Electronic Transactions on Numerical Analysis.
Volume 26, pp. 421-438, 2007.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

AN AUGMENTED MIXED FINITE ELEMENT METHOD FOR LINEAR ELASTICITY WITH NON-HOMOGENEOUS DIRICHLET CONDITIONS*

GABRIEL N. GATICA†

Abstract. We have recently developed a new augmented mixed finite element method for plane linear elasticity, which is based on the introduction of suitable Galerkin least-squares type terms. The corresponding analysis makes use of the first Korn inequality, and hence only null Dirichlet conditions, either on the whole boundary or on part of it, are considered. In the present paper we extend these results to the case of non-homogeneous Dirichlet boundary conditions. To this end, we incorporate additional consistent terms and then apply a slight extension of the classical Korn inequality. We show that the resulting augmented formulation and the associated Galerkin scheme are well posed. Finally, several numerical examples illustrating the good performance of the method are provided.

Key words. mixed-FEM, augmented formulation, linear elasticity

AMS subject classifications. 65N30, 65N12, 65N15, 74B05

1. Introduction. In [8] we present and analyze a new augmented mixed finite element method for plane linear elasticity. Our approach is based on the introduction of the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and from the relation defining the rotation in terms of the displacement. We show there that the continuous and discrete versions of the augmented variational formulation are well posed, and that the latter becomes locking-free and asymptotically locking-free for Dirichlet and mixed boundary conditions, respectively. In the case of pure Dirichlet conditions, the augmented formulation becomes strongly coercive, and hence arbitrary finite element subspaces can be employed in the associated discrete scheme, which constitutes one of its main advantages with respect to other methods. In particular, a non-feasible choice of subspaces for the Galerkin scheme of the usual mixed formulation, namely Raviart-Thomas spaces of lowest order for the stress tensor, piecewise linear elements for the displacement, and piecewise constants for the rotation, can be used in our augmented method. Moreover, these subspaces yield a total number of unknowns behaving asymptotically as 5 times the number of triangles of the triangulation, whereas this factor becomes 7.5 when the well known $PEERS$ from [1] is employed. In other words, the discrete system using $PEERS$ introduces at large $50\,\%$ more degrees of freedom than our approach at each mesh, and therefore the augmented method becomes a much cheaper alternative. Similarly the augmented scheme also becomes more economical than BDM; see, e.g., [6]. On the other hand, a residual based a posteriori error analysis yielding a reliable and efficient estimator for the augmented method from [8], is provided in the recent work [4]. Nevertheless, we remark that the analysis in [8], and hence in [4], requires the application of the first Korn inequality (see, e.g., [10, Theorem 10.1] or [5, Corollaries 9.2.22 and 9.2.25]), and therefore only null Dirichlet boundary conditions can be considered.

According to the above, the purpose of this paper is to extend the results from [8] to the plane linear elasticity problem with non-homogeneous Dirichlet boundary conditions, while keeping the same advantages from [8] in the resulting augmented formulation. The development of the corresponding a posteriori error analysis, as done in [4], will be reported in a separate work. The remainder of this paper is organized as follow. In Section 2 we establish

*Received August 17, 2006. Accepted for publication July 10, 2007. Recommended by Y. Achdou. This research was partially supported by CONICYT-Chile through the FONDAP Program in Applied Mathematics, and by the Dirección de Investigación of the Universidad de Concepción through the Advanced Research Groups Program.

†GI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile (ggatica@ing-mat.udec.cl).

ETNA
Kent State University
etna@mcs.kent.edu

the dual-mixed variational formulation of our boundary value problem. Then, in Sections 3 and 4 we introduce the continuous and discrete augmented formulations, respectively, and show that they are well posed. In particular, we prove a slight extension of the classical Korn inequality, which is used to conclude, similarly as in [8], that the associated bilinear form becomes strongly coercive. Finally, several numerical results are provided in Section 5.

We end this section by introducing some notation to be used throughout the paper. For each Hilbert space U, we let U^2 and $U^{2\times 2}$ be, respectively, the space of vectors and square matrices of order 2 with entries in U. In addition, given $\boldsymbol{\tau} := (\tau_{ij}),\ \boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we define the transpose tensor $\boldsymbol{\tau}^{\mathbf{t}} := (\tau_{ji})$, the trace $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}$,, the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$, and the deviator $\boldsymbol{\tau}^{\mathbf{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$. Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, and use C, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The dual-mixed variational formulation. Let Ω be a simply connected domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma := \partial\Omega$. Since we are interested in the mixed method of Hellinger and Reissner, our goal is to determine the displacement $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω. In other words, given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$ and a Dirichlet datum $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we seek a symmetric tensor field $\boldsymbol{\sigma}$ and a vector field $\mathbf{u}$ such that

$$\boldsymbol{\sigma} = \mathcal{C}\,\mathbf{e}(\mathbf{u}), \quad \mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \quad \text{in} \quad \Omega, \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma. \tag{2.1}$$

Hereafter, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{t}}\right)$ is the strain tensor of small deformations, $\mathbf{div}$ is the usual divergence operator div acting along each row of the tensor, and $\mathcal{C}$ is the elasticity tensor determined by Hooke's law, that is

$$\mathcal{C}\,\boldsymbol{\zeta} := \lambda\operatorname{tr}(\boldsymbol{\zeta})\,\mathbf{I} + 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta} \in [L^2(\Omega)]^{2\times 2}, \tag{2.2}$$

where $\lambda,\ \mu > 0$ denote the corresponding Lamé constants. It is easy to see from (2.2) that $\mathcal{C}$ is invertible and the tensor $\mathcal{C}^{-1}$ reduces to

$$\mathcal{C}^{-1}\,\boldsymbol{\zeta} := \frac{1}{2\,\mu}\,\boldsymbol{\zeta} - \frac{\lambda}{4\,\mu\,(\lambda+\mu)}\operatorname{tr}(\boldsymbol{\zeta})\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta} \in [L^2(\Omega)]^{2\times 2}. \tag{2.3}$$

We now follow the classical stress-displacement-rotation formulation (see [1] and [11]) and impose weakly the symmetry of $\boldsymbol{\sigma}$ through the introduction of the rotation $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathbf{t}})$ as a further unknown. In this way, multiplying by test functions and then integrating the equilibrium equation $\mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f}$ and the constitutive equation $\mathcal{C}^{-1}\boldsymbol{\sigma} = \mathbf{e}(\mathbf{u}) = \nabla\mathbf{u} - \boldsymbol{\gamma}$, we end up with the following dual-mixed variational formulation of (2.1): Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H \times Q$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) &= \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle \qquad \forall\,\boldsymbol{\tau} \in H, \\ b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,(\mathbf{v}, \boldsymbol{\eta}) \in Q, \end{aligned} \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing of $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$, with respect to the $[L^2(\Gamma)]^2$-inner product,

$$H = H(\mathbf{div};\,\Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2 \right\},$$

$$Q \;:=\; [L^2(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}}\,, \quad [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}} \;:=\; \left\{\boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2}: \quad \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} \;=\; 0\right\},$$

and the bilinear forms $a : H \times H \to \mathbb{R}$ and $b : H \times Q \to \mathbb{R}$ are defined by

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) \;:=\; \int_\Omega \mathcal{C}^{-1}\boldsymbol{\zeta} : \boldsymbol{\tau} \;=\; \frac{1}{2\,\mu}\int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} \;-\; \frac{\lambda}{4\,\mu\,(\lambda+\mu)}\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \tag{2.5}$$

and

$$b(\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\eta})) \;:=\; \int_\Omega \mathbf{v}\cdot \mathbf{div}(\boldsymbol{\tau}) \;+\; \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}, \tag{2.6}$$

for all $\boldsymbol{\zeta},\, \boldsymbol{\tau} \;\in\; H$ and for all $(\mathbf{v}, \boldsymbol{\eta}) \;\in\; Q$. It follows easily from (2.5) and (2.6) that for any $(\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\eta}), c) \in [L^2(\Omega)]^{2\times 2} \times Q \times \mathbb{R}$ there holds

$$a(c\,\mathbf{I}, \boldsymbol{\tau}) \;=\; \frac{c}{2\,(\lambda+\mu)}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \quad \text{and} \quad b(c\,\mathbf{I}, (\mathbf{v}, \boldsymbol{\eta})) \;=\; 0. \tag{2.7}$$

