $\Gamma_1 := \partial\Omega_1 \setminus \Gamma$ and $\Gamma_2 := \partial\Omega_2 \setminus \Gamma$.

The Poincaré–Steklov operator is then defined in the following way. Let us choose an arbitrary function $\gamma \in H_{00}^{1/2}(\Gamma)$. (For the definition of $H_{00}^{1/2}(\Gamma)$ and other related Sobolev spaces, see also [23].) Let $H_1\gamma$ and $H_2\gamma$ denote the harmonic extensions of γ in Ω_1 and Ω_2, respectively, with zero boundary condition on $\partial\Omega$, i.e., $H_i\gamma$, $i = 1, 2$, is the solution of the problem

$$\left\{ \begin{array}{l} -\Delta\, H_i\gamma = 0, \quad \text{in } \Omega_i, \\ H_i\gamma_{\,|\Gamma_i} = 0, \\ H_i\gamma_{\,|\Gamma} = \gamma. \end{array} \right. \tag{5.2}$$

Then the Poincaré–Steklov operator is $R : H_{00}^{1/2}(\Gamma) \to H_{00}^{-1/2}(\Gamma)$, which assigns to γ the jump of the normal derivatives of its harmonic extensions on Γ, i.e.,

$$R\gamma := \frac{\partial}{\partial n} H_1\gamma + \frac{\partial}{\partial n} H_2\gamma, \qquad \text{on } \Gamma. \tag{5.3}$$

The plus sign represents the jump with the convention that the outward normal vector n of Ω_1 is opposite to n of Ω_2 on Γ, which will be understood throughout this paper. That is, for a smooth function on Ω, the two normal derivatives are the opposite of each other and hence the jump on Γ equals zero.

REMARK 5.1. Problem (5.1) can then be reduced to equation

$$R\gamma = \psi \tag{5.4}$$

with ψ defined as follows. Let $T_i f$, $i = 1, 2$, respectively, denote the solutions of the problems

$$\left\{ \begin{array}{l} -\Delta\, T_i f = f, \quad \text{in } \Omega_i, \\ T_i f_{\,|\partial\Omega_i} = 0, \end{array} \right. \tag{5.5}$$

and let

$$\psi := -\frac{\partial}{\partial n} T_2 f - \frac{\partial}{\partial n} T_1 f, \qquad \text{on } \Gamma,$$

which represents the negative jump of the corresponding normal derivatives. Then $u := H_i\gamma + T_i f$ on Ω_i ($i = 1, 2$) satisfies $-\Delta\, u = f$ on both Ω_1 and Ω_2 and is continuous on

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Ω. Hence u solves (5.1) if and only if its normal derivative has zero jump on Γ, which is equivalent to (5.4).

REMARK 5.2. Green's formula implies that the bilinear form of the Poincaré–Steklov operator R is

$$\langle R\gamma,\mu\rangle = \int_{\Omega_1} \nabla H_1\gamma\cdot\nabla H_1\mu + \int_{\Omega_2} \nabla H_2\gamma\cdot\nabla H_2\mu, \qquad \forall\gamma,\mu\in H_{00}^{1/2}(\Gamma), \tag{5.6}$$

whence R is a symmetric and strictly positive operator.

On the discrete level, let us now consider a FEM discretization of problem (5.1) and let us decompose the stiffness matrix as

$$A = \begin{bmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix}, \tag{5.7}$$

corresponding to the node points in Ω_1, in Ω_2 and on Γ, respectively. The linear system can be reduced to the Schur complement

$$\Sigma := A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}, \tag{5.8}$$

i.e., Σ is the Schur complement for Γ with respect to both Ω_1 and Ω_2. Then, as pointed out in [22], Σ is the discrete analogue of the Poincaré–Steklov operator (5.3). Essentially, the term $A_{\Gamma\Gamma}$ is responsible for the boundary values of the considered function and the two other terms represent the procedures involving the two harmonic extensions.

REMARK 5.3. The generalization of the above notions to the case of more (say, k) subdomains is straightforward. Then the Poincaré–Steklov operator involves harmonic extensions from the union of interfaces to all subdomains, and its bilinear formulation will contain a sum of k terms, e.g., for $k = 3$ the form (5.6) is replaced by

$$\langle R\gamma,\mu\rangle = \int_{\Omega_1} \nabla H_1\gamma\cdot\nabla H_1\mu + \int_{\Omega_2} \nabla H_2\gamma\cdot\nabla H_2\mu + \int_{\Omega_3} \nabla H_3\gamma\cdot\nabla H_3\mu\,. \tag{5.9}$$

Similarly, the stiffness matrix (5.7) and the corresponding Schur complement (5.8) will include k interior blocks A_{ii}, e.g., for the above example $k = 3$, we have

$$\Sigma := A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma} - A_{\Gamma 3}A_{33}^{-1}A_{3\Gamma} \tag{5.10}$$

as in (3.5).

