Electronic Transactions on Numerical Analysis.
Volume 26, pp. 1-33, 2007.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

A STRUCTURED STAIRCASE ALGORITHM FOR SKEW-SYMMETRIC/SYMMETRIC PENCILS*

RALPH BYERS†, VOLKER MEHRMANN‡, AND HONGGUO XU§

Abstract. We present structure preserving algorithms for the numerical computation of structured staircase forms of skew-symmetric/symmetric matrix pencils along with the Kronecker indices of the associated skew-symmetric/symmetric Kronecker-like canonical form. These methods allow deflation of the singular structure and deflation of infinite eigenvalues with index greater than one. Two algorithms are proposed: one for general skew-symmetric/symmetric pencils and one for pencils in which the skew-symmetric matrix is a direct sum of 0 and $\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. We show how to use the structured staircase form to solve boundary value problems arising in control applications and present numerical examples.

Key words. structured staircase form, linear-quadratic control, H_∞ control, structured Kronecker canonical form, skew-symmetric/symmetric pencil, skew-Hamiltonian/Hamiltonian pencil

AMS subject classifications. 65F15, 15A21, 93B40

1. Introduction.

In this paper we study structure preserving numerical methods for the computation of the structural information associated with the singular and infinite eigenvalue parts of the Kronecker canonical form of real skew-symmetric/symmetric matrix pencils

$$\alpha N - \beta H, \tag{1.1}$$

where $N = -N^T, H = H^T \in \mathbb{R}^{n,n}$ and $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$. Here by $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices. In the following we adopt the notation of [31] and call pencils of this form *even pencils*, since replacing (α, β) by $(-\alpha, \beta)$ and transposing yields the same pencil.

Even pencils occur in the context of linear quadratic optimal control problems (see, e.g., [34, 38, 39, 45]), H_∞ control problems, (see, e.g., [4, 18, 37, 46]), and other applications (see, e.g., [31, 35]).

For control problems of the form

$$E\dot{x} = Ax + Bu, \; y = Cx, \tag{1.2}$$

it has been shown in [34] that the solution of the linear quadratic optimal control problem leads to the boundary value problem

$$N \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix} = H \begin{bmatrix} x \\ \mu \\ u \end{bmatrix} \tag{1.3}$$

(μ is an auxiliary vector, typically it is a vector of Lagrange multipliers) with boundary conditions

$$x(t_0) = x_0, \; \lim_{t\to\infty} E^T \mu(t) = 0, \tag{1.4}$$

*Received April 11, 2005. Accepted for publication June 27, 2006. Recommended by P. Van Dooren.

†Department of Mathematics, University of Kansas, Lawrence, KS 44045, USA. This author was partially supported by National Science Foundation grants 0098150, 0112375, 9977352 and by *Deutsche Forschungsgemeinschaft* through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

‡Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG (mehrmann@math.tu-berlin.de). This author was partially supported by *Deutsche Forschungsgemeinschaft*, through the DFG Research Center MATHEON *Mathematics for Key Technologies* in Berlin.

§Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA (xu@math.ukans.edu). This author was partially supported by *National Science Foundation grant 0314427, and the University of Kansas General Research Fund allocation # 2301717.*

ETNA
Kent State University
etna@mcs.kent.edu

where the matrix pencil associated with the boundary value problem

$$\alpha N - \beta H = \alpha \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} Q & A^T & S \\ A & W & B \\ S^T & B^T & R \end{bmatrix}, \tag{1.5}$$

is even, see [34]. (In particular, $Q = Q^T$, $W = W^T$ and $R = R^T$.)

The solution of the boundary value problem can be obtained via the computation of a structured Schur form of (1.5). Similar matrix pencils arise in the solution of optimal H_∞ control problems; see [4, 46]. If the control problem comes from an ordinary differential equation, then $E = I$ and if it comes from a differential-algebraic equation, then E is a singular matrix.

For both theoretical and computational purposes, the pencil (1.5) should be regular and of index at most 1. In order to check this property numerically and to remove singular parts and higher index infinite eigenvalue parts we need a staircase form. We discuss this topic in detail in Section 5.

We derive numerical methods to compute the characteristic quantities of the Kronecker canonical form of $\alpha N - \beta H$ under structure preserving congruence transformations

$$\alpha \tilde{N} - \beta \tilde{H} = \alpha P^T N P - \beta P^T H P.$$

The motivation for preserving the even structure comes from the special properties of such pencils. For example, even pencils have the *Hamiltonian eigensymmetry*, i.e., the finite eigenvalues occur in $\lambda, -\bar{\lambda}$ pairs and $\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$ quadruples for non-real eigenvalues of real pencils; see, e.g., [33, 34, 35].

As suggested by having eigenvalues with Hamiltonian symmetry, even pencils are closely related to skew-Hamiltonian/Hamiltonian pencils. Let $\mathcal{J}_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. (We leave off the subscript n, if the dimension is clear from the context.) A matrix $\mathcal{H} \in \mathbb{R}^{2n,2n}$ is called *Hamiltonian* if $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$. A matrix $\mathcal{N} \in \mathbb{R}^{2n,2n}$ is called *skew-Hamiltonian* if $(\mathcal{N}\mathcal{J})^T = -\mathcal{N}\mathcal{J}$. A matrix pencil $\alpha\mathcal{N} - \beta\mathcal{H}$ is called *skew-Hamiltonian/Hamiltonian* if $\mathcal{N}$ is skew-Hamiltonian and $\mathcal{H}$ is Hamiltonian. If the dimension of the even pencil $\alpha N - \beta H$ is even, then the pencil is equivalent to the skew-Hamiltonian/Hamiltonian pencil $\alpha\mathcal{N} - \beta\mathcal{H} = \alpha N \mathcal{J}^T - \beta H \mathcal{J}^T$.

Furthermore, if $N = \mathcal{J}$, then $N\mathcal{J}^T = I$ and we have a standard eigenvalue problem for the Hamiltonian matrix $\mathcal{H} = H\mathcal{J}^T$. It is well-known (see [28, 34]) that similarity transformations with symplectic matrices preserve the Hamiltonian and skew-Hamiltonian structure. (A matrix $\mathcal{S} \in \mathbb{R}^{2n,2n}$ is called *symplectic* if $\mathcal{S}\mathcal{J}\mathcal{S}^T = \mathcal{J}$.) It was shown in [29], that if the Hamiltonian matrix possesses a *Hamiltonian Jordan form* under symplectic similarity, then it also admits a *Hamiltonian Schur form* under orthogonal symplectic transformations. This work has been extended in [33] to skew-Hamiltonian/Hamiltonian pencils. For even pencils there exist well-known structured Kronecker forms; see, e.g., [43]. We briefly review these forms in Section 2.

It is the topic of this paper to construct a structured staircase form for even pencils that displays the invariants of the structured Kronecker form, while working only with unitary (orthogonal) transformations.