Also, it is important to remark that a can be rewritten as

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) \;=\; \frac{1}{2\,\mu}\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \;+\; \frac{1}{4\,(\lambda+\mu)}\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}),$$

which implies that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \;\geq\; \frac{1}{2\mu}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} \qquad \forall\, \boldsymbol{\tau} \;\in\; [L^2(\Omega)]^{2\times 2}.$$

We now define $H_0 \;:=\; \left\{\boldsymbol{\tau} \in H : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;=\; 0\right\}$ and note that $H \;=\; H_0 \,\oplus\, \mathbb{R}\,\mathbf{I}$, that is for any $\boldsymbol{\tau} \;\in\; H$ there exist unique $\boldsymbol{\tau}_0 \;\in\; H_0$ and $d \;:=\; \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;\in\; \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$. In particular, we obtain from (2.1) and (2.2) that

$$\mathrm{tr}(\boldsymbol{\sigma}) \;=\; 2\,(\lambda+\mu)\,\mathrm{tr}\,\mathbf{e}(\mathbf{u}) \;=\; 2\,(\lambda+\mu)\,\mathrm{div}(\mathbf{u}),$$

which yields

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbf{I}\,, \text{ with } \boldsymbol{\sigma}_0 \in H_0 \text{ and } c \;:=\; \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;=\; \frac{(\lambda+\mu)}{|\Omega|}\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu}. \tag{2.8}$$

Hence, replacing $\boldsymbol{\sigma}$ by the expression $\boldsymbol{\sigma}_0 \,+\, c\,\mathbf{I}$ in (2.4), applying the identities given in (2.7), and denoting from now on the unknown $\boldsymbol{\sigma}_0 \;\in\; H_0$ simply by $\boldsymbol{\sigma}$, we find that the dual-mixed variational formulation of (2.1) reduces to: Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H_0 \times Q$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \;+\; b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) \;&=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle \qquad &\forall\, \boldsymbol{\tau} \in H_0, \\ b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad &\forall\,(\mathbf{v}, \boldsymbol{\eta}) \in Q. \end{aligned} \tag{2.9}$$

In addition, taking into account the new meaning of $\boldsymbol{\sigma}$, we deduce from (2.8) and (2.3) that the constitutive equation in (2.1) now becomes

$$\mathbf{e}(\mathbf{u}) \;-\; \mathcal{C}^{-1}(\boldsymbol{\sigma}) \;=\; \left\{\frac{1}{2\,|\Omega|}\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu}\right\}\,\mathbf{I} \quad \text{in} \quad \Omega, \tag{2.10}$$

whereas the equilibrium equation remains the same, that is $\mathbf{div}\,(\boldsymbol{\sigma}) \;=\; -\,\mathbf{f}$ in Ω.

We remark that the well-posedness of (2.9), whose proof follows from the classical Babuška-Brezzi theory (see, e.g., [6]), yields a continuous dependence result independently of the Lamé constant λ. We refer to [1] or [3] for details; see also [8, Section 2.1]. At this point we only recall for later use the following result concerning H_0, which plays a key role in that proof.

LEMMA 2.1. *There exists $c_1 > 0$, depending only on Ω, such that*

$$c_1\,\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} \;\le\; \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \qquad \forall\,\boldsymbol{\tau} \in H_0. \tag{2.11}$$

Proof. See[2, Lemma 3.1] or [6, Chapter IV, Proposition 3.1]. □

In other words, the inequality (2.11), being valid only in H_0, explains the need of replacing (2.4) by the variational formulation (2.9). Moreover, the fact that (2.9) is posed in the product space $H_0 \times Q$, instead of $H \times Q$, will also be crucial below in the analysis of the corresponding augmented formulation; see, e.g., inequality (3.4).

3. The augmented dual-mixed variational formulation. We now follow the approach from [8] and enrich the dual-mixed variational formulation (2.9) with Galerkin least squares type terms arising from the constitutive and equilibrium equations, and from the relation defining the rotation as a function of the displacement. Note that the constitutive equation is given now by (2.10). In addition, we also include a suitable boundary term to deal with the non-homogeneous Dirichlet boundary condition. More precisely, we first substract the second from the first equation of (2.9) and then add

$$\kappa_1 \int_\Omega \left(\mathbf{e}(\mathbf{u}) - \mathcal{C}^{-1}\,\boldsymbol{\sigma}\right) : \left(\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\,\boldsymbol{\tau}\right) \;=\; \kappa_1 \left\{ \frac{1}{2\,|\Omega|} \int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} \right\} \int_\Omega \mathbf{I} : \left(\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\,\boldsymbol{\tau}\right),$$

$$\kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) \;=\; -\,\kappa_2 \int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}),$$

$$\kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})\right) : \left(\boldsymbol{\eta} + \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}})\right) \;=\; 0,$$

and

$$\kappa_4 \int_\Gamma \mathbf{u}\cdot\mathbf{v} \;=\; \kappa_4 \int_\Gamma \mathbf{g}\cdot\mathbf{v},$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in H_0 \times [H^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}}$, where $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ is a vector of positive parameters to be specified later, independently of the Lamé constant λ. It is important to observe here that the above terms require now the displacement $\mathbf{u}$ to live in $[H^1(\Omega)]^2$. In addition, it follows easily from (2.3) that

$$\mathrm{tr}\,\left(\mathcal{C}^{-1}\,\boldsymbol{\tau}\right) \;=\; \frac{1}{2\,(\lambda+\mu)}\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in H,$$

and hence for each $\boldsymbol{\tau} \in H_0$ there holds

$$\int_\Omega \mathbf{I} : \left(\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\,\boldsymbol{\tau}\right) \;=\; \int_\Omega \mathrm{tr}\,\left(\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\,\boldsymbol{\tau}\right) \;=\; \int_\Omega \mathrm{div}(\mathbf{v}) \;=\; \int_\Gamma \mathbf{v}\cdot\boldsymbol{\nu}\,.$$

ETNA
Kent State University
etna@mcs.kent.edu

In this way, instead of (2.9) we propose the following augmented dual-mixed variational formulation: Find $(\boldsymbol{\sigma}, \mathbf{u}, \gamma) \in \mathbf{H}_0 := H_0 \times [H^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}}$ such that

$$A((\boldsymbol{\sigma}, \mathbf{u}, \gamma), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;=\; F(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0, \tag{3.1}$$

where the bilinear form $A : \mathbf{H}_0 \times \mathbf{H}_0 \to \mathbb{R}$ and the functional $F : \mathbf{H}_0 \to \mathbb{R}$ are defined by

$$A((\boldsymbol{\sigma}, \mathbf{u}, \gamma), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \gamma : \boldsymbol{\tau} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma}$$

$$+\; \kappa_1 \int_\Omega \left(\mathbf{e}(\mathbf{u}) - \mathcal{C}^{-1}\boldsymbol{\sigma}\right) : \left(\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\boldsymbol{\tau}\right) \;+\; \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{div}(\boldsymbol{\tau})$$

$$+\; \kappa_3 \int_\Omega \left(\gamma - \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})\right) : \left(\boldsymbol{\eta} + \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}})\right) \;+\; \kappa_4 \int_\Gamma \mathbf{u}\cdot\mathbf{v}\,, \tag{3.2}$$

and

$$F(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) := \int_\Omega \mathbf{f}\cdot(\mathbf{v} - \kappa_2\,\mathbf{div}(\boldsymbol{\tau})) + \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle + \kappa_4 \int_\Gamma \mathbf{g}\cdot\mathbf{v} \;+\; \kappa_1\, c_{\mathbf{g}} \int_\Gamma \mathbf{v}\cdot\boldsymbol{\nu}, \tag{3.3}$$

with

$$c_{\mathbf{g}} \;:=\; \left\{ \frac{1}{2\,|\Omega|} \int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} \right\}.$$