5.2. The modified Poincaré–Steklov operator. Let us consider again a FEM discretization of problem (5.1). We decompose the domain Ω in subdomains $\Omega_1,\ldots,\Omega_m$, such that, in addition to a corresponding portion of the original boundary $\partial\Omega$, each Ω_i has a common boundary only with its neighbours Ω_{i-1} and Ω_{i+1}. Denoting here these common boundaries by $\Gamma_{i-1,i}$ and $\Gamma_{i,i+1}$, respectively, we decompose the stiffness matrix as in (2.1), corresponding to the subdomains $\Omega_1,\ldots,\Omega_m$, such that the node points on $\Gamma_{i,i+1}$ are taken into account in A_{ii} (i.e., together with Ω_i). Our goal is to study the factorization (2.2). Since, in contrast to the idea of (5.7), the boundary node points are not considered here separately, the Schur complements in (2.2) are understood recursively as complements for Ω_i with respect to Ω_{i-1}. This is an important difference as compared to (5.8), and therefore the continuous analogues of the Schur complements in (2.2) will also be appropriate modifications of the Poincaré–Steklov operator (5.3). In fact, the proper operator takes only into account the previous subdomain Ω_{i-1}.

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First, for simplicity, let us consider the case of two subdomains Ω_1 and Ω_2, where one can follow more clearly how the operator in subsection 5.1 is modified. Similarly as therein, the common boundary of Ω_1 and Ω_2 is denoted by Γ, further, we let $\Gamma_1 := \partial\Omega_1 \setminus \Gamma$ and $\Gamma_2 := \partial\Omega_2 \setminus \Gamma$. We wish to define the continuous analogue of the Schur complement $S_2 := A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Let us take a function u_2 on Ω_2, such that $u_{2\,|\Gamma_2} = 0$. Applying the operator $-\Delta_{|\Omega_2}$ to u_2 (which corrresponds to the term A_{22} in S_2), we want it to equal f. Let us further consider the restriction $u_{2\,|\Gamma}$, and calculate its harmonic extension to Ω_1, i.e., let H_1u_2 be the solution of the problem

$$\left\{ \begin{array}{ll} -\Delta\, H_1 u_2 = 0, & \text{in } \Omega_1, \\ H_1 u_{2\,|\Gamma_1} = 0, & \\ H_1 u_{2\,|\Gamma} = u_2, & \end{array} \right. \tag{5.11}$$

which solves the analogue of (5.2) only on Ω_1. Accordingly, the modified Poincaré–Steklov operator P assigns to u_2 the jump of the normal derivative of its harmonic extension and of itself, i.e.,

$$Pu_2 := \frac{\partial}{\partial n} H_1 u_2 + \frac{\partial}{\partial n} u_2, \qquad \text{on } \Gamma. \tag{5.12}$$

REMARK 5.4. Similarly as in Remark 5.1, problem (5.1) can now be reduced to the equation

$$Pu_2 = \chi, \tag{5.13}$$

where $\chi := -\frac{\partial}{\partial n} T_1 f$ with $T_1 f$ defined in (5.5). Letting $u = u_1 := H_1u_2 + T_1 f$ on Ω_1 and $u = u_2$ on Ω_2, it is readily seen that u solves (5.1) if and only if $Lu_2 = f$ in Ω_2 and (5.13) holds on Γ.

REMARK 5.5. The analogue of Remark 5.2 holds if, according to our setting, we handle the operators $-\Delta_{|\Omega_2}$ and P together. Using Green's formula, the pair $\tilde{P}$ of these operators satisfies

$$\left\langle \tilde{P}\,(u_2,\, u_{2\,|\Gamma}),\, (\varphi,\, \varphi_{|\Gamma}) \right\rangle \equiv \left\langle \begin{pmatrix} -\Delta \\ P \end{pmatrix} (u_2,\, u_{2\,|\Gamma}),\, (\varphi,\, \varphi_{|\Gamma}) \right\rangle = \int_{\Omega_2} (-\Delta u_2)\varphi + \int_\Gamma (Pu_2)\varphi$$

$$= \int_{\Omega_1} \nabla H_1 u_2 \cdot \nabla H_1 \varphi + \int_{\Omega_2} \nabla u_2 \cdot \nabla \varphi, \tag{5.14}$$

for all $\varphi \in H^1_D(\Omega_2) := \{\varphi \in H^1(\Omega_2) : \ \varphi_{|\Gamma_2} = 0\}$, whence it is a symmetric and strictly positive operator.