We could in theory also use an unstructured numerical method like the QZ or the GUPTRI algorithm to obtain this information, but this would destroy the symmetry structure in even pencils and introduce unnecessary unstructured rounding errors. The following example illustrates how such unstructured rounding errors may give misleading or even mathematically impossible computed “eigenvalues” and Kronecker structure.

ETNA
Kent State University
etna@mcs.kent.edu

EXAMPLE 1.1. As mentioned above, eigenvalues of even pencils have Hamiltonian pairing. A 3×3 even pencil has at least one infinite eigenvalue. The other two eigenvalues may be either both infinite, form a $(\lambda, -\lambda)$ pair of finite, real eigenvalues or form a complex conjugate pair of finite eigenvalues with zero real part. If $\lambda = 0$ is an eigenvalue, then it has multiplicity two.

Consider a 3×3 even pencil with matrices

$$N = Q \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T, \qquad H = Q \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} Q^T,$$

where Q is a random real orthogonal matrix generated as described in [40]. The pencil is congruent to the pencil

$$\alpha \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so it has a triple eigenvalue at ∞ with geometric multiplicity 1 and algebraic multiplicity 3.

We calculated the eigenvalues of $\alpha N - \beta H$ for several different randomly generated orthogonal matrices Q using the QZ algorithm in MATLAB [32] version 6.0.0.88 (R12) with unit round roughly 2.22×10^{-16}. MATLAB returns strikingly different approximations of eigenvalues for different randomly generated orthogonal matrices Q. For each example, we took care that H was exactly symmetric and N exactly skew symmetric.

Out of 1000 examples, the `eig` function built into Matlab version 6.0.0.88 (R12) reports that 644 have no finite eigenvalues (which is the correct result), 75 have one finite eigenvalue of magnitude roughly 10^{15}, 120 have two finite eigenvalues of magnitude roughly 10^7, and 61 have three finite eigenvalues of magnitude roughly 10^5. None of the computed sets of approximate eigenvalues that included finite eigenvalues was the set of eigenvalues of an even pencil; none had Hamiltonian eigenvalue pairing. Often, there was a singleton finite eigenvalue.

The QZ algorithm is numerically stable in the sense that the computed eigenvalues are exactly correct for some rounding-error-small perturbation of the original data matrices. However, this rounding-error-small perturbation is not necessarily an even perturbation of an even pencil. The unstructured rounding errors are sufficient to destroy the Hamiltonian pairing and return entirely unrealistic sets of eigenvalue approximations and Kronecker structures that do not occur in even pencils.

Recently, in [5, 11], numerical methods were developed to compute the Hamiltonian Schur form for Hamiltonian matrices and the methods were extended to the regular pencil case with nonsingular matrix N in [4].

An important remaining issue is a structure preserving method to compute the structural invariants under congruence associated with the infinite eigenvalues and the singular part of the pencil. This is of particular importance in the case of optimal control problems for descriptor systems, where E is a singular matrix, [34], since typical numerical methods for computing optimal feedback controls require the pencils to be regular and of index at most one. If this is not the case, then the singular part and the part associated with higher index singular blocks must be deflated first; see Section 5.

In Section 3 we derive structure preserving algorithms for the computation of structured staircase forms for arbitrary even pencils. In particular we show how to determine the Kronecker indices associated with singular Kronecker blocks and with Kronecker blocks corresponding to the eigenvalue infinity. The staircase form also provides a structure preserving

ETNA
Kent State University
etna@mcs.kent.edu

way to deflate these blocks. Section 4 treats the computation of eigenvalues and deflating subspaces for regular even pencils of index 1.

If $E = I$, then $N = \mathcal{J}_n \oplus 0$ is the direct sum of $\mathcal{J}_n$ and 0. In this case it has been shown in [10] how to preserve not only skew-symmetry but the whole $\mathcal{J}_n \oplus 0$ structure.

The results and algorithms of this paper also adapt to symmetric/symmetric and Hermitian/Hermitian pencils for which a similar structured Kronecker canonical form is known; see, e.g., [36, 42]. In a similar way, the results and algorithms also adapt to skew-Hermitian/Hermitian pencils and to complex skew-symmetric/symmetric pencils. For brevity, however, we will not discuss such variations here.

It should be noted that some of the ideas presented in this paper have been observed and discussed for special cases in [12]. Similar forms for a special case of symmetric/symmetric pencils have recently been proposed in [30].

2. Kronecker and staircase forms. In this section we review the Kronecker canonical form and staircase forms for unstructured, asymmetric pencils.

THEOREM 2.1. **Kronecker Canonical Form** [19, 24]. *Let $E, A \in \mathbb{R}^{m,n}$. Then there exist nonsingular matrices $P \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned} &P(\alpha E - \beta A)Q \\ &= \operatorname{diag}(\mathcal{O}_\eta, \mathcal{L}_{\epsilon_1}, \dots, \mathcal{L}_{\epsilon_k}, \mathcal{L}^T_{\delta_1}, \dots, \mathcal{L}^T_{\delta_\ell}, \mathcal{N}_{\sigma_1}, \dots, \mathcal{N}_{\sigma_r}, \mathcal{J}_{\rho_1}, \dots, \mathcal{J}_{\rho_s}), \end{aligned} \tag{2.1}$$

where ...

1. *$\mathcal{O}_\eta = \alpha 0_\eta - \beta 0_\eta$ is an $\eta \times \eta$ block of zeros;*
2. *each $\mathcal{L}_{\epsilon_j}$ is an $\epsilon_j \times (\epsilon_j + 1)$ right singular block with right minimal index ϵ_j and form*

$$\alpha \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \beta \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix};$$

3. *each $\mathcal{L}^T_{\delta_j}$ is a $(\delta_j + 1) \times \delta_j$ left singular block with left minimal index δ_j and form*

$$\alpha \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} - \beta \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix},$$

4. *each $\mathcal{N}_{\sigma_j}$ is a $\sigma_j \times \sigma_j$ infinite eigenvalue block with index σ_j and form*

$$\alpha \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \beta \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix};$$

5. *each $\mathcal{J}_{\rho_j}$ is a $\rho_j \times \rho_j$ Jordan block with finite eigenvalue $\lambda_j \in \mathbb{C}$ and form*

$$\alpha \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \beta \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

ETNA
Kent State University
etna@mcs.kent.edu

The Kronecker canonical form is unique up to permutation of the blocks, i. e., the kind, size and number of the blocks are characteristic for the pencil $\alpha E - \beta A$. It is more common to express $\mathcal{O}_\eta$ as a combination of L_0 and L_0^T blocks. Here, we display $\mathcal{O}_\eta$ explicitly to emphasize the similarities between Theorem 2.1 and the structured form in Theorem 2.3 below.

There also exists a real version of the Kronecker canonical form, where the blocks $\mathcal{J}_{\rho_j}$ are in real Jordan form and the transformation matrices are real. A similar result also holds for complex pencils, [19, 20].