As in [8], the idea is to choose $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ so that A becomes strongly coercive and bounded in $\mathbf{H}_0$, with constants independent of λ, with respect to the norm $\|\cdot\|_{\mathbf{H}_0}$ defined by

$$\|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}_0} \;:=\; \left\{ \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} \;+\; \|\mathbf{v}\|^2_{[H^1(\Omega)]^2} \;+\; \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \right\}^{1/2}$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0$. Indeed, proceeding as in [8], using (2.5) and the inverse relation (2.3), and performing some algebraic manipulations, we find that

$$\begin{aligned} &A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \\ &= \frac{1}{2\,\mu}\left(1 - \frac{\kappa_1}{2\,\mu}\right) \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} \;+\; \frac{1}{4(\lambda+\mu)}\left(1 - \frac{\kappa_1}{2(\lambda+\mu)}\right) \int_\Omega \mathrm{tr}^2(\boldsymbol{\tau}) \\ &+\; \kappa_2\, \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \;+\; (\kappa_1 + \kappa_3)\, \|\mathbf{e}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} \;-\; \kappa_3\, |\mathbf{v}|^2_{[H^1(\Omega)]^2} \\ &+\; \kappa_4\, \|\mathbf{v}\|^2_{[L^2(\Gamma)]^2} \;+\; \kappa_3\, \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0\,. \end{aligned}$$

Hence, choosing κ_1 so that $0 < \kappa_1 < 2\,\mu$, which guarantees that $1 - \dfrac{\kappa_1}{2\,\mu} > 0$ and $1 - \dfrac{\kappa_1}{2(\lambda+\mu)} > 0$, and applying the estimate (2.11) (cf. Lemma 2.1), we deduce that

$$\begin{aligned} A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;\geq\; \alpha_2\, \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} \;+\; (\kappa_1 + \kappa_3)\, \|\mathbf{e}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} \;-\; \kappa_3\, |\mathbf{v}|^2_{[H^1(\Omega)]^2} \\ +\; \kappa_4\, \|\mathbf{v}\|^2_{[L^2(\Gamma)]^2} \;+\; \kappa_3\, \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0. \end{aligned} \tag{3.4}$$

where

$$\alpha_2 \,:=\, \min\left\{\alpha_1\, c_1, \frac{\kappa_2}{2}\right\} \quad \text{and} \quad \alpha_1 \,:=\min\left\{\frac{1}{2\,\mu}\left(1-\frac{\kappa_1}{2\,\mu}\right), \frac{\kappa_2}{2}\right\}.$$

We observe here that the only restriction on κ_2 and κ_4 is that both be positive. In particular, as in [8], we can take $\kappa_2 = \frac{1}{\mu}\left(1-\frac{\kappa_1}{2\,\mu}\right)$, whence $\alpha_1 = \frac{\kappa_2}{2}$ and $\alpha_2 = \frac{\kappa_2}{2}\min\{c_1, 1\}$.

In order to determine a suitable choice of κ_3, we need a slight extension of the classical Korn inequality; cf. [5, Theorem 9.2.16]. To this end, we first introduce the space of rigid body motions in Ω, that is

$$\mathbb{RM}(\Omega) \quad := \quad \left\{\mathbf{w} : \Omega \to \mathbb{R}^2 \ : \ \mathbf{w}(\mathbf{x}) = \begin{pmatrix} a \\ b \end{pmatrix} + c\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right.$$
$$\left. \forall\, \mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega\,, \quad a,\, b\,,\, c \,\in\, \mathbb{R} \right\}.$$

It is easy to show (see, e.g., [5, Chapter 9]) that $[H^1(\Omega)]^2 \,=\, Z \,\oplus\, \mathbb{RM}(\Omega)$, where

$$Z \,:=\, \left\{\mathbf{z} \,\in\, [H^1(\Omega)]^2 : \quad \int_\Omega \mathbf{z} \,=\, \mathbf{0} \quad \text{and} \quad \int_\Omega \operatorname{curl}(\mathbf{z}) \,=\, 0\right\}.$$

Also, there exists $C > 0$ such that for each $\mathbf{v} \,=\, \mathbf{z} + \mathbf{w} \,\in\, [H^1(\Omega)]^2$, with $\mathbf{z} \,\in\, Z$ and $\mathbf{w} \,\in\, \mathbb{RM}(\Omega)$, there holds (cf. [5, eq. (9.2.19)])

$$\|\mathbf{z}\|_{[H^1(\Omega)]^2} \,+\, \|\mathbf{w}\|_{[H^1(\Omega)]^2} \,\leq\, C\,\|\mathbf{v}\|_{[H^1(\Omega)]^2}. \tag{3.5}$$

In addition, the second Korn inequality (cf. Theorem 9.2.12 in [5]) establishes the existence of $C > 0$ such that

$$\|\mathbf{e}(\mathbf{z})\|_{[L^2(\Omega)]^{2\times 2}} \,\geq\, C\,\|\mathbf{z}\|_{[H^1(\Omega)]^2} \qquad \forall\,\mathbf{z} \,\in\, Z\,. \tag{3.6}$$

In this way, we are able to prove now the following result.

LEMMA 3.1. *Let $p : [H^1(\Omega)]^2 \to \mathbb{R}$ be a continuous mapping such that*
a) $p(\mathbf{v}) \,\geq\, 0$ *for each* $\mathbf{v} \,\in\, [H^1(\Omega)]^2$,
b) $p(\alpha\,\mathbf{v}) \,=\, \alpha\, p(\mathbf{v})$ *for each* $\alpha > 0$, *and*
c) $\{\, \mathbf{w} \,\in\, \mathbb{RM}(\Omega) : \quad p(\mathbf{w}) \,=\, 0\,\} \,=\, \{\mathbf{0}\}$.
Then, there exists $\kappa > 0$ such that

$$\|\mathbf{e}(\mathbf{v})\|_{[L^2(\Omega)]^{2\times 2}} \,+\, p(\mathbf{v}) \,\geq\, \kappa\,\|\mathbf{v}\|_{[H^1(\Omega)]^2} \qquad \forall\,\mathbf{v} \,\in\, [H^1(\Omega)]^2.$$

Proof. We employ analogue arguments to those given in the proof of [5, Theorem 9.2.16]. Indeed, we assume by contradiction that such a constant κ does not exist. Hence, we can construct a sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \,\subseteq\, [H^1(\Omega)]^2$ such that

$$\|\mathbf{v}_n\|_{[H^1(\Omega)]^2} \,=\, 1 \tag{3.7}$$

and

$$\|\mathbf{e}(\mathbf{v}_n)\|_{[L^2(\Omega)]^{2\times 2}} \,+\, p(\mathbf{v}_n) \,<\, \frac{1}{n}\,. \tag{3.8}$$

ETNA
Kent State University
etna@mcs.kent.edu

Note that the hypothesis b) is used here. Next, we let $\{\mathbf{z}_n\}_{n\in\mathbb{N}} \subseteq Z$ and $\{\mathbf{w}_n\}_{n\in\mathbb{N}} \subseteq \mathbb{RM}(\Omega)$ be such that $\mathbf{v}_n = \mathbf{z}_n + \mathbf{w}_n$ for each $n \in \mathbb{N}$. It follows that $\mathbf{e}(\mathbf{v}_n) = \mathbf{e}(\mathbf{z}_n)$, and then, thanks to the inequalities (3.6) and (3.8), and the hypothesis a), we obtain

$$C\,\|\mathbf{z}_n\|_{[H^1(\Omega)]^2} \;\le\; \|\mathbf{e}(\mathbf{v}_n)\|_{[L^2(\Omega)]^{2\times 2}} \;<\; \frac{1}{n}\,,$$

which shows that $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ converges to the null function in $[H^1(\Omega)]^2$. Now, it is clear from (3.5) and (3.7) that $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ is bounded, and hence there exist a subsequence $\{\mathbf{w}_n^{(1)}\}_{n\in\mathbb{N}} \subseteq \{\mathbf{w}_n\}_{n\in\mathbb{N}}$ and $\mathbf{w} \in \mathbb{RM}(\Omega)$ such that $\{\mathbf{w}_n^{(1)}\}_{n\in\mathbb{N}}$ converges to $\mathbf{w}$ in $[H^1(\Omega)]^2$. Thus, the subsequence $\{\mathbf{v}_n^{(1)}\}_{n\in\mathbb{N}} := \{\mathbf{z}_n^{(1)} + \mathbf{w}_n^{(1)}\}_{n\in\mathbb{N}}$ also converges to $\mathbf{w}$, whence (3.7), (3.8), and the continuity of p imply