REMARK 5.6. For more subdomains, one can define P_i in just an analogous way. Namely, for simplicity, let Γ_{i-1} denote the common boundary of Ω_{i-1} and Ω_i. Let u_i be defined on Ω_i, such that $u_{i\,|\partial\Omega_i \setminus \Gamma_{i-1}} = 0$. We consider $u_{i\,|\Gamma_{i-1}} = 0$ and solve the Dirichlet problem on $\Omega_1 \cup \cdots \cup \Omega_{i-1}$ with this boundary condition (which can be reduced to previous subproblems in a recursive way, just as is the Schur complement reduced to previous Schur complements), and finally calculate the jump of the corresponding normal derivatives on Γ. Here the bilinear form that replaces (5.14) will thus include a term on $\Omega_1 \cup \cdots \cup \Omega_{i-1}$ and a term on Ω_i. For instance, in the case of three subdomains, we have

$$\left\langle \tilde{P}_3\,(u_3,\, u_{3\,|\Gamma}),\, (\varphi,\, \varphi_{|\Gamma}) \right\rangle = \int_{\Omega_1 \cup \Omega_2} \nabla H_{12} u_3 \cdot \nabla H_{12}\, \varphi + \int_{\Omega_3} \nabla u_3 \cdot \nabla \varphi, \tag{5.15}$$

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for all $\varphi \in H^1_D(\Omega_3) := \{\varphi \in H^1(\Omega_3) : \varphi_{|\partial\Omega_3\setminus\Gamma_2} = 0\}$, where $H_{12}u_3$ denotes the harmonic extension of $u_{3\,|\Gamma_2}$ to $\Omega_1 \cup \Omega_2$.

REMARK 5.7. For problems with jumps in the diffusion coefficients, the conditioning properties observed in Section 3 are in accordance with their analogues on the continuous level. This will be outlined here. Namely, we have observed in Section 3 that the condition numbers of the Schur complements are sensitive to jumps in the first approach but not in the second approach. Accordingly, one can indicate for the same example that the standard Poincaré–Steklov operator is sensitive to the jumps whereas the modified Poincaré–Steklov operator is not.

Let us therefore consider the model problem of Section 3. The domain Ω is decomposed in three subdomains Ω_1, Ω_2 and Ω_3, such that there are common boundaries $\Gamma_1 := \overline{\Omega}_1 \cap \overline{\Omega}_2$ and $\Gamma_2 := \overline{\Omega}_2 \cap \overline{\Omega}_3$, but Ω_1 and Ω_3 have no common boundary. We consider an elliptic problem, formally as $-\mathrm{div}\,(w\,\nabla u) = f$ with $u_{|\partial\Omega} = 0$, with weak form (3.13), where w is a weight function on Ω, such that $w_{|\Omega_i} \equiv w_i$ $(i = 1, 2, 3)$. We assume $w_1 \ge w_2 \ge w_3$ and, varying the coefficients, we are interested in the case $w_1/w_2 \to \infty$.

The standard Poincaré–Steklov operator can be extended directly to such piecewise constant coefficient problems, such that one considers weighted normal derivatives on the interfaces with weights w_i. Considering the bilinear form for our model problem with three subdomains, the form (5.9) is replaced by

(5.16)
$$\langle R(w)\gamma, \mu\rangle = w_1 \int_{\Omega_1} \nabla H_1\gamma \cdot \nabla H_1\mu + w_2 \int_{\Omega_2} \nabla H_2\gamma \cdot \nabla H_2\mu + w_3 \int_{\Omega_3} \nabla H_3\gamma \cdot \nabla H_3\mu\,.$$

Factoring out w_2, we see that $R(w)$ is the constant multiple of an operator, where the first term is proportional to w_1/w_2 and the other two terms are bounded as $w_1/w_2 \to \infty$, i.e., $R(w)$ behaves similarly as $\Sigma(w)$ in Corollary 3.2.

The modified Poincaré–Steklov operator can be extended similarly to piecewise constant coefficient problems, using the same weighted normal derivatives as above. The bilinear form for our model problem with three subdomains is the proper modification of (5.15):

$$\left\langle \tilde{P}_3(w)\,(u_3,\, u_{3\,|\Gamma}),\, (\varphi,\, \varphi_{|\Gamma}) \right\rangle = \int_{\Omega_1\cup\Omega_2} w\,\nabla H_{12}u_3 \cdot \nabla H_{12}\,\varphi + w_3 \int_{\Omega_3} \nabla u_3 \cdot \nabla\varphi, \tag{5.17}$$

for all $\varphi \in H^1_D(\Omega_3) := \{\varphi \in H^1(\Omega_3) : \varphi_{|\partial\Omega_3\setminus\Gamma_2} = 0\}$, where $H_{12}u_3$ denotes the w-harmonic extension of $u_{3\,|\Gamma_2}$ to $\Omega_1 \cup \Omega_2$, that is, $H_{12}u_3 = v$ if and only if $v_{|\Gamma_2} = u_3$ and $v_{|\partial(\Omega_1\cup\Omega_2)\setminus\Gamma_2} = 0$, and further

$$\int_{\Omega_1\cup\Omega_2} w\,\nabla v\cdot\nabla\phi \equiv w_1 \int_{\Omega_1} \nabla v\cdot\nabla\phi + w_2 \int_{\Omega_2} \nabla v\cdot\nabla\phi = 0 \qquad \forall\phi \in H^1_0(\Omega_1\cup\Omega_2). \tag{5.18}$$

