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MATHEMATICAL PROPERTIES OF FLOWS OF INCOMPRESSIBLE POWER-LAW-LIKE FLUIDS THAT ARE DESCRIBED BY IMPLICIT CONSTITUTIVE RELATIONS*

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Abstract. We report on very recent developments concerning the modelling of the complex behaviour of materials within the framework of implicit constitutive theory due to K. R. Rajagopal. In this paper, we restrict ourselves to a hierarchy of power-law-like fluids. For such a class of fluids, we provide an overview of recent results concerning the mathematical analysis of the relevant boundary value problems. Mathematical results are presented for the (Rothe) time discretizations of evolutionary problems. The main purpose of this paper is to emphasize the mathematical tools involved in the theoretical analysis and to initiate the development of numerical methods for the problems presented here.

Key words. Power-law fluid, incompressible fluid, implicit constitutive theory, Rothe approximation, time discretization, weak solution, existence, regularity.

AMS subject classifications. 35D05, 35Q30, 35Q35, 76D03, 76D99.

1. Introduction. In continuum physics, three different concepts have been recently developed by K. R. Rajagopal and his co-authors in order to successfully model complex processes in materials, which in most cases are of mechanical type. While individual components of the framework were introduced with a different intent, Rajagopal and his co-workers generalized these ideas and melded them together to form a meaningful coherent theory. We will briefly characterize these concepts.

The first approach is based on the notion of a natural configuration, associated to the current configuration of the body. The natural configuration is the one that the body would take on the removal of all external stimuli. This notion was introduced by Eckart [16], who, however, did not recognize either the importance of the symmetry of the natural configuration evolving during the process, or several other related issues. Recall, for example, that the classical continuum mechanics framework, built upon the notions of current and reference configurations, is too narrow to enable one to model inelastic behaviour of solid-like materials or viscoelastic properties of materials. Thus, some artificial internal variable models have been introduced in order to explain these features. On the other hand, an extended methodology involving the concept of natural configurations provides a sufficiently robust framework, which is free of such a deficiency. We refer the reader to Rajagopal [43] and his article in the current volume for details.

A characteristic feature of the second approach is the application of the assumption of maximization of the rate of entropy production in order to determine the form of the constitutive relation between the Cauchy stress and relevant quantities, such as, for example, the shear-rate in the case of fluids. This method, which efficiently selects the appropriate form of the constitutive relations requires one to know how the body stores the energy and what are the relevant dissipative mechanisms; see Rajagopal and Srinivasa [53] for a detailed and transparent description of the method. Note that the notion of maximization of the rate of entropy production has been considered by several other authors earlier. For instance, Ziegler [62]

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used this idea within the context of plasticity, but the way in which the idea is enforced is quite different from that of Rajagopal and his co-workers. A detailed discussion of the same topic can be found in Rajagopal and Srinivasa [54].

Finally, the third approach, namely the so-called implicit constitutive theory, shares a similarity with the first approach as it expands standard continuum mechanics enormously, so that the framework is sufficiently robust to capture complicated nonlinear responses of materials. In addition, this approach can eliminate some internal variable theories; observe that internal variables have mostly a vague physical meaning and it is thus difficult to specify boundary conditions for them. We refer the interested reader to the original papers of Rajagopal [44] and [45] for details. Implicit constitutive relations have been used to describe a material response for a long time. However, the idea of obtaining such models by appealing to the evolution of natural configurations and the maximization of the rate of entropy production was first considered by Rajagopal and his co-authors.

The above-mentioned approaches turn out to be very efficient in predicting the response of a wide variety of materials, as they represent a sufficiently robust framework of continuum physics, suitable to capture complex behaviour of materials without any need to introduce internal variable theory, macro-meso or macro-micro-scopic models. To illustrate our point, see the application of these techniques to viscoelasticity [51, 52, 35], classical plasticity [48, 49, 28], twinning [46, 47], solid to solid phase transition [50], crystallization in polymers [55, 56], single crystal super alloys [42], inhomogeneous incompressible fluids [34], mixture theory [37], fluids with pressure dependent viscosities or Bingham fluids [33, 54], etc. Such new approaches are an inspiring source of ideas for the mathematical formulation and analysis of the relevant initial-boundary-value problems, and are also useful in the design and analysis of numerical methods.

The aim of this article is twofold. First of all, in Section 2 we would like to illustrate the efficiency and wide applicability of the implicit constitutive theory, and the selectivity role of the maximization of the rate of entropy production, focusing on a hierarchy of the so-called homogeneous power-law-like incompressible fluids. For the fluids considered here, the natural configuration coincides with the current one, and thus we refer to [51, 53, 35] and the paper of K. R. Rajagopal in this volume for the demonstration of the usefulness of the concept of natural (preferred) configuration. We present examples of the explicit and implicit constitutive relations, and identify those non-Newtonian phenomena that these models can capture.

The second aim of this paper is to discuss the mathematical properties of these models. Based on the results available for steady flows, for which the existence theory is essentially complete, we formulate directly in Section 3 the analogous results for the Rothe approximations (time discretization) of the evolutionary models. We proceed from the simplest explicit to fully implicit power-law-like fluid models, focusing on how the assumptions regarding the structure of the constitutive relations and the mathematical formulations of problems change. Concluding remarks and future directions are discussed in Section 4.

2. Mechanics of power-law-like incompressible materials. A standard point of departure for continuum mechanics dealing with processes, which take place at the uniform temperature, is the following set of equations (considered at any time t and any position x at the current configuration of the body):

$$
\begin{aligned}
\varrho_t + \operatorname{div} \varrho\boldsymbol{v} &= 0,\\
(\varrho\boldsymbol{v})_t + \operatorname{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) &= \operatorname{div}\boldsymbol{T}, \quad \boldsymbol{T}=\boldsymbol{T}^T,\\
\boldsymbol{T}\cdot\boldsymbol{D} - \rho\frac{d\psi}{dt} &= \xi \qquad \text{with } \xi\geq 0,
\end{aligned}
\tag{2.1}
$$

where ϱ is the density, ψ denotes the Helmholtz free energy, $\boldsymbol{T}\cdot\boldsymbol{D}$ is the stress power, ξ stands for the rate of dissipation, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity, $\boldsymbol{T} = (T_{ij})_{i,j=1}^3$ is the Cauchy stress (symmetric tensor of the second order) and $\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{v}) := (\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)/2$ is the symmetric part of the velocity gradient. We recall that the above equations (2.1) express the balance of mass, the balance of linear and angular momentum, and the equation that is a consequence of the balance of energy and the second law of thermodynamics if the temperature is constant; see for example [53] or [33] for details.

We are interested in describing flows of various fluid-like-materials that, while exhibiting many different and fascinating phenomena, nevertheless share one common feature: these materials are well approximated as *incompressible, homogeneous* fluids. This means that the density ϱ and the Helmholtz free energy ψ (which is supposed to be a function of density) are constant and

$$\operatorname{div}\boldsymbol{v} = \operatorname{tr}\boldsymbol{D} = \boldsymbol{D}\cdot\boldsymbol{I} = 0. \tag{2.2}$$

For the sake of simplicity, we multiply the first equation of (2.1) by ϱ^{-1} and write $\boldsymbol{T}$, instead of $\varrho^{-1}\boldsymbol{T}$, in what follows.