$$\|\mathbf{w}\| \;=\; 1 \quad \text{and} \quad p(\mathbf{w}) \;=\; 0\,.$$

The above certainly contradicts hypothesis c), which ends the proof. □

Next, we consider in particular $p : [H^1(\Omega)]^2 \to \mathbb{R}$ defined by $p(\mathbf{v}) = \|\mathbf{v}\|_{[L^2(\Gamma)]^2}$. It is easy to see that p satisfies the hypotheses a), b), and c) of Lemma 3.1. In addition, applying the trace theorem in $[H^1(\Omega)]^2$ we obtain

$$|\,p(\mathbf{v}) - p(\tilde{\mathbf{v}})\,| \;=\; \Big|\,\|\mathbf{v}\|_{[L^2(\Gamma)]^2} - \|\tilde{\mathbf{v}}\|_{[L^2(\Gamma)]^2}\,\Big| \;\le\; \|\mathbf{v} - \tilde{\mathbf{v}}\|_{[L^2(\Gamma)]^2} \;\le\; C\,\|\mathbf{v} - \tilde{\mathbf{v}}\|_{[H^1(\Omega)]^2}$$

for all $\mathbf{v}\,,\ \tilde{\mathbf{v}} \in [H^1(\Omega)]^2$, which guarantees the continuity of p. Therefore, a straightforward application of Lemma 3.1 yields the existence of $\kappa_0 > 0$ such that

$$\|\mathbf{e}(\mathbf{v})\|_{[L^2(\Omega)]^{2\times 2}} \;+\; \|\mathbf{v}\|_{[L^2(\Gamma)]^2} \;\ge\; \kappa_0\,\|\mathbf{v}\|_{[H^1(\Omega)]^2} \qquad \forall\,\mathbf{v} \in [H^1(\Omega)]^2. \tag{3.9}$$

Alternatively, though Lemma 3.1 is certainly more general than (3.9), in what follows we show that this inequality can also be obtained as a direct application of Peetre-Tartar Lemma (see, e.g., [9, Chapter I, Theorem 2.1]), which reads as follows.

LEMMA 3.2. *Let $(E_1, \|\cdot\|_1)$, $(E_2, \|\cdot\|_2)$, and $(E_3, \|\cdot\|_3)$ be Banach spaces, and let $A : E_1 \to E_2$ and $B : E_1 \to E_3$ be bounded linear operators such that B is compact. Assume that there exists $C > 0$ such that*

$$\|v\|_1 \;\le\; C\,\big\{\,\|A(v)\|_2 \;+\; \|B(v)\|_3\,\big\} \qquad \forall\, v \in E_1. \tag{3.10}$$

Then there holds:

1) *the null space $N(A)$ of A is finite dimensional, A is an isomorphism from $E_1/N(A)$ onto the range $R(A)$ of A, and $R(A)$ is a closed subspace of E_2, that is there exists $C_1 > 0$ such that*

$$dist(v, N(A)) \;:=\; \inf_{z\in N(A)} \|v - z\|_1 \;\le\; C_1\,\|A(v)\|_2 \qquad \forall\, v \in E_1. \tag{3.11}$$

2) *if $(E_4, \|\cdot\|_4)$ is a Banach space and $L : E_1 \to E_4$ is a bounded linear operator that vanishes on $N(A)$, then there exists $C_2 > 0$ such that*

$$\|L(v)\|_4 \;\le\; C_2\,\|L\|\,\|A(v)\|_2 \qquad \forall\, v \in E_1.$$

3) *if $(E_5, \|\cdot\|_5)$ is a Banach space and $M : E_1 \to E_5$ is a bounded linear operator that satisfies $M(v) \neq 0$ for each $v \in N(A) - \{0\}$, then there exists $C_3 > 0$ such that*

$$\|v\|_1 \;\le\; C_3\,\big\{\,\|A(v)\|_2 \;+\; \|M(v)\|_5\,\big\} \qquad \forall\, v \in E_1. \tag{3.12}$$

Actually, we prove next that (3.9) follows by applying either 1) or 3) from Lemma 3.2. In fact, we first let $E_1 := [H^1(\Omega)]^2$, $E_2 := [L^2(\Omega)]^{2\times 2} \times [L^2(\Gamma)]^2$, $E_3 := \mathbb{RM}(\Omega)$, and define the bounded linear operators $A : E_1 \to E_2$ and $B : E_1 \to E_3$ as

$$A(\mathbf{v}) := (\mathbf{e}(\mathbf{v}), \mathbf{v}|_\Gamma) \quad \text{and} \quad B(\mathbf{v}) := \mathbf{w} \quad \forall\, \mathbf{v} = \mathbf{z} + \mathbf{w} \in [H^1(\Omega)]^2 = Z \oplus \mathbb{RM}(\Omega).$$

The boundedness of B follows from (3.5) and it is clear that this operator is compact. Then, using triangle inequality, (3.6), and the fact that $\mathbf{e}(\mathbf{v}) = \mathbf{e}(\mathbf{z})$, we find that

$$\|\mathbf{v}\|_{[H^1(\Omega)]^2} \leq \|\mathbf{z}\|_{[H^1(\Omega)]^2} + \|\mathbf{w}\|_{[H^1(\Omega)]^2} \leq C \left\{ \|\mathbf{e}(\mathbf{v})\|_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{w}\|_{[H^1(\Omega)]^2} \right\}, \tag{3.13}$$

which yields hypothesis (3.10). In this way, noting that $N(A) = \{0\}$, the inequality (3.11) becomes (3.9).

On the other hand, in order to apply 3) we let $E_1 := [H^1(\Omega)]^2$, $E_2 := [L^2(\Omega)]^{2\times 2}$, $E_3 := \mathbb{RM}(\Omega)$, $E_5 := [L^2(\Gamma)]^2$, and define the bounded linear operators $A : E_1 \to E_2$ and $M : E_1 \to E_5$ as

$$A(\mathbf{v}) := \mathbf{e}(\mathbf{v}) \quad \text{and} \quad M(\mathbf{v}) := \mathbf{v}|_\Gamma \quad \forall\, \mathbf{v} \in [H^1(\Omega)]^2.$$

In addition, we let $B : E_1 \to E_3$ be as before. Then, the estimate (3.13) corresponds exactly to hypothesis (3.10) and M does not vanish in $N(A) - \{0\} = \mathbb{RM}(\Omega) - \{0\}$. Therefore, it is easy to see that in this case (3.12) is nothing but (3.9).

It is interesting to observe here that the contradiction argument used in the proof of Lemma 3.1, which is adapted from [5, Theorem 9.2.16], resembles the arguments employed to prove the statements 1) and 3) of Lemma 3.2; see [9, Chap. I, Theorem 2.1].

We now go back to the process of choosing parameters that yield a strongly coercive bilinear form A. It follows from (3.4) and (3.9) that

$$A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \geq \alpha_2 \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} (\alpha_3\, \kappa_0 - \kappa_3) \|\mathbf{v}\|^2_{[H^1(\Omega)]^2} + \kappa_3 \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \tag{3.14}$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0$, where $\alpha_3 := \min\{\kappa_1 + \kappa_3, \kappa_4\}$. Hence, we just take for simplicity $\kappa_4 \geq \kappa_1 + \kappa_3$ so that α_3 becomes $\kappa_1 + \kappa_3$ and, in this way, the choice of κ_3 is determined by the value of κ_0. More precisely, if $\kappa_0 \geq 1$ it suffices to take any $\kappa_3 > 0$, whereas if $\kappa_0 < 1$ we choose κ_3 so that $0 < \kappa_3 < \left(\dfrac{\kappa_0}{1 - \kappa_0}\right) \kappa_1$.

At this point we remark that the introduction of the equation

$$\kappa_4 \int_\Gamma \mathbf{u} \cdot \mathbf{v} = \kappa_4 \int_\Gamma \mathbf{g} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in [H^1(\Omega)]^2,$$

in the augmented formulation (3.1), allows us to employ the inequality (3.9), which yields the term $\|\mathbf{v}\|^2_{[H^1(\Omega)]^2}$ in the estimate (3.14).