Let us now consider an arbitrary test function $\varphi \in H^1_D(\Omega_3)$ as required for (5.17), and denote by $\tilde{\varphi}$ an extension of φ to Ω, such that $\tilde{\varphi}_{|\Omega_1} \equiv 0$ and $\tilde{\varphi}_{|\partial\Omega} \equiv 0$. Then $\tilde{\varphi}$ coincides with the w-harmonic extension $H_{12}\varphi$ on Γ_2, and also on $\partial(\Omega_1 \cup \Omega_2) \setminus \Gamma_2$ since both vanish on the latter. Hence $H_{12}\varphi - \tilde{\varphi}$ equals zero on the entire $\partial(\Omega_1\cup\Omega_2)$, i.e., $H_{12}\varphi - \tilde{\varphi} \in H^1_0(\Omega_1\cup\Omega_2)$. Setting $\phi := H_{12}\varphi - \tilde{\varphi}$ in (5.18) and using $\tilde{\varphi}_{|\Omega_1} \equiv 0$, we obtain

$$\int_{\Omega_1\cup\Omega_2} w\,\nabla v \cdot \nabla H_{12}\varphi = \int_{\Omega_1\cup\Omega_2} w\,\nabla v \cdot \nabla\tilde{\varphi} = w_2 \int_{\Omega_2} \nabla v \cdot \nabla\tilde{\varphi}\,.$$

Since by definition $H_{12}u_3 = v$, we have just obtained an equality for the first term of (5.17). Substituting this into the whole expression in (5.17), we obtain a form for $\tilde{P}_3(w)$ that contains integrals only on Ω_2 and Ω_3 with respective weights w_2 and w_3:

$$\left\langle \tilde{P}_3(w)\,(u_3,\, u_{3\,|\Gamma}),\, (\varphi,\, \varphi_{|\Gamma})\right\rangle = w_2 \int_{\Omega_2} \nabla v \cdot \nabla \tilde{\varphi} + w_3 \int_{\Omega_3} \nabla u_3 \cdot \nabla \varphi\,. \tag{5.19}$$

To sum up, the behaviour of the Schur complements under jumps in Section 3 follows that of their continuous analogues.

5.3. Approximate modified Poincaré–Steklov operator on a model problem. In this subsection we consider a continuous analogue of the procedure (2.5)-(2.6), and show on a model problem that it can be carried out in a similar way as on the discrete level. This gives an alternate illustration of the fact that the condition numbers in Theorem 2.1 are mesh independent.

Let us consider the 3D model problem

$$\begin{cases} -\Delta u = f, & \text{in } B, \\ u_{|\partial B} = 0, \end{cases} \tag{5.20}$$

where $B \subset \mathbb{R}^3$ is the unit ball. Let us fix a positive integer k and numbers

$$0 = R_0 < R_1 < \cdots < R_{k-1} < R_k = 1.$$

Using notation $r := |x|$ for the Euclidean norm of vectors $x \in \mathbb{R}^3$, we define annular subdomains

$$\Omega_j := \{x \in B : \; R_{k-j} < r < R_{k-j+1}\}, \qquad i = 1, \ldots, k. \tag{5.21}$$

First, for simplicity, let $k = 2$ and $R_1 = 1/2$. Then (2.6) becomes $D_2\mathbf{e} = A_{2,1}A_{11}^{-1}A_{1,2}\mathbf{e}$ for the constant vector $\mathbf{e} = (1, \ldots, 1)$. Its continuous analogue, with the notation of subsection 5.2, is to find an operator $\hat{D}_2$, such that

$$\hat{D}_2 e = Pe, \qquad \text{on } \Gamma, \tag{5.22}$$

for the constant function $e \equiv 1$. Here $\Gamma := \{x \in \mathbb{R}^3 \; : \; r = 1/2\}$, and P is defined in (5.12) and the procedure before that. We have

$$\Omega_1 = \{x \in B : \; 1/2 < r < 1\} \quad \text{and} \quad \Omega_2 = \{x \in B : \; 0 < r < 1/2\}. \tag{5.23}$$

Further, $\Gamma_1 := \partial\Omega_1 \setminus \Gamma = \partial B$ and $\Gamma_2 := \partial\Omega_2 \setminus \Gamma = \emptyset$. Then Pe can be calculated explicitly. First, the harmonic extension of e to Ω_1 is $H_1 e =: v$, where v is the solution of

$$\begin{cases} -\Delta\, v = 0, & \text{in } \Omega_1, \\ v_{|\partial B} = 0, \\ v_{|\Gamma} = 1\,. \end{cases} \tag{5.24}$$

Here we use the form of the Laplace operator in 3D spherical coordinates, which reduces to $\Delta v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial v}{\partial r})$ for radially symmetric functions. Then an elementary calculation yields

$$v(r) = \frac{1}{r} - 1.$$

Hence by (5.12) and using that now $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ on Γ, we obtain $Pe = -\big(\frac{\partial v}{\partial r} + \frac{\partial e}{\partial r}\big)_{|r=1/2} = 4$. That is, Pe is constant on Γ, i.e., we can write $Pe = 4e$ on Γ, which means that the operator required in (5.22) can be defined as

$$\hat{D}_2 := 4I, \tag{5.25}$$

where I is the identity operator on Γ.