Introducing the pressure as $p := -\frac{1}{3}\operatorname{tr}\boldsymbol{T}$ we can decompose the Cauchy stress $\boldsymbol{T}$ into its spherical part $-p\boldsymbol{I}$ and the deviatoric (traceless) part $\boldsymbol{S}$, i.e., $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}$. Replacing $\boldsymbol{v}_t(t,\cdot)$ by its discretization $1/h\big(\boldsymbol{v}(t,\cdot) - \boldsymbol{v}(t-h,\cdot)\big)$ and setting[1] $\boldsymbol{f}(t,x) = \frac{1}{h}\boldsymbol{v}(t-h,x)$, we obtain the so-called Rothe approximation of (2.1)-(2.2):

$$\operatorname{div}\boldsymbol{v} = 0 \quad\text{and}\quad \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\boldsymbol{S} + \frac{1}{h}\boldsymbol{v} = -\nabla p + \boldsymbol{f}, \tag{2.3}$$

$$\boldsymbol{T}\cdot\boldsymbol{D} = \xi \quad\text{with}\quad \xi \geq 0. \tag{2.4}$$

An incompressible fluid is said to be Newtonian if

$$\boldsymbol{S} = 2\mu^*\boldsymbol{D} \iff \boldsymbol{T} = -p\boldsymbol{I} + 2\mu^*\boldsymbol{D} \qquad (\mu^* \in (0,\infty)). \tag{2.5}$$

Inserting (2.5) into (2.3), we obtain the Navier-Stokes equations and (2.4) then leads to

$$\xi = 2\mu^*|\boldsymbol{D}|^2 = (2\mu^*)^{-1}|\boldsymbol{T}|^2 = (2\mu^*)^{-1}|\boldsymbol{S}|^2, \tag{2.6}$$

where the symbol $|\boldsymbol{A}|$ stands for $(\boldsymbol{A}\cdot\boldsymbol{A})^{1/2} = \big(\sum_{i,j=1}^3 (A_{ij}^2)\big)^{1/2}$.

2.1. Implicit constitutive relations. As the broadly used and popular Navier-Stokes model (2.5) suggests, the relationship between the shear stress (or more generally the Cauchy stress $\boldsymbol{T}$) and the shear rate (the symmetric part of the velocity gradient $\boldsymbol{D}(\boldsymbol{v})$) is well accepted for many fluid-like materials. A general point-wise relation of this type can be written in the form:

$$\boldsymbol{G}(t, x, \boldsymbol{T}(t,x), \boldsymbol{D}(t,x)) = \boldsymbol{0}.$$

In what follows, we restrict ourselves to the relation

$$\boldsymbol{G}(\boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0}. \tag{2.7}$$

Following Rajagopal [45], we can look at the consequences of the assumption that $\boldsymbol{G}$ in (2.7) is an isotropic function of the tensors $\boldsymbol{T}$ and $\boldsymbol{D}$. It then follows from the representation

[1] Note that $\operatorname{div}\boldsymbol{f} = 0$ due to (2.2). Although we could consider a more general $\boldsymbol{f}$ in what follows, for simplicity we restrict ourselves only to $\boldsymbol{f}$ that satisfies $\operatorname{div}\boldsymbol{f} = 0$.

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theorem of such functions (see [59]) that (2.7) takes the following most general form

$$\begin{aligned}\alpha_0\boldsymbol{I}+&\alpha_1\boldsymbol{T}+\alpha_2\boldsymbol{D}+\alpha_3\boldsymbol{T}^2+\alpha_4\boldsymbol{D}^2+\alpha_5(\boldsymbol{T}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{T})\\&+\alpha_6(\boldsymbol{T}^2\boldsymbol{D}+\boldsymbol{D}\boldsymbol{T}^2)+\alpha_7(\boldsymbol{T}\boldsymbol{D}^2+\boldsymbol{D}^2\boldsymbol{T})+\alpha_8(\boldsymbol{T}^2\boldsymbol{D}^2+\boldsymbol{D}^2\boldsymbol{T}^2)=\boldsymbol{0},\end{aligned}\tag{2.8}$$

where $\alpha_i, i=0,\ldots,8$, are functions of the invariants

$$\operatorname{tr}\boldsymbol{T},\ \operatorname{tr}\boldsymbol{D},\ \operatorname{tr}\boldsymbol{T}^2,\ \operatorname{tr}\boldsymbol{D}^2,\ \operatorname{tr}\boldsymbol{T}^3,\ \operatorname{tr}\boldsymbol{D}^3,\ \operatorname{tr}(\boldsymbol{T}\boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{T}^2\boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{D}^2\boldsymbol{T}),\ \operatorname{tr}(\boldsymbol{D}^2\boldsymbol{T}^2).$$

Since we deal with incompressible fluids, there is an interesting subclass of (2.7), namely those fluids for which the traceless part $\boldsymbol{S}$ and $\boldsymbol{D}$ are related implicitly. More precisely,

$$\boldsymbol{T}=-p\boldsymbol{I}+\boldsymbol{S}\quad\text{ with }\quad\boldsymbol{G}(\boldsymbol{S},\boldsymbol{D})=\boldsymbol{0}.\tag{2.9}$$

In the next section, we intend to point out several examples of the so-called power-law-like fluids that fall into the categories (2.7) and/or (2.9). We then look at these models from two different points of view: (i) as special cases of the general form (2.8) obtained by using the representation theory for the isotropic second order tensors $\boldsymbol{G}$ of the form (2.7), and (ii) as models obtained by using the thermomechanical framework based on the assumption that material response corresponds to the response that maximizes the rate of entropy production.

2.2. Explicitly constituted power-law-like fluids. Besides the Navier-Stokes fluid defined in (2.5), the subclass (2.9) also includes power-law fluids:

$$\boldsymbol{T}=-p\boldsymbol{I}+2\mu^*|\boldsymbol{D}|^{r-2}\boldsymbol{D}=:-p\boldsymbol{I}+2\mu_1(|\boldsymbol{D}|^2)\boldsymbol{D}=-p\boldsymbol{I}+\boldsymbol{S},\tag{2.10}$$

where $r\in[1,\infty)$ is the so-called power-law index and $\mu_1(s)=\mu^*\,s^{(r-2)/2}$, $\mu^*\in(0,\infty)$. It is easy to observe that the following relations hold for (2.10):

$$|\boldsymbol{S}|=2\mu^*|\boldsymbol{D}|^{r-1},\tag{2.11}$$

$$\boldsymbol{S}\cdot\boldsymbol{D}=2\mu^*|\boldsymbol{D}|^r=\frac{|\boldsymbol{S}|^{r/(r-1)}}{(2\mu^*)^{1/(r-1)}}=\mu^*|\boldsymbol{D}|^r+\frac{1}{2}\frac{|\boldsymbol{S}|^{r/(r-1)}}{(2\mu^*)^{1/(r-1)}},\tag{2.12}$$

and for all $\boldsymbol{D},\boldsymbol{E}\in\mathbb{R}^{3\times3}$ such that $\boldsymbol{D}\neq\boldsymbol{E}$

$$\big(\tilde{\boldsymbol{S}}(\boldsymbol{D})-\tilde{\boldsymbol{S}}(\boldsymbol{E})\big)\cdot(\boldsymbol{D}-\boldsymbol{E})>0,\qquad\text{where}\quad\tilde{\boldsymbol{S}}(\boldsymbol{B}):=2\mu^*|\boldsymbol{B}|^{r-2}\boldsymbol{B}.\tag{2.13}$$