On the other hand, applying the Cauchy-Schwarz inequality to each term on the right hand side of (3.2), and noting from (2.3) that $\|\mathcal{C}^{-1}\,\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}} \leq \dfrac{1}{2\,\mu} \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}$ for all $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$, we find that A is bounded with a constant depending only on μ and the parameters κ_1, κ_2, κ_3, and κ_4.

We have thus proved the following main result.

THEOREM 3.3. *Assume that* $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ *is independent of* λ *and such that* $0 < \kappa_1 < 2\,\mu$, $0 < \kappa_2$, $0 < \kappa_3 < \left(\dfrac{\kappa_0}{1 - \kappa_0}\right) \kappa_1$ *(if* $\kappa_0 < 1$*) or* $\kappa_3 > 0$ *(if* $\kappa_0 \geq 1$*), and* $\kappa_4 \geq \kappa_1 + \kappa_3$. *Then, there exist positive constants* M, α, *independent of* λ, *such that*

$$|\, A((\boldsymbol{\sigma}, \mathbf{u}, \gamma), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}))\,| \leq M\, \|(\boldsymbol{\sigma}, \mathbf{u}, \gamma)\|_{\mathbf{H}_0} \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}_0},$$

and

$$A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;\geq\; \alpha \, \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|^2_{\mathbf{H}_0}$$

for all $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})$, $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0$*. In particular, taking* $\kappa_1 = C_1\,\mu$*, with any* $C_1 \in]0, 2[$, $\kappa_2 = \dfrac{1}{\mu}\left(1 - \dfrac{\kappa_1}{2\,\mu}\right)$, $\kappa_3 = C_3\,\kappa_1$*, with any* $C_3 \in]0, \frac{\kappa_0}{1-\kappa_0}[$ *if* $\kappa_0 < 1$*, or* $\kappa_3 = \kappa_1$ *if* $\kappa_0 \geq 1$*, and* $\kappa_4 = \kappa_1 + \kappa_3$*, yields* M *and* α *depending only on* μ, $\dfrac{1}{\mu}$, κ_0*, and* c_1*.*

In addition, the well posedness of (3.1) is now easily established.

THEOREM 3.4. *Assume the same hypotheses of Theorem 3.3. Then the augmented variational formulation (3.1) has a unique solution* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$*, and there exists a positive constant* C*, independent of* λ*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})\|_{\mathbf{H}_0} \;\leq\; C\,\|F\| \;\leq\; C\,\left\{\, \|\mathbf{f}\|_{[L^2(\Omega)]^2} \,+\, \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2} \,\right\}.$$

Proof. The linear functional F (see (3.3)) is certainly bounded as indicated with a constant C depending only on κ_1, κ_2, and κ_4. Therefore, the present proof is a simple consequence of Theorem 3.3 and the well known Lax-Milgram Lemma. □

4. The augmented mixed finite element method.

Let κ_1, κ_2, κ_3, and κ_4 be the parameters employed in the formulation (3.1), which satisfy the assumptions of Theorem 3.3. Then, given a finite element subspace $\mathbf{H}_{h,0} \subseteq \mathbf{H}_0$, the Galerkin scheme associated with (3.1) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{h,0}$ such that

$$A((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) \;=\; F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \qquad \forall\,(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{h,0}, \tag{4.1}$$

where A and F are defined by (3.2) and (3.3), respectively. Since A is bounded and strongly coercive on the whole space $\mathbf{H}_0$ (cf. Theorem 3.3), the well-posedness of (4.1) is guaranteed with any arbitrary choice of the subspace $\mathbf{H}_{h,0}$. More precisely, we have the following main result.

THEOREM 4.1. *Assume that the parameters* κ_1, κ_2, κ_3*, and* κ_4 *satisfy the assumptions of Theorem 3.3 and let* $\{\,\mathbf{H}_{h,0}\,\}_{h>0}$ *be a family of finite element subspaces of* $\mathbf{H}_0$*. Then, for each* $h > 0$ *the Galerkin scheme (4.1) has a unique solution* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{h,0}$*, and there exist positive constants* C, $\tilde{C}$*, independent of* λ *and* h*, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \;\leq\; C \sup_{\substack{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{h,0} \\ (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \neq 0}} \frac{|\,F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\,|}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0}} \;\leq\; C\,\left\{\, \|\mathbf{f}\|_{[L^2(\Omega)]^2} \,+\, \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2} \,\right\},$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \;\leq\; \tilde{C} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{h,0}} \|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0}. \tag{4.2}$$

Proof. It follows straightforwardly from Theorem 3.3 and the Lax-Milgram Lemma. □

As usual, the Cea estimate (4.2) and the approximation properties of the subspaces involved are the key ingredients to derive the corresponding rate of convergence of the Galerkin scheme. In order to introduce a specific $\mathbf{H}_{h,0}$, we now let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T with mesh size $h := \max\{h_T : T \in \mathcal{T}_h\}$, and such that there holds $\bar{\Omega} := \cup\{\,T : \; T \in \mathcal{T}_h\,\}$. Also, given an integer $\ell \geq 0$ and $T \in \mathcal{T}_h$, we denote by $\mathbb{P}_\ell(T)$ the space of polynomials in two variables defined in T of total degree at most ℓ. In addition, we let $\mathbb{RT}_0(T)$ be the local Raviart-Thomas space of order zero, that is

$$\mathbb{RT}_0(T) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \subseteq [\mathbb{P}_1(T)]^2,$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbb{R}^2$. Then, we define the finite element subspaces

$$H_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in [\mathbb{RT}_0(T)^{\mathtt{t}}]^2 \qquad \forall\, T \in \mathcal{T}_h \right\},$$

$$H_{h,0}^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) = 0 \right\},$$

$$H_h^{\mathbf{u}} := \left\{ \mathbf{v}_h \in [C(\bar\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_1(T)]^2 \quad \forall\, T \in \mathcal{T}_h \right\},$$

and

$$H_h^{\boldsymbol{\gamma}} := \left\{ \boldsymbol{\eta}_h \in [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}} : \quad \boldsymbol{\eta}_h|_T \in [\mathbb{P}_0(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h \right\}.$$

The corresponding approximation properties are as follows (see [6, 7, 8]):

($\mathrm{AP}_{h,0}^{\boldsymbol{\sigma}}$) *For each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2} \cap H_0$ with $\mathbf{div}(\boldsymbol{\tau}) \in [H^1(\Omega)]^2$ there exists $\boldsymbol{\tau}_h \in H_{h,0}^{\boldsymbol{\sigma}}$ such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \le C\,h \left\{ \|\boldsymbol{\tau}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|_{[H^1(\Omega)]^2} \right\}.$$

($\mathrm{AP}_h^{\mathbf{u}}$) *For each $\mathbf{v} \in [H^2(\Omega)]^2$ there exists $\mathbf{v}_h \in H_h^{\mathbf{u}}$ such that*

$$\|\mathbf{v} - \mathbf{v}_h\|_{[H^1(\Omega)]^2} \le C\,h\,\|\mathbf{v}\|_{[H^2(\Omega)]^2}.$$

($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) *For each $\boldsymbol{\eta} \in [H^1(\Omega)]^{2\times 2}_{\mathtt{skew}}$ there exists $\boldsymbol{\eta}_h \in H_h^{\boldsymbol{\gamma}}$ such that*

$$\|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{[L^2(\Omega)]^{2\times 2}} \le C\,h\,\|\boldsymbol{\eta}\|_{[H^1(\Omega)]^{2\times 2}}.$$

Consequently, we are able to establish the following result.