Our goal now is to verify a continuous analogue of condition (2.7). According to the above, the operator $4I$ corresponds to D_2, further, as seen before, the analogue of A_{22} is the operator $-\Delta$, such that homogeneous Dirichlet boundary conditions are considered on $\partial\Omega_2 = \partial B$. Hence the required analogue of (2.7) reads as

$$4I \leq -\varrho\Delta, \quad \text{for some} \quad \varrho < 1/2. \tag{5.26}$$

Denoting by λ_1 the smallest eigenvalue of $-\Delta$ with the given boundary conditions, and taking into account the condition $\varrho < 1/2$, inequality (5.26) is equivalent to $8 < \lambda_1$. Here the eigenfunctions of $-\Delta$ are the restrictions to Ω_2 of the eigenfunctions on B with homogeneous Dirichlet boundary conditions on ∂B. The first eigenfunction is the first three-dimensional Bessel function $w(r) := \dfrac{\sin \pi r}{\pi r}$, with eigenvalue $\lambda_1 = \pi^2 > 8$. Therefore (5.26) is satisfied.

Now let us consider more subdomains. Here by (5.21),

$$\Omega_k := \{x \in B : \; 0 < r < R_1\}. \tag{5.27}$$

In order to determine the operator P_k, problem (5.24) has now to be solved with Γ replaced by $\Gamma_{k-1} = \{x \in \mathbb{R}^3 \; : \; r = R_1\}$. The solution is

$$v(r) = \frac{\frac{1}{r} - 1}{\frac{1}{R_1} - 1}.$$

Hence the constant 4 in (5.25) is replaced by $\frac{1}{R_1(1-R_1)}$, and accordingly, the above property $8 < \pi^2$ is replaced by condition

$$2 < \pi^2 R_1(1 - R_1). \tag{5.28}$$

If this holds then the operator $\hat{D}_k := \frac{1}{R_1(1-R_1)} I$ satisfies the required analogue of (2.7), i.e., $\hat{D}_k \leq -\varrho\Delta$ for some $\varrho < 1/2$. Analogous calculations can be carried out to find $\hat{D}_1, \ldots, \hat{D}_{k-1}$.

Inequality (5.28) is satisfied if, up to four digits, $0.2824 < R_1 < 0.7176$. Concerning the case of several subdomains, one may define for technical convenience $R_j := (\frac{j}{k})^{1/3}$ in (5.21) to have equal volume of the subdomains. Then the condition $0.2824 < R_1 = (\frac{1}{k})^{1/3}$ is satisfied up to $k = 44$, i.e., (5.28) is satisfied for any reasonable number of subdomains.

6. Element by element preconditioners for matrices partitioned in 2×2 block form.

As the method described in Section 3 uses a recursive computation, its parallelism is restricted; on the other hand, the method in Section 4 is parallelizable. Now a highly parallelizable method to construct preconditioners for the Schur complement matrix is presented, which has also shown nice results in numerical tests [10, 12]. First the method is described briefly, then its robustness with respect to coefficient jumps is shown.

6.1. Construction of the method.

Let us consider an elliptic problem with piecewise constant coefficients. We start with a coarse mesh of triangles (tetrahedra) or rectangles (cubes), which has been constructed such that all coefficient jumps occur across element edges only. Each of these macroelements is then subdivided in a number (say m^2) of minielements. The corresponding global matrix is then partitioned in 2×2 block form

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}, \tag{6.1}$$

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where A_{cc} corresponds to the coarse (macroelement) vertex nodes. To form a preconditioner B to A, we will be guided by the block matrix factorization of A,

$$A = \begin{bmatrix} A_{ff} & 0 \\ A_{cf} & S_c(A) \end{bmatrix} \begin{bmatrix} I_f & A_{ff}^{-1} A_{fc} \\ 0 & I_c \end{bmatrix}, \tag{6.2}$$

where $S_c(A) = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc}$ is the corresponding Schur complement matrix.

In general, $S_c(A)$ is a full matrix and A_{ff} has a very large size. Therefore, to form B, we replace A_{ff} and $S_c(A)$ by some approximations. These will be based on macroelement by element constructed matrices.

For the global assembled matrix A_{ff}, we first take the restrictions $A_{ff}^{(E)}$ to each macroelement, form their exact inverses ${A_{ff}^{(E)}}^{-1}$, and assemble them to a global matrix denoted by B_{ff}, which will replace A_{ff}^{-1} in (6.2). Similarly, each element version $S_2^{(E)}$ of $S_c(A)$ is computed exactly from the exact form of the corresponding element matrix

$$A^{(E)} = \begin{bmatrix} \tilde{A}_{ff}^{(E)} & \tilde{A}_{fc}^{(E)} \\ \tilde{A}_{cf}^{(E)} & \tilde{A}_{cc}^{(E)} \end{bmatrix}, \tag{6.3}$$

i.e., $S_2^{(E)} = \tilde{A}_{cc}^{(E)} - \tilde{A}_{cf}^{(E)} \widetilde{A_{ff}^{(E)}}^{-1} \tilde{A}_{fc}^{(E)}$. Then $S_c(A)$ in (6.2) is replaced by the assembly of $S_2^{(E)}$, denoted by S, and the preconditioner B to A takes the form

$$B = \begin{bmatrix} B_{ff}^{-1} & 0 \\ A_{cf} & S \end{bmatrix} \begin{bmatrix} I_f & B_{ff} A_{fc} \\ 0 & I_c \end{bmatrix}. \tag{6.4}$$

Note that the actions of B_{ff} can take place fully in parallel, from the local actions of $A_{ff}^{(E)}$.