DEFINITION 2.2.

i) An $n \times n$ matrix pencil $\alpha E - \beta A$ is called regular, *if* $\det(\alpha E - \beta A) \neq 0$ *for some* $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$. *Otherwise the pencil is called* singular. *(Singular pencils are those whose Kronecker canonical form has either an $\mathcal{O}_\eta$ block with $\eta > 0$ or an $\mathcal{L}_\epsilon$ block with $\epsilon > 0$ or an $\mathcal{L}_\epsilon^T$ block with $\epsilon > 0$.)*

ii) If $\alpha E - \beta A$ is regular, then a pair of complex numbers $(\alpha, \beta) \neq (0, 0)$ is an eigenvalue *of $\alpha E - \beta A$, if* $\det(\alpha E - \beta A) = 0$. *If $\alpha E - \beta A$ is a singular pencil, then, for our purposes in this paper, its eigenvalues are the eigenvalues of the regular blocks in its Kronecker canonical form, i.e., the union of the eigenvalues of the $\mathcal{N}_{\sigma_j}$ and $\mathcal{J}_{\rho_j}$ blocks in Theorem 2.1. We identify eigenvalues (α, β) with $\beta \neq 0$ with the finite eigenvalue $\lambda = \alpha/\beta$. Eigenvalues (α, β) with $\beta = 0$ are called* infinite *eigenvalues.*

iii) The index *of a regular matrix pencil $\alpha E - \beta A$ is the size of the largest block $\mathcal{N}_{\sigma_j}$ in Theorem 2.1. It is denoted by* $\text{ind}(E, A)$.

iv) The inertia index *of a symmetric matrix H is the triple $In(H) = (\pi, \nu, \xi)$, where π is the number of positive eigenvalues of H, ν is the number of negative eigenvalues, and ξ is the number of zero eigenvalues.*

Arbitrarily small rounding errors can radically change the kind and number of the Kronecker blocks. Consequently, it is problematic to compute the Jordan or Kronecker canonical form with a numerical algorithm in finite precision arithmetic [41]. Among the most successful compromises in the nearly-impossible problem of calculating Kronecker canonical forms are the staircase algorithms. Using a sequence of rank decisions, orthogonal matrix multiplications, and small perturbations, staircase algorithms transform a pencil into staircase or generalized upper triangular (GUPTRI) form [13, 14, 15, 44]. The rank decisions and perturbations have the effect of determining the essential invariants in the Kronecker canonical form of a "least generic" pencil within a tolerated perturbation. (A formal definition of the term "least generic" is surprisingly complicated. See [16, 17] for a detailed discussion and a recently developed interactive tool.) Since the GUPTRI form is built on a sequence of rank decisions and tolerated perturbations with a built-in bias toward a nearby least generic pencil, the computed invariants may not always agree with the invariants of the original pencil.

Example 1.1 demonstrates that otherwise excellent numerical methods can give unsatisfactory results when applied to even pencils, because the eigenvalues of even pencils have a special structure that is not necessarily preserved by unstructured rounding errors. In fact, even pencils have a special even Kronecker-like canonical form described by the following theorem.

THEOREM 2.3. [43] *If N, $H \in \mathbb{R}^{n,n}$ with $N = -N^T, H = H^T$, then there exists a nonsingular matrix $X \in \mathbb{C}^{n,n}$ such that*

$$X^T(\alpha N - \beta H)X = \text{diag}(\mathcal{B}_\mathcal{S}, \mathcal{B}_\mathcal{I}, \mathcal{B}_\mathcal{Z}, \mathcal{B}_\mathcal{F}), \tag{2.2}$$

where

$$\mathcal{B}_\mathcal{S} = \text{diag}(\mathcal{O}_\eta, \mathcal{S}_{\xi_1}, \dots, \mathcal{S}_{\xi_k}),$$

$$\begin{aligned}\mathcal{B}_{\mathcal{I}} &= \mathrm{diag}\left(\mathcal{I}_{2\epsilon_1+1}, \ldots, \mathcal{I}_{2\epsilon_l+1}, \mathcal{I}_{2\delta_1}, \ldots, \mathcal{I}_{2\delta_m}\right),\\ \mathcal{B}_{\mathcal{Z}} &= \mathrm{diag}\left(\mathcal{Z}_{2\sigma_1+1}, \ldots, \mathcal{Z}_{2\sigma_r+1}, \mathcal{Z}_{2\rho_1}, \ldots, \mathcal{Z}_{2\rho_s}\right),\\ \mathcal{B}_{\mathcal{F}} &= \mathrm{diag}(\mathcal{R}_{\phi_1}, \ldots, \mathcal{R}_{\phi_t}, \mathcal{C}_{\psi_1}, \ldots, \mathcal{C}_{\psi_u})\end{aligned}$$

and the blocks have the following properties.

1. $\mathcal{O}_\eta = \alpha 0_\eta - \beta 0_\eta$;
2. *each* $\mathcal{S}_{\xi_j}$ *is a* $(2\xi_j+1)\times(2\xi_j+1)$ *block that combines a right singular block and a left singular block, both of minimal index* ξ_j. *It has the form*

$$\alpha\left[\begin{array}{c|c} & \begin{array}{cccc} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & 0 & & \end{array}\\ \hline \begin{array}{ccc} & & -1\\ & \iddots & 0\\ -1 & \iddots & \\ 0 & & \end{array} & \end{array}\right] - \beta\left[\begin{array}{c|c} & \begin{array}{cccc} & & 0 & 1\\ & \iddots & \iddots & \\ 0 & 1 & & \end{array}\\ \hline \begin{array}{ccc} & & 0\\ & \iddots & 1\\ 0 & \iddots & \\ 1 & & \end{array} & \end{array}\right];$$

3. *each* $\mathcal{I}_{2\epsilon_j+1}$ *is a* $(2\epsilon_j+1)\times(2\epsilon_j+1)$ *block that contains a single block corresponding to the eigenvalue* ∞ *with index* $2\epsilon_j+1$. *It has the form*

$$\alpha\left[\begin{array}{c|c} & \begin{array}{cccc} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & 0 & & \end{array}\\ \hline \begin{array}{ccc} & & -1\\ & \iddots & 0\\ -1 & \iddots & \\ 0 & & \end{array} & \begin{array}{c} 0 \\ \\ \\ \\ \end{array} \end{array}\right] - \beta\left[\begin{array}{c|c} & \begin{array}{cccc} & & 0 & 1\\ & \iddots & \iddots & \\ 0 & 1 & & \end{array}\\ \hline \begin{array}{ccc} & & 0\\ & \iddots & 1\\ 0 & \iddots & \\ 1 & & \end{array} & \begin{array}{c} s \\ \\ \\ \\ \end{array} \end{array}\right],$$

where $s \in \{1, -1\}$ *is the* sign-index *or* sign-characteristic *of the block;*