THEOREM 4.2. *Let $\mathbf{H}_{h,0} := H_{h,0}^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}}$ and let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{h,0}$ be the unique solutions of the continuous and discrete augmented mixed formulations (3.1) and (4.1), respectively. Assume that $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, $\mathbf{u} \in [H^{r+1}(\Omega)]^2$, and $\boldsymbol{\gamma} \in [H^r(\Omega)]^{2\times 2}$, for some $r \in (0,1]$. Then there exists $C > 0$, independent of λ and h, such that*

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \\
&\le C\,h^r \left\{ \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^r(\Omega)]^2} + \|\mathbf{u}\|_{[H^{r+1}(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^r(\Omega)]^{2\times 2}} \right\}.
\end{aligned}$$

Proof. It is a consequence of Cea's estimate, approximation properties ($\mathrm{AP}_{h,0}^{\boldsymbol{\sigma}}$), ($\mathrm{AP}_h^{\mathbf{u}}$), and ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$), and suitable interpolation theorems in the corresponding function spaces. ☐

On the other hand, in order to deal with the mean value condition required by the traces of the elements in $H_{h,0}^{\boldsymbol{\sigma}}$, we proceed as in [8] and consider, instead of (4.1), the modified discrete scheme: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \rho_h) \in H_h^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ such that

$$\begin{aligned}
A((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) + \rho_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h), \\
\chi_h \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) &= 0,
\end{aligned} \tag{4.3}$$

ETNA
Kent State University
etna@mcs.kent.edu

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h, \chi_h) \in H_h^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}} \times \mathbb{R}$. In this way, the Lagrange multiplier $\rho_h \in \mathbb{R}$ and the corresponding test constants $\chi_h \in \mathbb{R}$ take care of the above mentioned mean value condition, whence (4.1) and (4.3) become equivalent, as it is established in the following theorem.

THEOREM 4.3. *Let* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{h,0}$ *be the solution of* (4.1) *and define*

$$\rho_h := \frac{1}{2\,|\Omega|} \left\{ \int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} - \frac{\kappa_1}{2\,(\lambda+\mu)} \int_\Gamma \mathbf{u}_h\cdot\boldsymbol{\nu} \right\}. \tag{4.4}$$

Then $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \rho_h)$ *is a solution of the system* (4.3). *Conversely, if* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \rho_h) \in H_h^{\boldsymbol{\sigma}} \times H_{h,0}^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ *is a solution of* (4.3)*, then* ρ_h *is given by* (4.4) *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ *becomes the solution of* (4.1).

Proof. According to the definitions of A and F (cf. (3.2) and (3.3)) we easily find that

$$A((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\mathbf{I},0,0)) = \frac{1}{2(\lambda+\mu)}\left(1-\frac{\kappa_1}{2(\lambda+\mu)}\right)\int_\Omega \operatorname{tr}(\boldsymbol{\tau}) + \frac{\kappa_1}{2(\lambda+\mu)}\int_\Gamma \mathbf{v}\cdot\boldsymbol{\nu}$$

for all $(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) \in H \times [H^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}}$, and

$$F(\mathbf{I},0,0) = \int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu}.$$

The rest of the proof, being based on the above identities and the fact that $\mathbf{I}$ belongs to $H_h^{\boldsymbol{\sigma}}$, is similar to the proof of [8, Theorem 4.3], and hence we omit further details. □

5. Numerical results. In this section we provide numerical results illustrating the performance of the augmented mixed finite element scheme (4.3) on a finite sequence of uniform triangulations of the domain. We begin with some notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (4.3), which behaves asymptotically as 5 times the number of triangles of each triangulation; see [8] for details. Also, the individual and total errors are denoted by

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}, \qquad e(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{[H^1(\Omega)]^2}, \quad e(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{[L^2(\Omega)]^{2\times 2}},$$

and

$$e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}) := \left\{ [e(\boldsymbol{\sigma})]^2 + [e(\mathbf{u})]^2 + [e(\boldsymbol{\gamma})]^2 \right\}^{1/2}.$$

Next, we recall that given the Young modulus E and the poisson ratio ν of a linear elastic material, the corresponding Lamé constants are defined as

$$\mu := \frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda := \frac{E\,\nu}{(1+\nu)\,(1-2\,\nu)}.$$

In the examples we fix $E = 1$ and take $\nu = 0.4900$ and $\nu = 0.4999$, which give the following values of μ and λ:

ν	μ	λ
0.4900	0.3356	16.4430
0.4999	0.3333	1666.4444

In addition, according to Theorem 3.3, we consider the parameters

$$\kappa_1 = \mu, \quad \kappa_2 = \frac{1}{2\,\mu}, \quad \kappa_3 = \frac{1}{8}\kappa_1 \quad \text{and} \quad \kappa_4 = \kappa_1 + \kappa_3, \tag{5.1}$$

which corresponds to choosing $C_1 = 1$ and $C_3 = \frac{1}{8}$. Certainly, since κ_0 is unknown, we have assumed here that $\frac{1}{8} < \frac{\kappa_0}{1-\kappa_0}$. As we observe in the tables and figures below, the present choice of κ_3 works fine. Otherwise, we would have to decrease the value of C_3.

We now specify the two examples to be presented here. We take Ω as the square domain $]0,1[\times]0,1[$ and chose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution $\mathbf{u} := (u_1, u_2)^{\mathtt{t}}$ is given as follows:

EXAMPLE	$u_1(x_1,x_2) = u_2(x_1,x_2)$
1	$(x_1^2+1)\,(x_2^2+1)\,e^{x_1+x_2}$
2	$20\,\sin(x_1+x_2) + x_1^3 + 2\,x_1\,x_2 + x_2^2 + 10$

The numerical results given below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a fortran code. The Galerkin scheme (4.3) is implemented in this code following [8, Section 4.3], and it is solved by a direct method. The individual errors are computed on each triangle using a Gaussian quadrature rule.

In Tables 5.1 – 5.4 we present the individual and total errors of each example for a sequence of uniform meshes going up to $N = 208515$. Then, in Figures 5.1 – 5.4 we display the log-log curves of the number of unknowns N versus the meshsize h and the errors. We observe there that the rate of convergence $O(h)$ predicted by Theorem 4.2 (when $r = 1$) is attained in both examples. We also notice from these figures that the convergence of $e(\mathbf{u})$ is even faster than $O(h)$, specially for Example 2, which, however, is just a special behaviour of these particular solutions $\mathbf{u}$. In addition, we deduce from the tables that the dominant component of the total error is given by $e(\boldsymbol{\sigma})$, which is actually a quite frequent fact in many mixed finite element schemes. This feature is also evident from the figures, where one sees that the curves corresponding to $e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$ and $e(\boldsymbol{\sigma})$ do not distinguish from each other. Finally, in Figures 5.5 – 5.8 we display some components of the approximate and exact solutions of both examples for $N = 41475$.

Summarizing, the numerical results presented here constitute enough support to consider our augmented mixed finite element scheme as a valid and competive alternative to solve the linear elasticity problem with non-homogeneous Dirichlet boundary conditions.

Acknowledgments. The author is thankful to Antonio Márquez for performing the computational code and running the numerical examples. Also, the author expresses his gratitude to one of the referees for pointing out the application of Peetre-Tartar Lemma to prove the modified Korn inequality (3.9).

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ETNA
Kent State University
etna@mcs.kent.edu

TABLE 5.1
*Degrees of freedom, meshsizes, and errors (*EXAMPLE 1*, $\nu = 0.4900$)*

N	h	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$
2691	0.06250	0.4838E+02	0.4377E+01	0.2889E+01	0.4866E+02
4163	0.05000	0.3870E+02	0.3117E+01	0.2433E+01	0.3890E+02
5955	0.04167	0.3225E+02	0.2353E+01	0.2108E+01	0.3241E+02
8067	0.03571	0.2765E+02	0.1851E+01	0.1862E+01	0.2777E+02
10499	0.03125	0.2419E+02	0.1502E+01	0.1667E+01	0.2430E+02
13251	0.02778	0.2150E+02	0.1249E+01	0.1509E+01	0.2159E+02
16323	0.02500	0.1935E+02	0.1059E+01	0.1378E+01	0.1943E+02
23427	0.02083	0.1613E+02	0.7958E+00	0.1173E+01	0.1619E+02
31811	0.01786	0.1382E+02	0.6262E+00	0.1020E+01	0.1388E+02
41475	0.01562	0.1210E+02	0.5097E+00	0.9020E+00	0.1214E+02
64643	0.01250	0.9677E+01	0.3633E+00	0.7317E+00	0.9711E+01
92931	0.01042	0.8064E+01	0.2774E+00	0.6150E+00	0.8092E+01
126339	0.00892	0.6912E+01	0.2220E+00	0.5301E+00	0.6936E+01
164867	0.00781	0.6048E+01	0.1838E+00	0.4657E+00	0.6069E+01
208515	0.00694	0.5376E+01	0.1563E+00	0.4151E+00	0.5394E+01