The above method can be extended to a multilevel version, but in this paper we only study the two-level version. Our goal is to show that the condition number of $B^{-1}A$ is bounded independently of coefficient jumps in the given elliptic operator.

We will use that

$$B^{-1}A = \begin{bmatrix} I_f & -B_{ff}A_{fc} \\ 0 & I_c \end{bmatrix} \begin{bmatrix} B_{ff}A_{ff} & 0 \\ S^{-1}A_{cf}(I_f - B_{ff}A_{ff}) & S^{-1}S_c(A) \end{bmatrix} \begin{bmatrix} I_f & A_{ff}^{-1}A_{fc} \\ 0 & I_c \end{bmatrix}. \tag{6.5}$$

6.2. Independence of coefficient jumps in a model problem. For simplicity, we follow the model problem in Section 3, and study the case of three macroelements E_1, E_2 and E_3. Accordingly, we have common boundaries $\Gamma_1 := \overline{E}_1 \cap \overline{E}_2$ and $\Gamma_2 := \overline{E}_2 \cap \overline{E}_3$, but E_1 and E_3 have no common boundary. These macroelements are defined to match the coefficient of problem (3.13), i.e.,

$$w_{|E_i} \equiv w_i, \qquad i = 1, 2, 3.$$

We assume as in Section 3 that the relations (3.14)-(3.16) hold. Our goal is to show that the condition number $\kappa(B^{-1}A)$ remains bounded as the ratio w_1/w_2 grows unboundedly.

The stiffness matrix then has a form as in (3.17), where the five rows/columns now correspond to the nodes in E_1, E_2 and E_3, on Γ_1 and on Γ_2:

$$A = \begin{bmatrix} w_1 A_{11} & 0 & 0 & w_1 A_{1,\Gamma_1} & 0 \\ 0 & w_2 A_{22} & 0 & w_2 A_{2,\Gamma_1} & w_2 A_{2,\Gamma_2} \\ 0 & 0 & w_3 A_{33} & 0 & w_3 A_{3,\Gamma_2} \\ w_1 A_{\Gamma_1,1} & w_2 A_{\Gamma_1,2} & 0 & \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & w_2 A_{\Gamma_2,2} & w_3 A_{\Gamma_2,3} & 0 & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} \end{bmatrix}. \tag{6.6}$$

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With the notation of (6.1), we have

$$A_{ff} = \begin{bmatrix} w_1 A_{11} & 0 & 0 \\ 0 & w_2 A_{22} & 0 \\ 0 & 0 & w_3 A_{33} \end{bmatrix}, \qquad A_{fc} = \begin{bmatrix} w_1 A_{1,\Gamma_1} & 0 \\ w_2 A_{2,\Gamma_1} & w_2 A_{2,\Gamma_2} \\ 0 & w_3 A_{3,\Gamma_2} \end{bmatrix},$$

$$A_{cf} = \begin{bmatrix} w_1 A_{\Gamma_1,1} & w_2 A_{\Gamma_1,2} & 0 \\ 0 & w_2 A_{\Gamma_2,2} & w_3 A_{\Gamma_2,3} \end{bmatrix}, \qquad A_{cc} = \begin{bmatrix} \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} \end{bmatrix}.$$

From (3.18), the Schur complement is

$$S_c(A) = W A_{\Gamma\Gamma} - w_1 A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - w_2 A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} - w_3 A_{\Gamma 3} A_{33}^{-1} A_{3\Gamma}, \tag{6.7}$$

where W is the diagonal matrix

$$W := \begin{bmatrix} \frac{w_1+w_2}{2} & 0 \\ 0 & \frac{w_2+w_3}{2} \end{bmatrix}. \tag{6.8}$$

First we observe that

$$A_{ff}^{-1} A_{fc} = \begin{bmatrix} A_{11}^{-1} A_{1,\Gamma_1} & 0 \\ A_{22}^{-1} A_{2,\Gamma_1} & A_{22}^{-1} A_{2,\Gamma_2} \\ 0 & A_{33}^{-1} A_{3,\Gamma_2} \end{bmatrix}$$

is independent of the w_i. Further, by construction, we have

$$B_{ff} = \begin{bmatrix} \frac{1}{w_1} B_{11} & 0 & 0 \\ 0 & \frac{1}{w_2} B_{22} & 0 \\ 0 & 0 & \frac{1}{w_3} B_{33} \end{bmatrix},$$

where $B_{ii} := A_{ff}^{(E_i)^{-1}}$, $i = 1, 2, 3$, since the latter act independently on the three macroelements. Hence

$$B_{ff} A_{fc} = \begin{bmatrix} B_{11} A_{1,\Gamma_1} & 0 \\ B_{22} A_{2,\Gamma_1} & B_{22} A_{2,\Gamma_2} \\ 0 & B_{33} A_{3,\Gamma_2} \end{bmatrix}$$

is also independent of the w_i. That is, the left and right matrices in the product in (6.5) are independent of w_i. It remains to study the matrix in the center.