The model (2.10) has the ability, at a simple shear flow, to shear-rate thicken (for $r>2$) or thin (if $r\in(1,2)$). Note that for $r=2$ this model coincides with (2.5). Note also that if $r>2$, then the generalized viscosity $\mu_1(|\boldsymbol{D}|^2)$ vanishes as $|\boldsymbol{D}|$ tends to zero, or if $r\in(1,2)$ the viscosity $\mu_1(|\boldsymbol{D}|^2)$ tends to $+\infty$ as $|\boldsymbol{D}|\to0^+$. While this degenerate or singular behaviour of the viscosity might be useful to approximate adequately the behaviour of certain materials, there are many other materials for which μ, considered as a function of $|\boldsymbol{D}|^2$, attains a proper limit as $|\boldsymbol{D}|\to0^+$. The same goes for the behaviour of μ_1 for large $\boldsymbol{D}$ as well. As an example, we can state two models suitable to model such a behaviour,

$$\begin{aligned}\boldsymbol{T}&=-p\boldsymbol{I}+2\mu_0^*\boldsymbol{D}+2\mu_1^*|\boldsymbol{D}|^{r-2}\boldsymbol{D}=-p\boldsymbol{I}+2(\mu_0^*+\mu_1^*|\boldsymbol{D}|^{r-2})\boldsymbol{D}\\&=:-p\boldsymbol{I}+2\mu_2(|\boldsymbol{D}|^2)\boldsymbol{D}\,,\qquad\qquad\mu_0^*,\mu_1^*\in(0,\infty)\,,\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}\boldsymbol{T}&=-p\boldsymbol{I}+2\mu_0^*(\varepsilon+\mu_1^*|\boldsymbol{D}|^2)^{(r-2)/2}\boldsymbol{D}\\&=:-p\boldsymbol{I}+2\mu_3(|\boldsymbol{D}|^2)\boldsymbol{D}\,,\qquad\qquad\mu_0^*,\mu_1^*,\varepsilon\in(0,\infty)\,,\end{aligned}\tag{2.15}$$

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which, for obvious reasons, are called the modified and generalized Navier-Stokes fluids, respectively.

It is worth mentioning that the fluids characterized by the relations (2.14) and (2.15) satisfy (2.13) and fulfil growth and coerciveness conditions similar to (2.11) and (2.12).

We can require that $\boldsymbol{T}$ or $\boldsymbol{S}$ behaves as (2.10) or (2.14) only asymptotically, for small or large values of $\boldsymbol{D}$. Such fluids, which we call power-law-like fluids, can be characterized by the constitutive relation

$$\boldsymbol{S} = \tilde{\boldsymbol{S}}(\boldsymbol{D}) = 2\nu(|\boldsymbol{D}|^2)\boldsymbol{D} \tag{2.16}$$

completed by the following assumption: there are $C_1, C_2 \in (0,\infty)$ and $\kappa \geq 0$ (usually $\kappa = 0$ or $\kappa = 1$) such that

$$\begin{aligned} \tilde{\boldsymbol{S}}(\boldsymbol{D}) \cdot \boldsymbol{D} &\geq C_1(\kappa + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{D}|^2 \\ |\tilde{\boldsymbol{S}}(\boldsymbol{D})| &\leq C_2(\kappa + |\boldsymbol{D}|^2)^{(r-1)/2}, \end{aligned} \tag{2.17}$$

and (2.13) holds for all $\boldsymbol{D}$, $\boldsymbol{E} \in \mathbb{R}^{3\times3}$, $\boldsymbol{D} \neq \boldsymbol{E}$.

Sometimes it is more suitable to require a stronger assumption, namely that there are $K_1, K_2 \in (0,\infty)$ and $\kappa \geq 0$ such that, for all $\boldsymbol{D}, \boldsymbol{B} \in \mathbb{R}^{3\times3}$,

$$K_1(\kappa + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2 \leq \frac{\partial \tilde{\boldsymbol{S}}(\boldsymbol{D})}{\partial \boldsymbol{D}} \cdot \boldsymbol{B} \otimes \boldsymbol{B} \leq K_2(\kappa + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2, \tag{2.18}$$

where we use the notation

$$\frac{\partial \tilde{\boldsymbol{S}}(\boldsymbol{D})}{\partial \boldsymbol{D}} \cdot \boldsymbol{B} \otimes \boldsymbol{B} = \sum_{i,j,k,\ell=1}^{3} \frac{\partial \tilde{S}_{ij}(\boldsymbol{D})}{\partial D_{kl}} B_{ij} B_{kl}.$$

It is shown, for example in [32], that (2.18) implies (2.17) and (2.13).

More generally, one can assume the constitutive relation of the form (2.16), in which the dependence of $\tilde{\boldsymbol{S}}$ on $\boldsymbol{D}$ (or ν on $|\boldsymbol{D}|$) is *not* polynomial. Such fluids are called fluids with shear-rate dependent viscosity. The case $\mu(s) = \mu^* \exp(s)$ or $\mu(s) = \mu^* \log(1+s)$ are just two possible examples.

The framework of power-law-like fluids presented above is sufficiently robust to model behaviour of various types of fluid-like materials. Models are frequently used in many areas of engineering and natural sciences: mechanics of colloids and suspensions, biological fluid mechanics, elastohydrodynamics, ice mechanics and glaciology, food processing, etc. We refer the reader to [38] for a representative list of relevant references.

Note that all the above-considered relations between (the deviatoric part $\boldsymbol{S}$ of) the Cauchy stress $\boldsymbol{T}$ and $\boldsymbol{D}$ are *explicit*. All of them have the potential to capture the non-Newtonian phenomena of shear-rate thickening or shear-rate thinning; see [33] for a more recent description of non-Newtonian characteristics.

2.3. Fluids with pressure (and shear-rate) dependent viscosity. In this subsection, we consider fluids with Cauchy stress of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu(p)\boldsymbol{D} \quad \text{or} \quad \boldsymbol{T} = -p\boldsymbol{I} + 2\mu(p, |\boldsymbol{D}|^2)\boldsymbol{D}. \tag{2.19}$$

These are fully *implicit* constitutive relations that generate a subclass of (2.7). Indeed, following Rajagopal [45] and making a very special choice of α_i in (2.8), namely

$$\alpha_0 = (\operatorname{tr}\boldsymbol{T})/3, \quad \alpha_1 = 1, \quad \alpha_2 = \mu\Big((\operatorname{tr}\boldsymbol{T})/3, |\boldsymbol{D}|^2\Big), \quad \alpha_i = 0 \quad (i = 3,\dots,8),$$

we obtain (2.19).