4. *each* $\mathcal{I}_{2\delta_j}$ *is a* $4\delta_j \times 4\delta_j$ *block that combines two* $2\delta_j \times 2\delta_j$ *infinite eigenvalue blocks of index* δ_j. *It has the form*

$$\alpha\left[\begin{array}{c|c} & \begin{array}{cccc} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{array}\\ \hline \begin{array}{cccc} & & -1 & 0\\ & \iddots & \iddots & \\ -1 & \iddots & & \\ 0 & & & \end{array} & \end{array}\right] - \beta\left[\begin{array}{c|c} & \begin{array}{ccc} & & 1\\ & \iddots & \\ 1 & & \end{array}\\ \hline \begin{array}{ccc} & & 1\\ & \iddots & \\ 1 & & \end{array} & \end{array}\right];$$

5. *each* $\mathcal{Z}_{2\sigma_j+1}$ *is a* $(4\sigma_j+2)\times(4\sigma_j+2)$ *block that combines two* $(2\sigma_j+1)\times(2\sigma_j+1)$ *Jordan blocks corresponding to the eigenvalue* 0. *It has the form*

$$\alpha\left[\begin{array}{c|c} & \begin{array}{ccc} & & 1\\ & \iddots & \\ 1 & & \end{array}\\ \hline \begin{array}{ccc} & & -1\\ & \iddots & \\ -1 & & \end{array} & \end{array}\right] - \beta\left[\begin{array}{c|c} & \begin{array}{cccc} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{array}\\ \hline \begin{array}{cccc} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{array} & \end{array}\right];$$

ETNA
Kent State University
etna@mcs.kent.edu

6. *each $\mathcal{Z}_{2\rho_j}$ is a $2\rho_j \times 2\rho_j$ block that contains a single Jordan block corresponding to the eigenvalue 0. It has the form*

$$
\alpha \left[\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & & \cdot^{\cdot^{\cdot}} & \\
 & & & 1 & & \\
\hline
 & & -1 & & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \\
-1 & & & & &
\end{array}\right]
- \beta \left[\begin{array}{cccc|cccc}
 & & & & & & 1 & 0 \\
 & & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\
 & & & & 1 & \cdot^{\cdot^{\cdot}} & & \\
 & & & s & 0 & & & \\
\hline
 & & 1 & 0 & & & & \\
 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & \\
1 & \cdot^{\cdot^{\cdot}} & & & & & & \\
0 & & & & & & &
\end{array}\right],
$$

where $s \in \{1, -1\}$ is the sign characteristic of this block;

7. *each $\mathcal{R}_{\phi_j}$ is a $2\phi_j \times 2\phi_j$ block that combines two $\phi_j \times \phi_j$ Jordan blocks corresponding to nonzero real eigenvalues a_j and $-a_j$. It has the form*

$$
\alpha \left[\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & & \cdot^{\cdot^{\cdot}} & \\
 & & & 1 & & \\
\hline
 & & -1 & & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \\
-1 & & & & &
\end{array}\right]
- \beta \left[\begin{array}{cccc|cccc}
 & & & & & & 1 & a_j \\
 & & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\
 & & & & 1 & \cdot^{\cdot^{\cdot}} & & \\
 & & & & a_j & & & \\
\hline
 & & 1 & a_j & & & & \\
 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & \\
1 & \cdot^{\cdot^{\cdot}} & & & & & & \\
a_j & & & & & & &
\end{array}\right].
$$

8. *The entries $\mathcal{C}_{\psi_j}$ take two slightly different forms.*
(a) One possibility is that $\mathcal{C}_{\psi_j}$ is a $2\psi_j \times 2\psi_j$ block combining two $\psi_j \times \psi_j$ Jordan blocks with purely imaginary eigenvalues $ib_j, -ib_j$ ($b_j > 0$). In this case it has the form

$$
\alpha \left[\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & & \cdot^{\cdot^{\cdot}} & \\
 & & & 1 & & \\
\hline
 & & -1 & & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \\
-1 & & & & &
\end{array}\right]
- \beta\, s \left[\begin{array}{cccc|cccc}
 & & 1 & b_j & & & & \\
 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & \\
1 & \cdot^{\cdot^{\cdot}} & & & & & & \\
b_j & & & & & & & \\
\hline
 & & & & & & 1 & b_j \\
 & & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\
 & & & & 1 & \cdot^{\cdot^{\cdot}} & & \\
 & & & & b_j & & &
\end{array}\right],
$$

where $s \in \{1, -1\}$ is the sign characteristic.
(b) The other possibility is that $\mathcal{C}_{\psi_j}$ is a $4\psi_j \times 4\psi_j$ block combining $\psi_j \times \psi_j$ Jordan blocks for each of the complex eigenvalues $a_j + ib_j, a_j - ib_j, -a_j + ib_j, -a_j - ib_j$ (with $a_j \neq 0$ and $b_j \neq 0$). In this case it has form

$$
\alpha \left[\begin{array}{ccc|ccc}
 & & & & & \Omega \\
 & & & & \cdot^{\cdot^{\cdot}} & \\
 & & & \Omega & & \\
\hline
 & & -\Omega & & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \\
-\Omega & & & & &
\end{array}\right]
- \beta \left[\begin{array}{cccc|cccc}
 & & & & & & \Omega & \Lambda_j \\
 & & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\
 & & & & \Omega & \cdot^{\cdot^{\cdot}} & & \\
 & & & & \Lambda_j & & & \\
\hline
 & & \Omega & \Lambda_j & & & & \\
 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & \\
\Omega & \cdot^{\cdot^{\cdot}} & & & & & & \\
\Lambda_j & & & & & & &
\end{array}\right]
$$

with $\Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\Lambda_j = \begin{bmatrix} -b_j & a_j \\ a_j & b_j \end{bmatrix}$.

This structured Kronecker canonical form is unique up to permutation of the blocks, i.e., the kind, size and number of the blocks as well as the sign characteristics are characteristic of the pencil $\alpha N - \beta H$.

A corresponding structured Kronecker form is also known for complex even pencils $\alpha N - \beta H$ with $N, H \in \mathbb{C}^{n,n}$ and $N = -N^H, H = H^H$, see [43].

It was shown in [33] that the existence of the canonical form in Theorem 2.3 guarantees that corresponding condensed forms under orthogonal transformations also exist, see also [29].

The computation of the canonical form in Theorem 2.3 faces similar difficulties to those discussed above for the general Kronecker canonical form. Example 1.1 and experience with the unstructured Kronecker canonical form suggest that a successful numerical method for computing the characteristic indices and sign characteristics should use a staircase-like condensed form under unitary transformations that preserve the even structure of the pencil. This is the topic of this paper.

3. Staircase algorithms for even pencils.

In this section we discuss staircase algorithms for even pencils of the form (1.1). One may distinguish two cases. The first method that we discuss here deals with pencils where N is a general skew-symmetric matrix and the second method which is discussed in [10] treats the important special case that $N = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$ with $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

The procedures for computing staircase forms are built on a sequence of numerical rank decisions. This is also true for the procedures for even pencils that we present below. For general matrices the rank can be determined by the rank revealing QR factorization [21, Sec. 5.4] or the singular value decomposition (SVD) [21, Sec. 8.6]. For more details on determining numerical ranks, see, for example, [6, 21].