TABLE 5.2
*Degrees of freedom, meshsizes, and errors (*EXAMPLE 1, $\nu = 0.4999$*)*

N	h	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$
2691	0.06250	0.4738E+04	0.4286E+03	0.2817E+03	0.4766E+04
4163	0.05000	0.3791E+04	0.3033E+03	0.2341E+03	0.3810E+04
5955	0.04167	0.3159E+04	0.2274E+03	0.2014E+03	0.3174E+04
8067	0.03571	0.2708E+04	0.1775E+03	0.1772E+03	0.2719E+04
10499	0.03125	0.2369E+04	0.1429E+03	0.1583E+03	0.2379E+04
13251	0.02778	0.2106E+04	0.1178E+03	0.1431E+03	0.2114E+04
16323	0.02500	0.1895E+04	0.9896E+02	0.1305E+03	0.1903E+04
23427	0.02083	0.1580E+04	0.7299E+02	0.1110E+03	0.1585E+04
31811	0.01786	0.1354E+04	0.5630E+02	0.9649E+02	0.1359E+04
41475	0.01562	0.1185E+04	0.4488E+02	0.8531E+02	0.1189E+04
64643	0.01250	0.9477E+03	0.3065E+02	0.6919E+02	0.9508E+03
92931	0.01042	0.7898E+03	0.2240E+02	0.5816E+02	0.7922E+03
126339	0.00892	0.6770E+03	0.1715E+02	0.5013E+02	0.6790E+03
164867	0.00781	0.5923E+03	0.1361E+02	0.4404E+02	0.5941E+03
208515	0.00694	0.5265E+03	0.1108E+02	0.3926E+02	0.5281E+03

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ETNA
Kent State University
etna@mcs.kent.edu

TABLE 5.3
*Degrees of freedom, meshsizes, and errors (*EXAMPLE 2, $\nu = 0.4900$*)*

N	h	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$
2691	0.06250	0.1443E+02	0.2299E+01	0.3582E+01	0.1504E+02
4163	0.05000	0.1154E+02	0.1644E+01	0.3028E+01	0.1204E+02
5955	0.04167	0.9615E+01	0.1249E+01	0.2615E+01	0.1004E+02
8067	0.03571	0.8241E+01	0.9896E+00	0.2296E+01	0.8611E+01
10499	0.03125	0.7210E+01	0.8092E+00	0.2043E+01	0.7537E+01
13251	0.02778	0.6408E+01	0.6778E+00	0.1839E+01	0.6701E+01
16323	0.02500	0.5767E+01	0.5787E+00	0.1671E+01	0.6032E+01
23427	0.02083	0.4806E+01	0.4406E+00	0.1411E+01	0.5028E+01
31811	0.01786	0.4119E+01	0.3504E+00	0.1220E+01	0.4310E+01
41475	0.01562	0.3604E+01	0.2877E+00	0.1074E+01	0.3772E+01
64643	0.01250	0.2883E+01	0.2077E+00	0.8661E+00	0.3017E+01
92931	0.01042	0.2402E+01	0.1597E+00	0.7250E+00	0.2514E+01
126339	0.00892	0.2059E+01	0.1283E+00	0.6233E+00	0.2155E+01
164867	0.00781	0.1802E+01	0.1064E+00	0.5465E+00	0.1886E+01
208515	0.00694	0.1601E+01	0.9046E-01	0.4864E+00	0.1676E+01

TABLE 5.4
*Degrees of freedom, meshsizes, and errors (*EXAMPLE 2, $\nu = 0.4999$*)*

N	h	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$
2691	0.06250	0.1424E+04	0.2207E+03	0.3373E+03	0.1480E+04
4163	0.05000	0.1139E+04	0.1566E+03	0.2853E+03	0.1185E+04
5955	0.04167	0.9494E+03	0.1181E+03	0.2465E+03	0.9880E+03
8067	0.03571	0.8137E+03	0.9286E+02	0.2165E+03	0.8472E+03
10499	0.03125	0.7120E+03	0.7536E+02	0.1928E+03	0.7414E+03
13251	0.02778	0.6328E+03	0.6265E+02	0.1736E+03	0.6592E+03
16323	0.02500	0.5695E+03	0.5308E+02	0.1578E+03	0.5933E+03
23427	0.02083	0.4745E+03	0.3980E+02	0.1333E+03	0.4945E+03
31811	0.01786	0.4067E+03	0.3118E+02	0.1153E+03	0.4239E+03
41475	0.01562	0.3559E+03	0.2522E+02	0.1015E+03	0.3709E+03
64643	0.01250	0.2847E+03	0.1766E+02	0.8189E+02	0.2967E+03
92931	0.01042	0.2372E+03	0.1319E+02	0.6856E+02	0.2473E+03
126339	0.00892	0.2033E+03	0.1029E+02	0.5895E+02	0.2119E+03
164867	0.00781	0.1779E+03	0.8298E+01	0.5168E+02	0.1854E+03
208515	0.00694	0.1581E+03	0.6859E+01	0.4601E+02	0.1648E+03

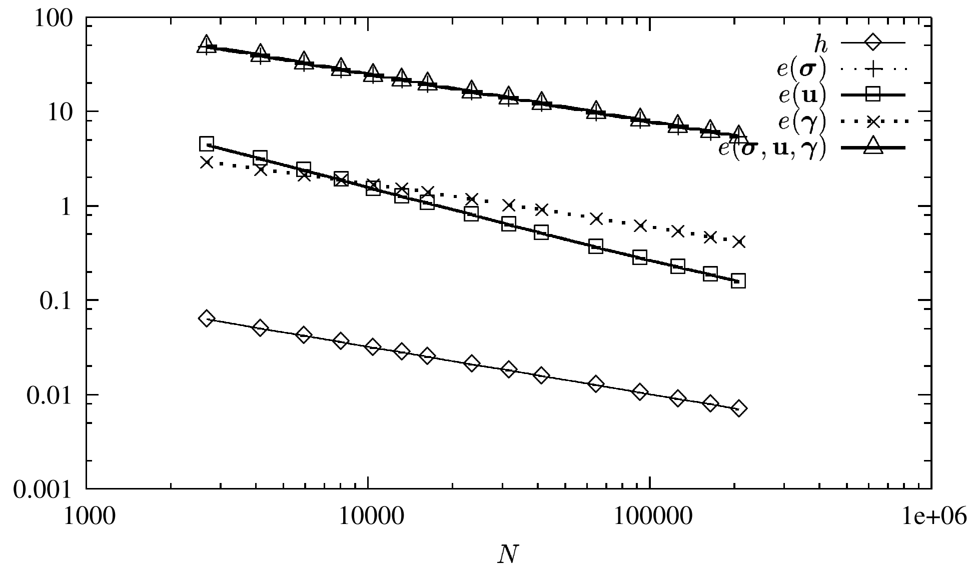

FIG. 5.1. *Meshsizes and errors vs. degrees of freedom (*EXAMPLE 1*, $\nu = 0.4900$)*