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Fluids with pressure-dependent viscosity have broad applications not only in elastohydrodynamics (see [61]), but they also play an important role in processes involving high pressures. The fact that the viscosity should depend on the pressure is carefully discussed by Stokes in [60]. A typical dependence of the viscosity on the pressure is exponential, as documented in the experimental reports by Andrade in the 1920's; see the book by Bridgman [9] and the survey article [36] for further references.

For the sake of completeness we add three examples of the viscosities depending on the pressure and the shear rate:

$$\mu(p,|\boldsymbol{D}|^2) = \left(\eta_\infty + \frac{\eta_0-\eta_\infty}{1+\delta|\boldsymbol{D}|^{2-r}}\right)\exp(\gamma\, p) \qquad \eta_0,\eta_\infty,\delta,\gamma\in(0,\infty),\ r=1.43, \tag{2.20}$$

$$\mu(p,|\boldsymbol{D}|^2) = c_0\frac{p}{|\boldsymbol{D}|} \qquad c_0\in(0,\infty),\ r=1, \tag{2.21}$$

$$\mu(p,|\boldsymbol{D}|^2) = (A+(1+\exp(\alpha\, p))^{-q}+|\boldsymbol{D}|^2)^{\frac{r-2}{2}} \qquad \alpha, A\in(0,\infty),\ r\in(1,2),\ q\in\left(0,\frac{1}{2\alpha}\frac{r-1}{2-r}A^{(2-r)/2}\right). \tag{2.22}$$

The fluid (2.20) was numerically treated in [25], the second model (2.21) was proposed by Schaeffer [57] in order to model flows of sand in silos. The last example (2.22), proposed in [31], approximates a pure exponential relationship between the viscosity and the pressure so that the assumptions (**A1**)-(**A2**) required by the current mathematical methods are fulfilled.

Although we have clarified the implicit character of the fluid with pressure-dependent viscosity (the relationship between $\boldsymbol{T}$ and $\boldsymbol{D}$ is implicit), it is also possible to look at these models as those with the explicit relations between $\boldsymbol{S}$ and $\boldsymbol{D}$ parameterized by p, i.e.,

$$\boldsymbol{S} = \tilde{\boldsymbol{S}}(p,\boldsymbol{D}) = 2\mu(p,|\boldsymbol{D}|^2)\boldsymbol{D}. \tag{2.23}$$

In the next subsection we discuss the fluids where the relation between $\boldsymbol{S}$ and $\boldsymbol{D}$ is implicit, thus being of the form (2.9).

2.4. Fluids with activation criteria. The characteristic feature of the fluid considered in this part is a dramatic change in the material response once the critical value for the stress or the shear rate is reached. We shall distinguish two cases: (i) the Bingham and Herschel-Bulkley fluids where the critical value for the modulus of the shear rate is zero, (ii) the fluids with activation criteria that takes place if $|\boldsymbol{D}| = d^*$ with d^* positive. One should take into account that these are just two prototypes of implicitly constituted power-law-like fluids, as we shall illustrate in what follows, and it is possible to generate plenty of other models described by the implicit constitutive relationships.

Referring the reader to (2.10), (2.14) and (2.15) for the definition of μ_i, $i = 1,2,3$, that are all of the power-law type, the description of the Bingham [7] and the Herschel-Bulkley [26] fluids reads

$$\begin{aligned} |\boldsymbol{S}| > 2\tau^* \quad &\text{if and only if} \quad \boldsymbol{S} = 2\tau^*\frac{\boldsymbol{D}}{|\boldsymbol{D}|} + 2\mu_i(|\boldsymbol{D}|^2)\boldsymbol{D}, \\ |\boldsymbol{S}| \le 2\tau^* \quad &\text{if and only if} \quad \boldsymbol{D} = \boldsymbol{0}. \end{aligned} \tag{2.24}$$

If the fluid behaves as the Navier-Stokes one once the threshold has been reached (by that we mean that $r = 2$ in μ_i in (2.24)), the model is called a Bingham fluid. If the fluid response is that of a power-law-like fluid, we talk about a Herschel-Bulkley fluid. We refer the reader to [39] for further references dealing with the Bingham and Herschel-Bulkley fluid models.

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For our purpose, it is suitable to follow Rajagopal and Srinivasa [54] and to write (2.24) as

$$2\mu_i(|\boldsymbol{D}|^2)\boldsymbol{D}\left(2\tau^* + (|\boldsymbol{S}| - 2\tau^*)^+\right) = \left(|\boldsymbol{S}| - 2\tau^*\right)^+\boldsymbol{S}. \tag{2.25}$$

Clearly, (2.25) captures all features of (2.24).

Note that (2.25) is an implicit relation[2] between $\boldsymbol{S}$ and $\boldsymbol{D}$. From the point of view of the mathematical analysis, such a formulation has several advantages. For example, it does not require one to think in terms of variational inequalities, the penalty method, etc.

Another class of models is used to take into account radical changes of properties of the fluids (such as the viscosity, for instance) when a certain critical value $d^* \in (0, \infty)$ of the modulus of the shear rate $|\boldsymbol{D}|$ is met. Inspired by Anand and Rajagopal [2], who were modelling the changes in the blood flow due to the blood platelets activation[3], Gwiazda et al. [23] considered the following model:

$$\begin{aligned} \boldsymbol{S} &= \mu_\alpha(|\boldsymbol{D}|^2)\boldsymbol{D} && \text{if} \quad |\boldsymbol{D}| < d^*, \\ \boldsymbol{S} &= \mu_\beta(|\boldsymbol{D}|^2)\boldsymbol{D} && \text{if} \quad |\boldsymbol{D}| > d^*, \\ \boldsymbol{S} &= \mu^*\boldsymbol{D} && \text{if} \quad |\boldsymbol{D}| = d^*, \end{aligned} \tag{2.26}$$

where μ^* takes any value between $\mu^*_\alpha := \lim_{s\to d^*_-} \mu_\alpha(s)$ and $\mu^*_\beta := \lim_{s\to d^*_+} \mu_\beta(s)$. Here, μ_α, μ_β are viscosities of any of the power-law-like models discussed above. To motivate a response described in (2.26), we could consider a biological fluid that at a particular shear rate or shear stress undergoes chemical processes at a different (much shorter) time scale. During such a quick process the viscosity of the fluid can change so dramatically that it is reasonable to model it as in (2.26). Another possible scenario for such a response is the flow of a granular material wherein at a certain shear rate the material continues flowing at the same shear rate even though the stress is increasing. Only after reaching a higher threshold of stress does the shear rate start to increase.