For symmetric and skew-symmetric matrices the rank can be determined via the appropriate Schur forms [21, Chapter 8]. An inexpensive way is the following modified rank revealing QR–factorization method. Let A be symmetric or skew-symmetric. Compute the rank revealing QR factorization

$$Q^T A \Pi = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where R_1 is of full row rank, Q is real orthogonal, and Π is a permutation. Compute

$$Q^T A Q = \begin{bmatrix} R_1 \Pi^T \\ 0 \end{bmatrix} Q = \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

The zero $(1, 2)$ block follows from the symmetry or skew-symmetry of A. Note also that R_{11} must be nonsingular. When A is skew-symmetric, R_{11} must have even order.

3.1. Even staircase form.

For a general even pencil we construct a symmetric variation of the staircase form of [44]. The staircase form displays the regular, index 1 part of the pencil. Moreover, we show below that the staircase form also displays the characteristic

ETNA
Kent State University
etna@mcs.kent.edu

quantities describing the singular part and the eigenvalue infinity of Theorem 2.3.

THEOREM 3.1. ***Even staircase form.*** *For a matrix pencil $\alpha N - \beta H$ with $N = -N^T, H = H^T \in \mathbb{R}^{n,n}$, there exists a real orthogonal matrix $U \in \mathbb{R}^{n,n}$, such that*

$$U^T N U =$$

$$\left[\begin{array}{cccc|c|cccc} N_{11} & \cdots & \cdots & N_{1,m} & N_{1,m+1} & N_{1,m+2} & \cdots & N_{1,2m} & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\ \vdots & & \ddots & \vdots & \vdots & N_{m-1,m+2} & \iddots & & \\ -N_{1,m}^T & \cdots & \cdots & N_{m,m} & N_{m,m+1} & 0 & & & \\ \hline -N_{1,m+1}^T & \cdots & \cdots & -N_{m,m+1}^T & N_{m+1,m+1} & & & & \\ \hline -N_{1,m+2}^T & \cdots & -N_{m-1,m+2}^T & 0 & & & & & \\ \vdots & \iddots & \iddots & & & & & & \\ -N_{1,2m}^T & \iddots & & & & & & & \\ 0 & & & & & & & & \end{array}\right] \begin{array}{c} n_1 \\ \vdots \\ \vdots \\ n_m \\ \hline l \\ \hline q_m \\ \vdots \\ q_2 \\ q_1 \end{array}$$

$$U^T H U =$$

$$\left[\begin{array}{cccc|c|cccc} H_{11} & \cdots & \cdots & H_{1,m} & H_{1,m+1} & H_{1,m+2} & \cdots & \cdots & H_{1,2m+1} \\ \vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\ H_{1,m}^T & \cdots & \cdots & H_{m,m} & H_{m,m+1} & H_{m,m+2} & & & \\ \hline H_{1,m+1}^T & \cdots & \cdots & H_{m,m+1}^T & H_{m+1,m+1} & & & & \\ \hline H_{1,m+2}^T & \cdots & \cdots & H_{m,m+2}^T & & & & & \\ \vdots & & \iddots & & & & & & \\ \vdots & \iddots & & & & & & & \\ H_{1,2m+1}^T & & & & & & & & \end{array}\right] \begin{array}{c} n_1 \\ \vdots \\ \vdots \\ n_m \\ \hline l \\ \hline q_m \\ \vdots \\ \vdots \\ q_1 \end{array}, \tag{3.1}$$

where $q_1 \geq n_1 \geq q_2 \geq n_2 \geq \ldots \geq q_m \geq n_m$,

$$\begin{aligned} &N_{j,2m+1-j} \in \mathbb{R}^{n_j, q_{j+1}}, \qquad 1 \leq j \leq m-1, \\ &N_{m+1,m+1} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = -\Delta^T \in \mathbb{R}^{2p,2p}, \\ &H_{j,2m+2-j} = \begin{bmatrix} \Gamma_j & 0 \end{bmatrix} \in \mathbb{R}^{n_j, q_j}, \quad \Gamma_j \in \mathbb{R}^{n_j, n_j}, \qquad 1 \leq j \leq m, \\ &H_{m+1,m+1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}, \quad \Sigma_{11} = \Sigma_{11}^T \in \mathbb{R}^{2p,2p}, \quad \Sigma_{22} = \Sigma_{22}^T \in \mathbb{R}^{l-2p,l-2p}, \end{aligned}$$

and the blocks Σ_{22} and Δ and Γ_j, $j = 1, \ldots, m$ are nonsingular.

Proof. A formal, constructive proof is given by Algorithm 1 in Appendix A, but for ease of explication, we present a less formal construction here. Both the formal algorithm and the less formal construction described here are explicit but recursive procedures. During the construction, we note the inertias of certain symmetric submatrices that will be used by Theorem 3.3. Note also that some blocks in the partitioned matrices may be void, i.e., they may have zero rows or zero columns or both.

Let $\alpha N - \beta H$ be an even pencil. If $N = H = 0$, then the pencil is singular and trivially in even staircase form. If N is nonsingular, then this is a regular pencil of index 0 and thus trivially in even staircase form. If N is singular, then determine an rank revealing factorization or skew-symmetric Schur decomposition $U_1^T N U_1 = \left[\begin{smallmatrix} \Delta & 0 \\ 0 & 0 \end{smallmatrix}\right]$, with U_1 orthogonal and Δ

ETNA
Kent State University
etna@mcs.kent.edu

nonsingular. Perform a pencil equivalence

$$U_1^T(\alpha N - \beta H)U_1 = \alpha \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ \hat{\mathcal{H}}_{12}^T & \hat{\mathcal{H}}_{22} \end{bmatrix}.$$

If $\hat{\mathcal{H}}_{22}$ is nonsingular, then the pencil is regular, of index at most 1 with rank $\hat{\mathcal{H}}_{22}$ infinite eigenvalues and the even staircase form is complete. If $\hat{\mathcal{H}}_{22}$ is singular, determine a rank revealing factorization or symmetric Schur decomposition $U_2^T\hat{\mathcal{H}}_{22}U_2 = \left[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with U_2 orthogonal and Σ nonsingular.