PROPOSITION 6.1. *The condition number of $S^{-1} S_c(A)$ is bounded as $\frac{w_1}{w_2} \to \infty$.*

Proof. Let $\tilde{\Sigma} := \frac{1}{w_2} S_c(A)$ and $\tilde{S} := \frac{1}{w_2} S$. Then $\kappa(S^{-1} S_c(A)) = \kappa(\tilde{S}^{-1} \tilde{\Sigma})$. Here, following (3.20),

$$\tilde{\Sigma} = \begin{bmatrix} \frac{w_1}{w_2} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & \frac{1}{2}(1 + \frac{w_3}{w_2}) A_{\Gamma_2,\Gamma_2} \end{bmatrix} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} - \frac{w_3}{w_2} A_{\Gamma 3} A_{33}^{-1} A_{3\Gamma}, \tag{6.9}$$

where $\Sigma_1 := \frac{1}{2} A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}$. As seen after (3.20), here Σ_1 is a Schur complement, corresponding to the positive definite matrix $\tilde{A}_{11}$ from (3.7), modified by setting the weights $w_1 = 1$ and $\frac{w_1+w_2}{2} = \frac{1}{2}$ in the integrals. Hence Σ_1 is still positive definite. Denoting by R_Σ the second to fourth terms of (6.9), we have

$$\tilde{\Sigma} = \frac{w_1}{w_2} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + R_\Sigma,$$

where R_Σ has bounded coefficients as $\frac{w_1}{w_2} \to \infty$.

Let us now similarly rewrite $\tilde{S}$. By definition, S is the assembly of the Schur complements $S_2^{(E_i)}$ for the element matrices $A^{(E_i)}$ $(i = 1, 2, 3)$. To form the latter, we note that by assumption E_1 and E_3 only have interior vertices on Γ_1 or Γ_3, respectively, whereas E_2 has vertices on both of Γ_1 and Γ_3. Therefore the element matrices take the following form:

$$A^{(E_1)} = \begin{bmatrix} w_1 A_{11} & w_1 A_{1,\Gamma_1} \\ w_1 A_{\Gamma_1,1} & \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} \end{bmatrix}, \qquad A^{(E_3)} = \begin{bmatrix} w_3 A_{33} & w_3 A_{3,\Gamma_2} \\ w_3 A_{\Gamma_2,3} & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} \end{bmatrix},$$

$$A^{(E_2)} = \begin{bmatrix} w_2 A_{22} & w_2 A_{2c} \\ w_2 A_{c2} & W A_{cc} \end{bmatrix}, \quad \text{where} \quad A_{c2} := A_{2c}^T := \begin{bmatrix} A_{\Gamma_1,2} \\ A_{\Gamma_2,2} \end{bmatrix},$$

and W is from (6.8). Then

$$\begin{aligned} S_2^{(E_1)} &= \tfrac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} - w_1 A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}, \; S_2^{(E_3)} = \tfrac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} - w_3 A_{\Gamma_2,3} A_{33}^{-1} A_{3,\Gamma_2}, \\ S_2^{(E_2)} &= W A_{cc} - w_2 A_{c2} A_{22}^{-1} A_{2c} \\ &= \begin{bmatrix} \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} & 0 \\ 0 & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} \end{bmatrix} - w_2 \begin{bmatrix} A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_1} & A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_2} \\ A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_1} & A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{w_1+w_2}{2} A_{\Gamma_1,\Gamma_1} - w_2 A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_1} & -w_2 A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_2} \\ -w_2 A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_1} & \frac{w_2+w_3}{2} A_{\Gamma_2,\Gamma_2} - w_2 A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_2} \end{bmatrix}. \end{aligned}$$

The assembly of $S_2^{(E_i)}$ is a 2×2 block matrix, where the first and second rows/columns correspond to the boundaries Γ_1 and Γ_2, respectively. That is, $S_2^{(E_1)}$ and $S_2^{(E_3)}$ are added to the (1,1) and (2,2) blocks of $S_2^{(E_2)}$:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \qquad \text{where}$$

$$\begin{aligned} S_{11} &= (w_1 + w_2)\, A_{\Gamma_1,\Gamma_1} - w_1 A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1} - w_2 A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_1}, \\ S_{12} &= -w_2 A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_2}, \\ S_{21} &= -w_2 A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_1}, \\ S_{22} &= (w_2 + w_3)\, A_{\Gamma_2,\Gamma_2} - w_2 A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_2} - w_3 A_{\Gamma_2,3} A_{33}^{-1} A_{3,\Gamma_2}. \end{aligned}$$