As in the case of Herschel-Bulkley fluids, we can rewrite (2.26) implicitly in the form

$$|\,|\boldsymbol{D}| - d^*\,|\boldsymbol{S} = M(|\boldsymbol{D}|^2)(|\boldsymbol{D}| - d^*)\boldsymbol{D}$$

with $M(s) := \max\{\mu_\alpha(s)\text{sgn}(s - d^*); \mu_\beta(s)\text{sgn}(s - d^*)\}$, where we assume (just for simplicity) that $\mu_\alpha(s) \le \mu_\beta(s)$ for all $s \in \mathbb{R}^+_0$.

2.5. An application of the maximal rate of the entropy production in determining the constitutive relations. The aim of this subsection is to show, following Rajagopal and Srinivasa [54], how all the power-law like models can be obtained using the assumption that the material responses to the external stimuli in such a way that the entropy production (in our case the rate of dissipation), considered as a function of the flux of the molecular momentum $\boldsymbol{S}$, is maximal provided that the thermomechanical equation (2.4) is fulfilled.

We start with the observation that the decomposition of $\boldsymbol{T}$ into its spherical and deviatoric part $-p\boldsymbol{I}$ and $\boldsymbol{S}$, and the constraint of incompressibility applied to (2.4), lead to the equation

$$\boldsymbol{S} \cdot \boldsymbol{D} = \xi \qquad \text{with } \xi \ge 0. \tag{2.27}$$

Inspired by (2.6) we start considering ξ of the form

$$\xi = \frac{g(|\boldsymbol{D}|)}{m(p, |\boldsymbol{D}|^2)}|\boldsymbol{S}|^2\,. \tag{2.28}$$

[2] In fact, it is an example of the relation where $\boldsymbol{D}$ is a function of $\boldsymbol{S}$.

[3] For the full treatment of the modelling of blood that takes into account more details of biochemistry; see Anand et al. [3, 4].

Assuming that $g(s) \geq 0$ and $m(s) > 0$ then ξ is non-negative and the second-law of thermodynamics is satisfied.

Maximizing ξ of the form (2.28) with respect to all admissible $\boldsymbol{S}$ fulfilling (2.27) results in (observe that the auxiliary Lagrange function is of the form $\xi + \lambda(\xi - \boldsymbol{S} \cdot \boldsymbol{D})$)

$$\frac{1+\lambda}{\lambda} \frac{\partial \xi}{\partial \boldsymbol{S}} = \boldsymbol{D}.$$

Using the constraint (2.27) together with (2.28) leads directly to $\frac{1+\lambda}{\lambda} = \frac{1}{2}$. Consequently

$$g(|\boldsymbol{D}|)\boldsymbol{S} = m(p, |\boldsymbol{D}|^2)\boldsymbol{D}.$$

Special choices give particular models. For example, considering the case $g \equiv 1$ and taking $m(p,s) = 2\mu^*$, $m(p,s) = 2\mu_i^*(s)$H, for $i = 1,2,3$, and $m(p,s) = 2\mu(p,s)$, we obtain the Navier-Stokes fluid (2.5), the classical, modified and generalized power-law fluids (2.10), (2.14) and (2.15), and finally the fluid with pressure and shear-rate dependent viscosity (2.23), respectively.

It is obvious that there are many other choices of ξ that satisfy the second-law of thermodynamics; they lead to new classes of models. For example, as the structure of ξ generated by the power-law-fluids suggests (see (2.12)), the choices

$$\xi = \beta(\kappa + |\boldsymbol{S}|^2)^{\frac{1}{2}\frac{(2-r)}{(r-1)}} |\boldsymbol{S}|^2 \qquad \text{(with } \beta > 0 \text{ and } \kappa > 0)$$

lead to interesting classes of power-law-like models which have not been studied so far in this form.

3. Analysis of PDEs and existence of weak solutions. Models of continuum mechanics that are built on the principles of classical mechanics form huge classes of models designed to capture motions/deformations of bodies. The main task is then to determine these motions/deformations from the knowledge of the initial and boundary conditions. Thus, independently of how precisely these models approximate reality, mathematical analysts ask about the mathematical consistency (well-posedness) of these problems. What is the appropriate notion of solution to the considered initial-boundary-value problems? Is it then possible to establish its existence and to analyze its further qualitative properties, such as uniqueness, smoothness for smooth data, long-time behaviour, stability of special flows or deformations, or to investigate various singular limits? Can one use this information to investigate shape optimization problems or to control the flows optimally? How should one design the numerical schemes that would take advantage of theoretical results so that we know exactly which object we approximate?

With this aim in mind, we need the appropriate notion of a solution. Since the balance equations of continuum physics are formulated for any subset of a body, the principles (assumptions) of classical mechanics are required to hold for average quantities, and so the notions such as the *weak solution* seem to be a sound choice; see the book by Feireisl [17] or the introductory part of one of his more recent papers [18]. The fact that the notion of a weak solution is the point of departure of the Finite Element Method (which has a close relation to the Finite Volume Method) gives even stronger support to the notion of a weak solution.

It is of interest to mention, referring to [34] for details, that the procedure of maximization of the rate of entropy production that is based on the knowledge of the constitutive equations for ψ and ξ, as appeared in (2.17), also provides the function spaces in which the weak solution should be constructed.

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The aim of this section is to present results regarding the existence of weak solutions to the boundary-value problem

$$\operatorname{div}\boldsymbol{v}=0 \quad \text{and} \quad \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})-\operatorname{div}\boldsymbol{S}+\frac{1}{h}\boldsymbol{v}=-\nabla p+\boldsymbol{f} \ \text{ in } \Omega, \tag{3.1}$$

$$\boldsymbol{v}=\boldsymbol{0} \ \text{ on } \partial\Omega, \qquad \frac{1}{|\Omega|}\int_\Omega p\,dx=p^*, \tag{3.2}$$

where $\boldsymbol{S}$ and $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{v})$ are related through the power-law-like relationships discussed in the previous section, and $\Omega\subset\mathbb{R}^3$ is a bounded open connected set that is, for simplicity, assumed to have Lipschitz boundary $\partial\Omega$, and $p^*\in\mathbb{R}$ is a constant that denotes the mean value of the pressure.

The scheme of this section is the following: first we shall restrict ourselves to Navier-Stokes fluids and instead of mentioning any existence results[4], we outline the standard existence scheme. Then we shall focus on classical power-law fluids and explain where the critical values of the power-law index appear and emphasize the differences in the analysis of Navier-Stokes fluids and power-law fluids. Finally, we shall discuss existence results for power-law-like fluids, for fluids with the pressure and shear-rate dependent viscosity, and, finally, for power-law-like fluids with activation criteria.