Record the inertia $(\pi, \nu, 0)$ of Σ for use in Theorem 3.3 below and perform a further pencil equivalence

$$\begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix}^T \left(\alpha \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ \hat{\mathcal{H}}_{12}^T & \hat{\mathcal{H}}_{22} \end{bmatrix}\right) \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix}$$
$$= \alpha \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} & \tilde{\mathcal{H}}_{13} \\ \tilde{\mathcal{H}}_{12}^T & \Sigma & 0 \\ \tilde{\mathcal{H}}_{13}^T & 0 & 0 \end{bmatrix}.$$

Determine a rank revealing factorization or singular value decomposition

$$U_3^T\tilde{\mathcal{H}}_{13}V_3 = \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}$$

with U_3 and V_3 orthogonal and Γ nonsingular. Perform another pencil equivalence

$$\begin{bmatrix} U_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & V_3 \end{bmatrix}^T \left(\alpha \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} & \tilde{\mathcal{H}}_{13} \\ \tilde{\mathcal{H}}_{12}^T & \Sigma & 0 \\ \tilde{\mathcal{H}}_{13}^T & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & V_3 \end{bmatrix}$$

$$(3.2)\qquad = \alpha \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} & \mathcal{N}_{13} & 0 & 0 \\ -\mathcal{N}_{12}^T & \mathcal{N}_{22} & 0 & 0 & 0 \\ -\mathcal{N}_{13}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{H}_{13} & \Gamma & 0 \\ \mathcal{H}_{12}^T & \mathcal{H}_{22} & \mathcal{H}_{23} & 0 & 0 \\ \mathcal{H}_{13}^T & \mathcal{H}_{23}^T & \Sigma & 0 & 0 \\ \Gamma^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\Delta = \left[\begin{smallmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{12}^T & \mathcal{N}_{22} \end{smallmatrix}\right]$ and $\mathcal{N}_{13} = 0$. The $\mathcal{N}_{13}$ block may fill with nonzero entries later in the process, so we do not distinguish it from other blocks that may be nonzero.

Recursively apply the even staircase reduction to the central subpencil $\alpha \left[\begin{smallmatrix} \mathcal{N}_{22} & 0 \\ 0 & 0 \end{smallmatrix}\right] - \beta \left[\begin{smallmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{23}^T & \Sigma \end{smallmatrix}\right]$ recording the inertias of the submatrices Σ as they occur. This corresponds to performing another pencil equivalence to (3.2) that modifies rows and columns 2 and 3 typically modifying $\mathcal{N}_{12}, \mathcal{N}_{13}, \mathcal{H}_{12}$ and $\mathcal{H}_{13}$ along with the central subpencil. At that point the pencil is in even staircase form. □

REMARK 3.2. It should be noted that the rank decisions in the recursive procedure described in the proof of Theorem 3.1 have to be carried out with great care. Ideally one would need a structured version of the procedure for general pencils in [16, 17]. The development of such a procedure is currently under investigation.

The recursive construction of the even staircase form also generates a sequence of inertias of certain ephemeral symmetric submatrices that appear briefly during the construction. The following theorem shows that the characteristic quantities describing the singular part and the eigenvalue infinity of $\alpha N - \beta H$ are determined by the integer sequences $\{q_j\}_{j=1}^m$, $\{n_j\}_{j=1}^m$, $\{\pi_j\}_{j=1}^{m+1}$, $\{\nu_j\}_{j=1}^{m+1}$ and $\{r_j = \pi_j + \mu_j\}_{j=1}^{m+1}$.

THEOREM 3.3. *Suppose that an even pencil $\alpha N - \beta H$ has been reduced to the condensed form (3.1) by Algorithm 1 with integer sequences $\{\pi_j\}$, $\{\nu_j\}$, and $\{r_j = \pi_j + \nu_j\}$. Then $\alpha N - \beta H$ has the following block structures associated with the singular part and the eigenvalue ∞ in the even Kronecker canonical form (2.2) of Theorem 2.3.*

1. *For every $j = 1, \ldots, m$, the pencil has $\frac{1}{2}[n_j - q_{j+1} - (r_{j+1} - r_j)]$ blocks $\mathcal{I}_{2j}$ corresponding to the eigenvalue ∞. (Here we set $q_{m+1} = 0$).*
2. *For every $j = 1, \ldots, m+1$, the pencil has $r_j - r_{j-1}$ odd-sized blocks $\mathcal{I}_{2j-1}$ corresponding to the eigenvalue ∞,among which $\pi_j - \pi_{j-1}$ blocks have sign index 1 and $\nu_j - \nu_{j-1}$ blocks have sign index -1. (Here we set $\pi_0 = \nu_0 = r_0 = 0$.)*
3. *The pencil has a singular block $\alpha 0_{q_1 - n_1} - \beta 0_{q_1 - n_1}$.*
4. *For every $j = 2, \ldots, m$, the pencil has $q_j - n_j$ singular blocks $\mathcal{S}_{j-1}$.*
5. *The subpencil $\alpha N_{m+1,m+1} - \beta H_{m+1,m+1}$ is a regular pencil of index at most 1. It contains the Jordan structure associated with all finite eigenvalues of $\alpha N - \beta H$.*

Proof. See Appendix B ∎

EXAMPLE 3.4. This example demonstrates the effect of rank decisions and the ability of the even staircase algorithm to determine a nearby even pencil with non-generic structure.

Our experimental MATLAB implementation of the even staircase algorithm makes rank decisions using a singular value drop tolerance $\tau > 0$, i.e. singular values of magnitude less than an absolute threshold $\tau > 0$ are taken to be zero. In this experiment, the threshold τ varied from 10^{-16} to 10^{-1}. (The unit round is approximately 2.22×10^{-16}.) Oversimplifying slightly, the algorithm searches for a "most non-generic" even pencil in the cloud of pencils that lie within a distance of roughly τ of then nominal input pencil.

We constructed even pencils $\alpha N - \beta H$ where

$$N = Q \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} Q^T + \epsilon \Delta N$$

$$H = Q \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} Q^T + \epsilon \Delta H$$

where ϵ is a positive real number varying from 10^{-16} to 10^{-1}, Q is a random real orthogonal matrix generated as described in [40], and ΔN and ΔH are skew-symmetric and symmetric matrices, respectively, whose nontrivial entries are normal $(0, 1)$ random variables.

If $\epsilon = 0$, then these unperturbed pencils have simple finite eigenvalues $\pm 2i$ and an index 2 infinite eigenvalue. If $\epsilon > 0$, then the perturbed pencils typically lie at a distance of roughly ϵ from the $\epsilon = 0$ unperturbed even pencil.

In this experiment, for each value of the singular value drop tolerance τ, we chose ten random equivalence matrices Q. For each τ and Q, we varied the perturbation magnitude ϵ logarithmically as $\epsilon = 10^{-16}, 10^{-15.9}, 10^{-15.8}, \ldots, 10^{-1}$ and recorded the smallest value of the selected ϵ's for which the algorithm did *not* find an even pencil with an index 2 infinite eigenvalue (and two finite eigenvalues) with a distance of roughly τ of the test pencil. We plotted the recorded points (τ, ϵ) in Figure 3.1. As expected the points fall near the line $\epsilon = \tau$.