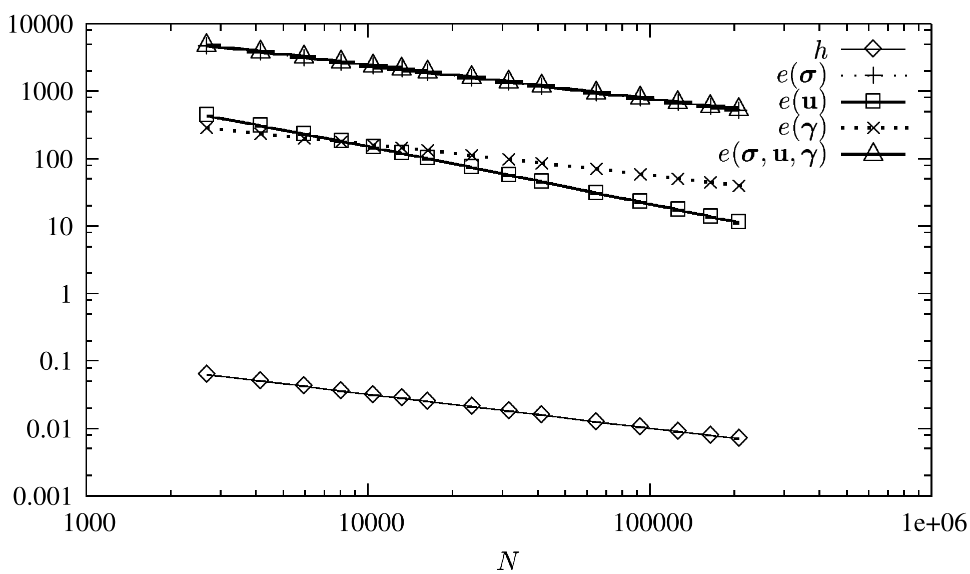

FIG. 5.2. *Meshsizes and errors vs. degrees of freedom (*EXAMPLE 1*, $\nu = 0.4999$)*

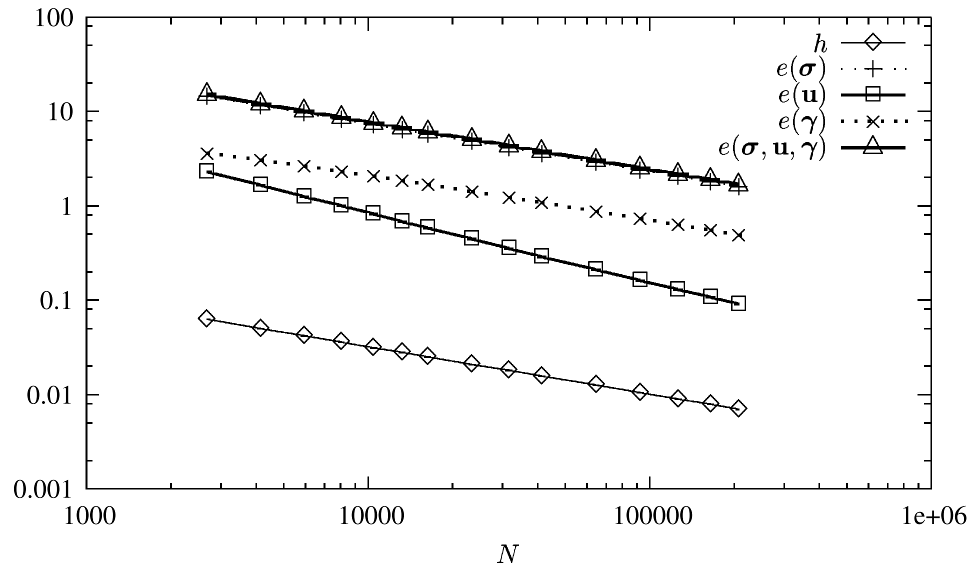

FIG. 5.3. *Meshsizes and errors vs. degrees of freedom (*EXAMPLE 2*, $\nu = 0.4900$)*

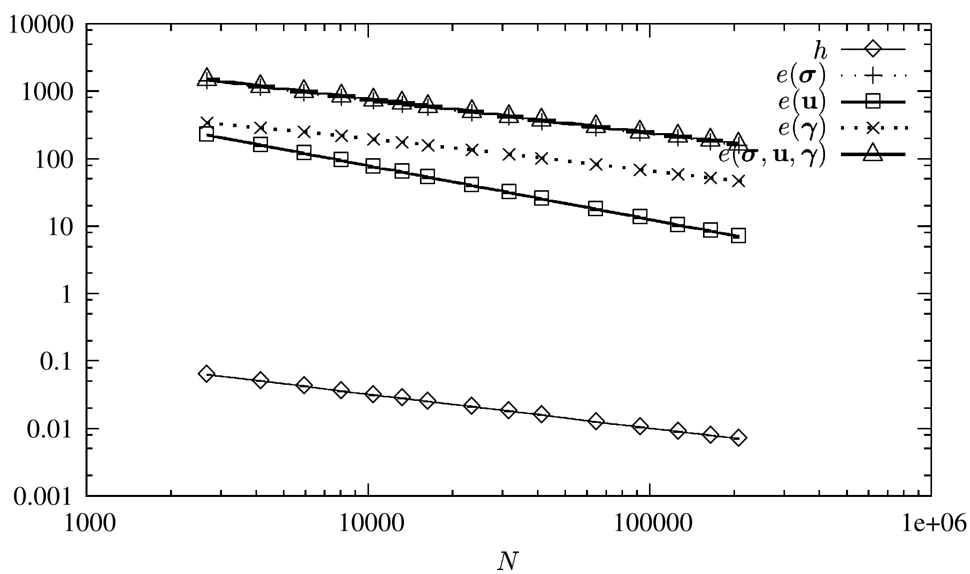

FIG. 5.4. *Meshsizes and errors vs. degrees of freedom (*EXAMPLE 2*, $\nu = 0.4999$)*

ETNA
Kent State University
etna@mcs.kent.edu

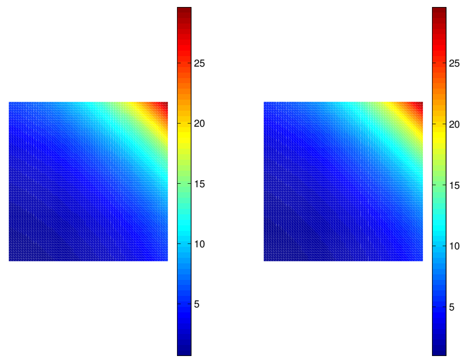

FIG. 5.5. *Approximate (left) and exact* u_2 *(*EXAMPLE 1*,* $\nu = 0.4900$*,* $N = 41475$*)*

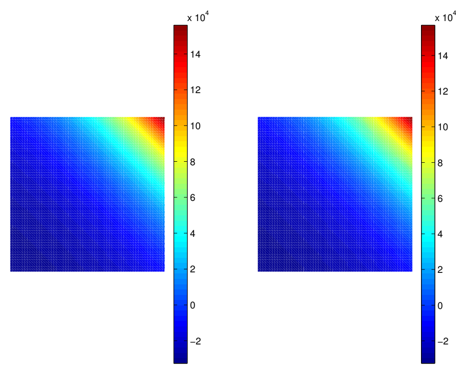

FIG. 5.6. *Approximate (left) and exact* σ_{11} *(*EXAMPLE 1*,* $\nu = 0.4999$*,* $N = 41475$*)*

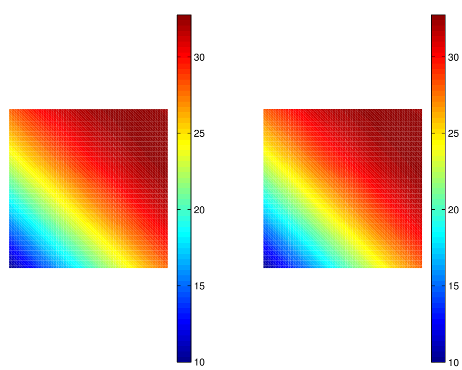

FIG. 5.7. *Approximate (left) and exact* u_1 *(*EXAMPLE 2*,* $\nu = 0.4900$*,* $N = 41475$*)*

ETNA
Kent State University
etna@mcs.kent.edu

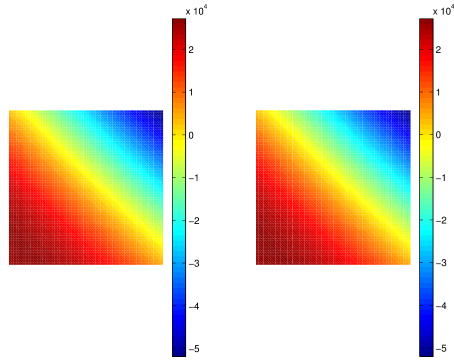

FIG. 5.8. *Approximate (left) and exact σ_{22} (EXAMPLE 2, $\nu = 0.4999$, $N = 41475$)*