We use a standard notation for the Lebesgue and Sobolev spaces[5] $\big(L^r(\Omega),\|\cdot\|_r\big)$ and $\big(W^{1,r}(\Omega),\|\cdot\|_{1,r}\big)$. We also define

$$W_0^{1,r}(\Omega):=\{u;u\in W^{1,r}(\Omega),u|_{\partial\Omega}=0\},$$
$$W_{0,\operatorname{div}}^{1,r}(\Omega):=\{\boldsymbol{v};\boldsymbol{v}\in W_0^{1,r}(\Omega)^3,\operatorname{div}\boldsymbol{v}=0\}.$$

By the symbol $\cdot$ we mean the scalar product in $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$.

3.1. Navier-Stokes fluids. Upon inserting (2.5) into (3.1), we obtain the Navier-Stokes equations

$$\operatorname{div}\boldsymbol{v}=0, \qquad -\mu^*\Delta\boldsymbol{v}+\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})+\frac{1}{h}\boldsymbol{v}=-\nabla p+\boldsymbol{f} \ \text{ in } \Omega. \tag{3.3}$$

When completed by boundary conditions, such as those in (3.2), a constructive (i.e., numerically realizable) proof of the existence of a weak solution proceeds via the following steps:
Step 1. The construction of finite-dimensional approximations $(\boldsymbol{v}^N,p^N)$, such as, for example, Galerkin or finite element approximations, and a proof of existence for fixed N (that characterizes the dimension of the finite-dimensional space) using one of the variants of the Brower fixed-point theorem.

Frequently, one needs some intermediate continuous systems between the Galerkin finite-dimensional approximations and the original system. To give an example, one can consider, for $\varepsilon\in(0,\infty)$, the quasicompressible approximation to (3.3) of the form

$$-\varepsilon\Delta p+\operatorname{div}\boldsymbol{v}=0, \quad -\mu^*\Delta\boldsymbol{v}+\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})+\frac{1}{h}\boldsymbol{v}=-\nabla p+\boldsymbol{f} \ \text{ in } \Omega. \tag{3.4}$$

Even more, with $\eta\in(0,\infty)$, the system (3.4) can be approximated by a variant of the Oseen approximation, where the term $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$ is replaced by $\operatorname{div}(\boldsymbol{v}_\eta\otimes\boldsymbol{v})$ and $\boldsymbol{v}_\eta$ is a suitable smooth, divergence-free function that approximates $\boldsymbol{v}$.

[4] Note that it has been recently proved that, for any smooth data, there is a smooth solution to the Rothe approximation of the Navier-Stokes system independently of dimension; see [22].
[5] If X denotes a space of scalar functions, then X^3 or $X^{3\times 3}$ denotes the corresponding space of vector-valued or tensor-valued functions, respectively.

Step 2. The establishment of the uniform estimates $\sup_N \|\nabla \boldsymbol{v}^N\|_2^2 + \sup_N \|p^N\|_2^2 < \infty$, by taking $\boldsymbol{v}^N$ as a "test" function in the Galerkin system.
Step 3. The identification of a candidate $(\boldsymbol{v}, p)$ for the solution by using the fact that the balls in the infinite-dimensional reflexive separable spaces are weakly compact: for selected subsequences (for which we use the same index) $\nabla \boldsymbol{v}^N \rightharpoonup \boldsymbol{v}$ and $p^N \rightharpoonup p$ weakly in L^2 as $N \to \infty$.
Step 4. The nonlinear terms require additional analysis, since the weak convergence is not alone sufficient to guarantee the convergence of these terms. The Navier-Stokes equations include one quadratic nonlinearity $\boldsymbol{v} \otimes \boldsymbol{v}$. The fact that $\boldsymbol{v}^N \to \boldsymbol{v}$ strongly in L^2 follows from the compact embedding $W_0^{1,2}(\Omega)^3$ into $L^2(\Omega)^3$ and this is sufficient to identify the limit in the convective term.
Step 5. The limit in the approximations as $N \to \infty$ that leads to the conclusion that $(\boldsymbol{v}, p)$ is indeed a week solution of the problem.

When dealing with power-law-like fluids, we have to identify the limit of another nonlinear term, namely, to show that, at almost all points of Ω, the relationship between $\boldsymbol{S}$ and $\boldsymbol{D}$ of the type (2.9) holds. To identify this relationship in the limit requires other tools, such as, for example, some kind of a monotone operator theory, or a higher-differentiability method, etc. Below, instead of the proofs, we just list the tools that are involved in the analysis of these problems; and we refer the reader to the original papers, where the details and the complete proofs are given. Before this, we first point out where the critical exponents arise.

3.2. Critical values for the power-law index. Upon inserting (2.10) into (3.1), we obtain the system

$$\operatorname{div} \boldsymbol{v} = 0, \quad -\mu^* \operatorname{div}(|\boldsymbol{D}(\boldsymbol{v})|^{r-2}\boldsymbol{D}(\boldsymbol{v})) + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \frac{1}{h}\boldsymbol{v} = -\nabla p + \boldsymbol{f} \ \text{ in } \Omega.$$

If we proceed as in the case of the Navier-Stokes equations and construct finite-dimensional approximations, we conclude that

$$\sup_N \|\nabla \boldsymbol{v}^N\|_r^r < \infty,$$

which leads us to expect that the weak solution (weak limit generated by $\boldsymbol{v}^N$) would belong to the space $W^{1,r}_{0,div}(\Omega)$.

Next, applying the divergence to the second equation in (3.3), and using the fact that $\operatorname{div} \boldsymbol{f} = 0$ in our slightly simplified setting, we obtain the equation for the pressure:

$$p = (-\Delta)^{-1} \operatorname{div} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v} - \mu^* |\boldsymbol{D}(\boldsymbol{v})|^{r-2}\boldsymbol{D}(\boldsymbol{v})).$$

Since, in three spatial dimensions, $W^{1,r} \hookrightarrow L^{3r/(3-r)}$, we observe that $\boldsymbol{v} \otimes \boldsymbol{v} \in L^{\frac{3r}{2(3-r)}}$. We also know that $|\boldsymbol{D}(\boldsymbol{v})|^{r-2}\boldsymbol{D}(\boldsymbol{v}) \in L^{r'}$ with $r' := r/(r-1)$. Thus, the pressure belongs to the intersection of these two Lebesgue spaces (which form a chain since we are in bounded domain). Since

$$\frac{r}{r-1} \leq \frac{3r}{2(3-r)} \iff r \geq \frac{9}{5},$$

we conclude that $p \in L^{r'}(\Omega)$ if $r \geq \frac{9}{5}$ and $p \in L^{\frac{3r}{2(3-r)}}(\Omega)$ if $r < \frac{9}{5}$.

Moreover, as a consequence of this observation, one can see that: (i) if $\boldsymbol{v}, \phi \in W^{1,r}$ then $\boldsymbol{v} \otimes \boldsymbol{v} \cdot \nabla \phi \in L^1$ only for $r \geq 9/5$; (ii) for $r \geq 9/5$, the energy equality holds and higher differentiability techniques can be applied. These tools make the analysis of the problem easier. Consequently, the case $r < 9/5$ is more complicated.