ETNA
Kent State University
etna@mcs.kent.edu

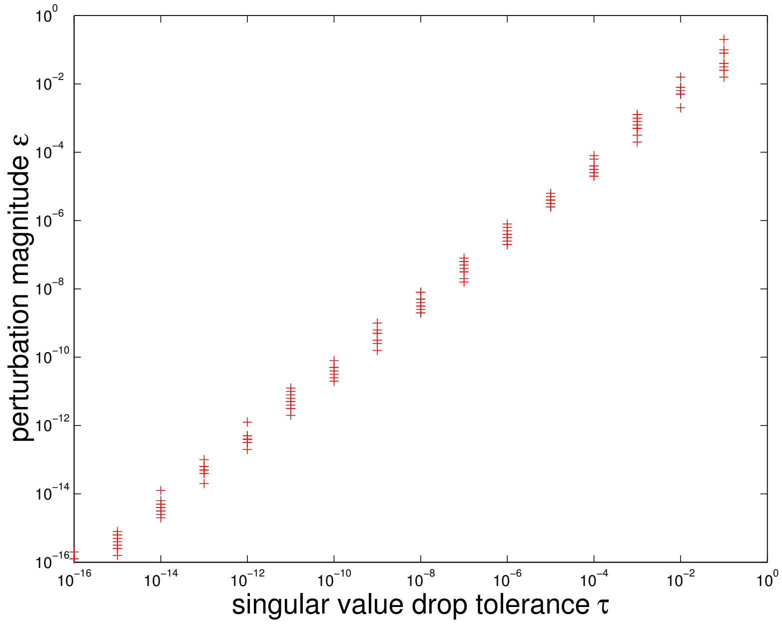

FIG. 3.1. *Figure from Example 3.4. For each selected value of the singular value drop tolerance τ and each of ten random pencil equivalences, the graph plots the smallest of the selected perturbation magnitudes ϵ for which the even pencil staircase algorithm did* not *find an index 3 pencil. As expected, the points line near the line $\epsilon = \tau$. In every test case, for which ϵ was significantly smaller than τ, the experimental staircase algorithm successfully found a nearby index 3 even pencil.*

In every test case with perturbation magnitude ϵ significantly smaller than the singular value drop tolerance τ, the staircase algorithm successfully located a nearby index 3 even pencil.

If the skew-symmetric matrix N in the pencil (1.5) is of the special form $\begin{bmatrix} J_n & 0 \\ 0 & 0 \end{bmatrix}$ as in applications from linear quadratic optimal control or H_∞ control, then from a perturbation theory point of view it is advisable to preserve this structure as much as possible, i.e., we would like to compute a staircase form, where the middle block associated with the finite eigenvalues and the infinite-eigenvalue-index-1 part is again of the same form as the original pencil with a (possibly smaller) skew-symmetric part $\begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix}$. An algorithm to compute a variant even staircase form while preserving the $\begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix}$ structure of the skew-symmetric part has been presented in [10].

4. The regular, index one case. It remains to determine the finite eigenvalues and index 1 infinite eigenvalues contained in the central block of the even staircase form (3.1). To avoid the hazards of introducing asymmetric rounding errors demonstrated above, a structure preserving numerical method is necessary. In this section we outline how to modify a skew-Hamiltonian/Hamiltonian structure preserving algorithm from [3] for regular even pencils of index at most 1. For ease of notation, in this section we assume that the even pencil is regular of index at most 1.

In order to use the skew-Hamiltonian/Hamiltonian algorithm, we must transform the skew-symmetric/symmetric pencil into skew-Hamiltonian/Hamiltonian form. For this we

ETNA
Kent State University
etna@mcs.kent.edu

assume that the even pencil already has the form

$$\alpha N - \beta H = \alpha \begin{bmatrix} 0 & D^2 & 0 & 0 \\ -D^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^T & H_{22} & H_{23} & H_{24} \\ H_{13}^T & H_{23}^T & H_{33} & H_{34} \\ H_{14}^T & H_{24}^T & H_{34}^T & H_{44} \end{bmatrix}, \tag{4.1}$$

where D^2 is positive diagonal, $D^2, H_{11}, H_{22} \in \mathbb{R}^{p,p}$, and $H_{33}, H_{44} \in \mathbb{R}^{r,r}$. For a general even pencil this may be achieved, for example, by computing and reordering the real Schur form of N_{11}.

The pencil (4.1) has even size. If the size of the original pencil is odd, then add one more index 1 eigenvalue infinity by appending one row and column as

$$\alpha \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} H & 0 \\ 0 & \Theta \end{bmatrix},$$

where Θ is a nonzero scalar. The extended pencil is simply a direct sum of the original pencil and the scalar pencil $\alpha 0 - \beta\Theta$. So the eigen-structure of the original pencil will not be affected by this adding.

Let $S = \operatorname{diag}(D, 0, D, 0)$ and $n = p + r$. By interchanging the 2nd and 3rd columns and rows and then multiplying with J_n^T from the right, the pencil (4.1) is equivalent to the skew-Hamiltonian/Hamiltonian pencil

$$\alpha Z - \beta M = \alpha S J_n S^T J_n^T - \beta \left[\begin{array}{cc|cc} H_{12} & H_{14} & -H_{11} & -H_{13} \\ H_{23} & H_{34} & -H_{13}^T & -H_{33} \\ \hline H_{22} & H_{24} & -H_{12}^T & -H_{23}^T \\ H_{24}^T & H_{44} & -H_{14}^T & -H_{34} \end{array}\right].$$

We then have the following structured Schur form.

THEOREM 4.1. *Let $S = \operatorname{diag}(D, 0, D, 0)$, $T = J_n S^T J_n^T$, and let $M \in \mathbb{R}^{2n,2n}$ be Hamiltonian. Then there exist orthogonal matrices Q_1, Q_2 and orthogonal symplectic matrices U_1, U_2 such that*

$$Q_1^T M Q_2 = \left[\begin{array}{cc|cc} M_{11} & M_{12} & M_{13} & M_{14} \\ 0 & M_{22} & M_{23} & M_{24} \\ \hline 0 & 0 & M_{33} & 0 \\ 0 & 0 & M_{43} & M_{44} \end{array}\right],$$

$$Q_1^T S U_1 = \left[\begin{array}{cc|cc} 0 & S_{12} & 0 & S_{14} \\ 0 & S_{22} & 0 & S_{24} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{array}\right],$$

$$U_2^T T Q_2 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & T_{22} & T_{23} & T_{24} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & T_{43} & T_{44} \end{array}\right], \tag{4.2}$$

where $M_{11}, M_{33}^T \in \mathbb{R}^{r,r}$ are upper triangular $M_{22}, S_{22}, T_{22}, M_{44}, S_{44}^T, T_{44}^T \in \mathbb{R}^{p,p}$ are upper triangular, and M_{44} is lower quasi-triangular. Furthermore, $S_{22}, S_{44}, T_{22}, T_{44}$ are nonsingular.

The finite eigenvalues of $\alpha ST - \beta M$, and $\alpha N - H$ in (4.1) as well, are the same as the finite eigenvalues of the index 0 pencil

$$\alpha A - \beta B = \alpha \begin{bmatrix} S_{22} S_{44}^T & 0 \\ 0 & T_{44}^T T_{22} \end{bmatrix} - \beta \begin{bmatrix} 0 & M_{22} \\ -M_{44}^T & 0 \end{bmatrix}.$$

Proof. The proof appears in Appendix C, where (4.2) is proved constructively by Algorithm 4. □

5. Application to optimal control. Consider the linear quadratic control problem described by (1.2)–(1.5). It is well known that if the pencil is regular then the boundary value problem is uniquely solvable [8]. We therefore assume that the pencil (1.5) is regular. (For the singular case, see [9].)