Therefore

$$\begin{aligned} \tilde{S} &= \begin{bmatrix} \frac{w_1}{w_2} \left(A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1} \right) & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_1} & -A_{\Gamma_1,2} A_{22}^{-1} A_{2,\Gamma_2} \\ -A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_1} & (1 + \frac{w_3}{w_2})\, A_{\Gamma_2,\Gamma_2} - A_{\Gamma_2,2} A_{22}^{-1} A_{2,\Gamma_2} - \frac{w_3}{w_2} A_{\Gamma_2,3} A_{33}^{-1} A_{3,\Gamma_2} \end{bmatrix} \\ &= \frac{w_1}{w_2} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + R_S, \end{aligned}$$

where $S_1 := A_{\Gamma_1,\Gamma_1} - A_{\Gamma_1,1} A_{11}^{-1} A_{1,\Gamma_1}$ is a Schur complement of the matrix $\tilde{A}_{11}$ in (3.7). Hence S_1 is positive definite, further, R_S denotes the second matrix above, which has bounded coefficients as $\frac{w_1}{w_2} \to \infty$. (Recall that, by assumption, $0 < \alpha \le \frac{w_3}{w_2} \le 1$.)

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Now it is easy to derive the spectral equivalence of $\tilde{S}$ and $\tilde{\Sigma}$. Let us consider vectors in the form $x = \binom{x_1}{x_2}$, where the decomposition into the vectors x_1 and x_2 corresponds to the block form of $\tilde{S}$. Then

$$\tilde{S}x \cdot x \;=\; \frac{w_1}{w_2}\,(S_1 x_1 \cdot x_1) + R_S\, x \cdot x \quad \text{and} \quad \tilde{\Sigma}x \cdot x \;=\; \frac{w_1}{w_2}\,(\Sigma_1 x_1 \cdot x_1) + R_\Sigma\, x \cdot x\,.$$

Hence

$$\frac{\tilde{\Sigma}x \cdot x}{\tilde{S}x \cdot x} \;=\; \frac{(\Sigma_1 x_1 \cdot x_1) + \frac{w_2}{w_1}\,(R_\Sigma\, x \cdot x)}{(S_1 x_1 \cdot x_1) + \frac{w_2}{w_1}\,(R_S\, x \cdot x)}\,.$$

Here, by assumption, $0 \le \frac{w_2}{w_1} \le 1$. Hence

$$\frac{(\Sigma_1 x_1 \cdot x_1)}{(S_1 x_1 \cdot x_1) + (R_S\, x \cdot x)} \;\le\; \frac{\tilde{\Sigma}x \cdot x}{\tilde{S}x \cdot x} \;\le\; \frac{(\Sigma_1 x_1 \cdot x_1) + (R_\Sigma\, x \cdot x)}{(S_1 x_1 \cdot x_1)}\,. \tag{6.10}$$

Here the matrices Σ_1 and S_1 were seen to be positive definite. Let us now define

$$\tilde{\Sigma}_1 := \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + R_\Sigma \qquad \text{and} \qquad \tilde{S}_1 := \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + R_S\,.$$

These matrices are also positive definite since they coincide with $\tilde{\Sigma}$ and $\tilde{S}$, respectively, in the case $w_1 = w_2$. Then (6.10) implies

$$\lambda_{min}(\tilde{S}_1^{-1}\,\Sigma_1) \le \lambda_{min}(\tilde{S}^{-1}\,\tilde{\Sigma}), \qquad \lambda_{max}(\tilde{S}^{-1}\,\tilde{\Sigma}) \le \lambda_{max}(S_1^{-1}\,\tilde{\Sigma}_1),$$

which yields the desired boundedness of $\kappa(\tilde{S}^{-1}\tilde{\Sigma})$ as $\frac{w_1}{w_2} \to \infty$. □

COROLLARY 6.2. *The condition number of $B^{-1}A$ is bounded as $\frac{w_1}{w_2} \to \infty$.*

Proof. We have seen just before Proposition 6.1 that the left and right matrices in the product in (6.5) are independent of the w_i. Hence it remains to prove that the matrix in the center has bounded condition number as $\frac{w_1}{w_2} \to \infty$. Since this matrix is block diagonal, its eigenvalues coincide with those of its diagonal blocks, therefore it suffices that $\kappa(B_{ff}A_{ff})$ and $\kappa(S^{-1}S_c(A))$ have bounded condition numbers as $\frac{w_1}{w_2} \to \infty$. The latter has been proved in Proposition 6.1, whereas in the former case

$$B_{ff}A_{ff} = \begin{bmatrix} B_{11}A_{11} & 0 & 0 \\ 0 & B_{22}A_{22} & 0 \\ 0 & 0 & B_{33}A_{33} \end{bmatrix}$$

is even independent of the w_i. □

REMARK 6.3. The above result can be extended to the case of more than three subdomains under corresponding assumptions on the coefficients w_i.

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