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There is another bound that comes from the compactness required for the quadratic nonlinearity $\boldsymbol{v} \otimes \boldsymbol{v}$. It follows from the compact embedding theorem that

$$W^{1,r} \hookrightarrow\hookrightarrow L^2 \quad \text{if and only if } r > \frac{6}{5}.$$

To summarize, the analysis of the problems for power-law-like fluid models should in general be easier for $r \geq 9/5$ than for $r \in (6/5, 9/5)$; in both cases it is more difficult than the proof of the existence of a weak solution to the Navier-Stokes equations. This type of analysis is completely open for $r \in [1, 6/5]$ in three-dimensional domains.

3.3. Results for explicitly constituted power-law like fluids. Consider the problem (3.1) with the constitutive relation (2.16), where $\tilde{\boldsymbol{S}}$ is assumed to satisfy (2.13) and (2.17)[6]. This means that we are interested in finding a (weak) solution of

$$\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0 \quad \text{and} \quad \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\tilde{\boldsymbol{S}}(\boldsymbol{D}(\boldsymbol{v}))) + \frac{1}{h}\boldsymbol{v} = -\nabla p + \boldsymbol{f} \ \text{ in } \Omega, \\ &\boldsymbol{v} = \boldsymbol{0} \ \text{ on } \partial\Omega, \qquad \frac{1}{|\Omega|}\int_\Omega p dx = p^*. \end{aligned} \tag{3.5}$$

The following results have been established so far.

THEOREM 3.1. *Let $\tilde{\boldsymbol{S}}(\cdot)$ fulfil (2.13) and (2.17) with $r \geq 9/5$. Assume that*

$$\boldsymbol{f} \in \left(W_0^{1,r}(\Omega)^3\right)^* \qquad \text{and} \qquad p^* \in R. \tag{3.6}$$

Then there is a weak solution $(\boldsymbol{v}, p)$ to (3.5) such that

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \text{and} \qquad p \in L^{r'}(\Omega).$$

Proof. See Ladyzhenskaya [29] and Lions [30]. Tools: monotone operator theory and the Minty method (the energy equality) to show the almost everywhere convergence of $\boldsymbol{D}(\boldsymbol{v}^N)$, and a compact embedding for $\boldsymbol{v}^N$. □

THEOREM 3.2. *Let $\tilde{\boldsymbol{S}}(\cdot)$ fulfil (2.13) and (2.17) with $r \in (6/5, 9/5)$ and suppose that (3.6) holds. Then there is a weak solution $(\boldsymbol{v}, p)$ to (3.5) such that*

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \text{and} \qquad p \in L^{3r/(2(r-1))}(\Omega).$$

Proof. See Frehse et al. [21] and Diening et al. [13]. Tools: Lipschitz approximations of Sobolev functions (a strengthened version) in order to replace $\boldsymbol{v}^N - \boldsymbol{v}$, which is not admissible as test function, by its Lipschitz approximation; a strictly monotone operator theory; a compact embedding for $\boldsymbol{v}^N$. □

As already observed in the previous section, one could assume a stronger condition, namely (2.18), which implies both (2.13) and (2.17). A condition very similar to (2.18) will appear in the next subsection.

3.4. Results for fluids with pressure and shear-rate dependent viscosities. Having in mind $\tilde{\boldsymbol{S}}(p, \boldsymbol{D}(\boldsymbol{v}))$ of the form $\mu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})$, we are interested in finding a (weak) solution to the problem

$$\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0 \quad \text{and} \quad \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\tilde{\boldsymbol{S}}(p, \boldsymbol{D}(\boldsymbol{v}))) + \frac{1}{h}\boldsymbol{v} = -\nabla p + \boldsymbol{f} \ \text{ in } \Omega, \\ &\boldsymbol{v} = \boldsymbol{0} \ \text{ on } \partial\Omega, \qquad \frac{1}{|O|}\int_O p \, dx = p^*, \end{aligned} \tag{3.7}$$

[6] This is exactly the characterization of what we mean by power-law-like fluids.

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where $O \subset \Omega$ is a typically small open subset, in which we fix the pressure.
We impose the following structural assumptions on $\tilde{\boldsymbol{S}}(p, \boldsymbol{D})$:

(A1) given $r \in (1, 2)$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$ and all p

$$C_1(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \le \frac{\partial \left[(\mu(p, |\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \le C_2(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2;$$

(A2) for all symmetric matrices $\boldsymbol{D}$ and all p

$$\left|\frac{\partial[\mu(p, |\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial p}\right| = \left|\boldsymbol{D}\frac{\partial \mu(p, |\boldsymbol{D}|^2)}{\partial p}\right| \le \gamma_0(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{4}} \le \gamma_0$$

and

$$\gamma_0 < \frac{1}{C_{div,2}} \frac{C_1}{C_1 + C_2},$$

where the constant $C_{div,q}$ occurs in the following problem: find $\mathbf{z} \in W_0^{1,q}(\Omega)$ which solves

$$\operatorname{div} \mathbf{z} = g \ \text{ in } \Omega, \quad \mathbf{z} = \mathbf{0} \ \text{ on } \partial\Omega \quad \text{and} \quad \|\mathbf{z}\|_{1,q} \le C_{div,q}\|g\|_q\,,$$

where $g \in L^q(\Omega)$, fulfilling $\int_\Omega g\,dx = 0$, is given. The solvability of such a problem is discussed in Bogovskiĭ [8], Amrouche, Girault [1] or in the recent book by Novotný and Straškraba [40].

Note that (**A1**) expresses the fact that although $\tilde{\boldsymbol{S}}$ depends on p, it behaves as a power-law-like fluid uniformly with respect to p; see (2.18) for a comparison.

THEOREM 3.3. *Take $r \in (9/5, 2)$ and let (3.6) hold. Assume that* **(A1)-(A2)** *are fulfilled. Then there is a weak solution $(\boldsymbol{v}, p)$ to (3.7) such that*

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \textit{and} \qquad p \in L^{r'}(\Omega).$$

Proof. See Franta et al. [20]. Tools: quasi compressible approximations in order to identify the pressure at the early stage of the proof; the structure of the viscosities; the solvability of the equation $\operatorname{div} \boldsymbol{z} = g$; strictly monotone operator theory in the $\boldsymbol{D}$-variable; compactness for the velocity gradient; compactness for the pressure; the energy equality; a compact embedding for the velocity. □

THEOREM 3.4. *Take $r \in (6/5, 9/5)$ and let (3.6) hold. Assume that* **(A1)-(A2)** *are fulfilled. Then there is a weak solution $(\boldsymbol{v}, p)$ to (3.7) such that*

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \textit{and} \qquad p \in L^{3r/(2(r-1))}(\Omega).$$

Proof. See Bulíček, Fišerová [11]. Tools: quasi compressible approximations in order to identify the pressure at an early stage of the proof; Lipschitz approximations of Sobolev functions (a strengthened version [13]); the structure of the viscosities; the solvability of the equation $\operatorname{div} \boldsymbol{z} = g$; strictly monotone operator theory in the $\boldsymbol{D}$-variable; compactness for the velocity gradient; compactness for the pressure; the decomposition of the pressure; a compact embedding for the velocity. □

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3.5. Implicit power-law-like fluids. For the power-law index $r \in (1, \infty)$, we set $r' := r/(r-1)$ and $\tilde{r} := \min\{r', r^*/2\}$, where $r^* = 3r/(3-r)$ if $r \in [1, 3)$, and $r^* = \infty$ otherwise. We look for the triplet $(\boldsymbol{v}, p, \boldsymbol{S}) : \Omega \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{3\times 3}$ satisfying

$$\begin{aligned} &\boldsymbol{v} \in W^{1,r}_{0,\mathrm{div}}(\Omega), \ \ p \in L^{\tilde{r}}(\Omega), \quad \boldsymbol{S} \in L^{r'}(\Omega)^{3\times 3}, \\ &\mathrm{div}\,(\boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I} - \boldsymbol{S}) + \frac{1}{h}\boldsymbol{v} = \boldsymbol{f} \quad \text{in } \mathcal{D}'(\Omega), \\ &(\boldsymbol{D}\boldsymbol{v}(x), \boldsymbol{S}(x)) \in \mathcal{A} \quad \text{for almost all } x \in \Omega, \end{aligned} \tag{3.8}$$

where $\mathcal{A}$ shares the following properties:

(B1) $(\boldsymbol{0}, \boldsymbol{0}) \in \mathcal{A}$.

(B2) For all $(\boldsymbol{D}_1, \boldsymbol{S}_1), (\boldsymbol{D}_2, \boldsymbol{S}_2) \in \mathcal{A}$

$$(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) \geq 0 \quad (\mathcal{A} \text{ is a monotone graph}).$$

Moreover, if $\boldsymbol{D}_1 \neq \boldsymbol{D}_2$ and $\boldsymbol{S}_1 \neq \boldsymbol{S}_2$ then the inequality is strict (it means that $\mathcal{A}$ is a strictly monotone graph in a generalized sense).

(B3) If $(\boldsymbol{D}, \boldsymbol{S}) \in R^{3\times 3}_{\mathrm{sym}} \times R^{3\times 3}_{\mathrm{sym}}$ fulfils

$$(\overline{\boldsymbol{S}} - \boldsymbol{S}) \cdot (\overline{\boldsymbol{D}} - \boldsymbol{D}) \geq 0 \quad \text{for all } (\overline{\boldsymbol{D}}, \overline{\boldsymbol{S}}) \in \mathcal{A},$$

then $(\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A}$ ($\mathcal{A}$ is a maximal monotone graph).

(B4) There are non-negative $m \in L^1(\Omega)$ and $C > 0$ such that for all $(\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A}$

$$\boldsymbol{S} \cdot \boldsymbol{D} \geq -m(x) + c(|\boldsymbol{D}|^r + |\boldsymbol{S}|^{r'}) \qquad (\mathcal{A} \text{ is a } r\text{-graph})^7.$$

One could assume (see [12]) that $\mathcal{A}$ is an x-dependent graph without any essential difficulties. Here, we assume that $\mathcal{A}(x)$ is the same at all points $x \in \Omega$.

Before formulating the results that have been recently established, we would like to emphasize the symmetric role of $\boldsymbol{S}$ and $\boldsymbol{D}$ in the assumptions **(B1)-(B4)** that thus fully reflects the implicit character of the constitutive relationships $(3.8)_3$ or (2.9).

THEOREM 3.5. *Take $r \in (9/5, 2)$ and let (3.6) hold. Assume that* **(B1)-(B4)** *are fulfilled and consider Herschel-Bulkley or Bingham fluids. Then there is a weak solution $(\boldsymbol{v}, p, \boldsymbol{S})$ to (3.8).*

Proof. See Málek et al. [39]. Tools: the local regularity method performed in such a way that the whole analysis does not involve the pressure; higher-differentiability; uniform monotone operator properties; a compact embedding for the velocity and the velocity gradient; Shelukhin's approach [58] to identify jumps in the constitutive equations. □

THEOREM 3.6. *Take $r \geq 9/5$ and let (3.6) hold. Assume that* **(B1)-(B4)** *are fulfiled. Then there is a weak solution $(\boldsymbol{v}, p, \boldsymbol{S})$ to (3.8).*

Proof. See Gwiazda et al. [23]. Tools: Young measures (generalized version); the energy equality; a strictly monotone operator; properties of approximations of discontinuous functions; a compact embedding for the velocity. □

THEOREM 3.7. *Take $r > 6/5$ and let (3.6) hold. Assume that* **(B1)-(B4)** *are fulfilled. Then there is a weak solution $(\boldsymbol{v}, p, \boldsymbol{S})$ to (3.8).*

Proof. See Bulíček et al. [12]. Tools: the characterization of maximal monotone graphs in terms of 1-Lipschitz continuous mappings due to Francfort et al. [19]; Young measures; the biting lemma [5, 10]; Lipschitz approximations of Sobolev functions (a strengthened version [13]); a compact embedding for the velocity. □

[7] Compare with (2.12).

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4. Concluding remarks. We have considered a hierarchy of so-called power-law-like (homogeneous, incompressible) fluids within the framework of implicit constitutive theory, developed recently by K. R. Rajagopal in [44, 45] and [54]. Following these studies, we have shown that the class of fluids considered is thermomechanically consistent. We have also briefly mentioned that the models have the ability to capture three types of non-Newtonian characteristics: shear thinning and thickening, pressure thickening and "jumps" in the stress due to activation. Focusing on how the definition of the (weak) solution and the structural assumptions change when going from explicit to implicit relations between the shear rate and the shear stress, we have presented available mathematical results concerning the existence of weak solutions to the Rothe approximations of the relevant evolutionary problem. While for the explicit relations the unknown functions are only the velocity and the pressure, for the problems with implicit relations the set of unknown functions also includes the deviatoric part of the Cauchy stress. Most of the results are recent; their proof required a development of new tools such as the strengthened properties of the Lipschitz approximations of Sobolev functions [13], or the characterization of maximal monotone graphs in terms of 1-Lipschitz continuous mappings [19]. The tools involved differ depending on whether we deal with models that are characterized by the power-law index r being in the subcritical regime (that is $r \geq 9/5$ in three spatial dimensions), or in the supercritical regime (when $r < 9/5$).

There are studies dealing with numerical schemes for some of the problems discussed, starting from the notion of a weak solution, and also many reports on the results of computational simulations; see for example [14, 15, 6] and [41, 27]. One of the aims of this article is to call for a unified view of these numerical and computational methods in the spirit of the available theoretical results summarized here.

The framework presented here is not complete. One should consider fluids with shear-rate dependent viscosity that are not of a power-law type (see [24] for a very recent result in this direction) include different types of boundary conditions, including those that describe inflow or outflow, and treat unsteady flows; we refer the reader to [33] for a survey concerning the mathematical results available prior to 2005. In order to include viscoelastic properties of fluid-like materials, attention should be devoted to the investigation of rate type and integral type fluids with material coefficients depending on the pressure, the shear rate, etc.

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