The following proposition shows how the even staircase form (3.1) characterizes consistency of boundary conditions for the special problem (1.3). (The general theory of linear differential-algebraic equations of [25, 26] uses a normal form to characterize consistent initial conditions.)

THEOREM 5.1. *Consider the boundary value problem (1.3) with a regular matrix pencil. Transform the boundary value problem to the staircase form (3.1). With U as in (3.1), partition*

$$z = U^T \begin{bmatrix} x \\ \mu \\ u \end{bmatrix} = \begin{bmatrix} z_1^T & \dots & z_m^T & z_{m+1}^T & z_{m+2}^T & \dots & z_{2m+1}^T \end{bmatrix}^T \tag{5.1}$$

conformally. Then $z_1, \dots, z_m = 0$, and $\dot z_1, \dots, \dot z_m = 0$. The solution of the boundary value problem

$$N_{m+1,n+1}\dot z_{m+1} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \dot z_{m+1} = H_{m+1,m+1} z_{m+1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} z_{m+1},$$

(which is of index at most 1) uniquely determines the remaining components, z_{m+2}, …, z_{2m+1}.

Proof. Because the pencil is regular, we have $n_j = q_j$ and $H_{j,2m+2-j} = \Gamma_j$, $j = 1, \dots, m$. Hence, for $j = 1, \dots, m$, we obtain recursively that

$$z_{m+j+1} = \Gamma_{m-j+1}^{-1} \left(\sum_{i=m+1}^{m+j} N_{m-j+1,i} \dot z_i - \sum_{i=m+1}^{m+j} H_{m-j+1,i} z_i \right). \quad \square$$

The consistency of the boundary conditions in (1.3) may be checked by using the recursion formulas for $z_{m+2}, \dots, z_{2m+1}$, the explicit solution representation

$$z_{m+1} = \begin{bmatrix} I \\ -\Sigma_{22}^{-1}\Sigma_{12}^T \end{bmatrix} e^{W(t-t_0)} \begin{bmatrix} I & 0 \end{bmatrix} z_{m+1}(t_0),$$

with $W = \Delta^{-1}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)$, and (5.1) with $z_j = 0$ for $j = 1, \dots, m$. A similar observation was made about more general pencils in [9].

In this way we may reduce the general linear differential-algebraic boundary value problem (1.3) in an even structured way to a smaller linear differential-algebraic boundary value problem of index at most 1, to which appropriate methods may be applied. See for example, [2, 1, 27].

6. Conclusion. Even pencils have paired eigenvalues and a structured Kronecker-like canonical form with paired blocks. Even otherwise numerically stable numerical methods that allow asymmetric rounding errors can return computed "eigenvalues" that are unrealistic in the sense that they do not have proper pairing and, hence, are not eigenvalues of an even pencil. Numerical procedures including asymmetric staircase forms for determining Kronecker indices do not calculate the sign indices of the even Kronecker-like form and, if they allow asymmetric rounding errors, can return unrealistic results.

This paper presents an even staircase form for even pencils that displays the structure and characteristic indices of the singular and infinite eigenvalue structure of even Kronecker-like canonical form. Using only orthogonal matrix multiplications and rank decisions, the accompanying numerically stable numerical method preserves even structure throughout and introduces only even rounding errors.

The use of the even staircase form is illustrated using an application to boundary value problems arising from optimal control of differential-algebraic systems. As outlined in Section 4, the even staircase form may be the first step in a method for calculating eigenvalues of an even pencil.

REFERENCES

[1] U. ASCHER, R. MATTHEIJ, AND R. RUSSELL, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, Philadelphia, PA, second ed., 1995.

[2] U. M. ASCHER AND L. R. PETZOLD, *Computer Methods for Ordinary Differential and Differential-Algebraic Equations*, SIAM, Philadelphia, PA, 1998.

[3] ———, *Numerical computation of deflating subspaces of skew Hamiltonian/Hamiltonian pencils*, SIAM J. Matrix Anal. Appl., 24 (2002), pp. 165–190.

[4] P. BENNER, R. BYERS, V. MEHRMANN, AND H. XU, *A robust numerical method for the γ-iteration in H_∞-control*, to appear in: Linear Algebra Appl., (2007).

[5] P. BENNER, V. MEHRMANN, AND H. XU, *A new method for computing the stable invariant subspace of a real Hamiltonian matrix*, J. Comput. Appl. Math., 86 (1997), pp. 17–43.

[6] C. BISCHOF AND G. QUINTANA-ORTÍ, *Algorithm 782: codes for rank-revealing QR factorizations of dense matrices*, ACM Trans. Math. Software, 24 (1998), pp. 254–257.

[7] A. BOJANCZYK, G. GOLUB, AND P. VAN DOOREN, *The periodic Schur decomposition; algorithms and applications*, in Proc. SPIE Conference, vol. 1770, 1992, pp. 31–42.

[8] K. BRENAN, S. CAMPBELL, AND L. PETZOLD, *Numerical Solution of Initial–Value Problems in Differential–Algebraic Equations*, SIAM, Philadelphia, PA, second ed., 1996.

[9] R. BYERS, T. GEERTS, AND V. MEHRMANN, *Descriptor systems without controllability at infinity*, SIAM J. Cont. Optim., 35 (1997), pp. 462–479.

[10] R. BYERS, V. MEHRMANN, AND H. XU, *A structured staircase algorithm for skew-symmetric/symmetric pencils*, technical report, DFG Research Center MATHEON, *Mathematics for key technologies* in Berlin, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany, 2005. url: http://www.matheon.de/ .

[11] D. CHU, X. LIU, AND V. MEHRMANN, *A numerical method for computing the Hamiltonian Schur form*, Numer. Math., 105 (2007), pp. 375–412.

[12] D. CLEMENTS, *Private communication*, 2002.

[13] J. DEMMEL AND B. KÅGSTRÖM, *Computing stable eigendecompositions of matrix pencils*, Linear Algebra Appl., 88 (1987), pp. 139–186.

[14] ———, *The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: robust software with error bounds and applications. Part I: theory and algorithms*, ACM Trans. Math. Software, 19 (1993), pp. 160–174.

[15] ———, *The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: robust software with error bounds and applications. Part II: software and applications*, ACM Trans. Math. Software, 19 (1993), pp. 175–201.

[16] A. EDELMANN, E. ELMROTH, AND B. KÅGSTRÖM, *A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 653–692.

[17] ———, *A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm.*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 667–699.

[18] P. GAHINET AND A. J. LAUB, *Numerically reliable computation of optimal performance in singular H_∞ control*, SIAM J. Cont. Optim., 35 (1997), pp. 1690–1710.

[19] F. GANTMACHER, *Theory of Matrices*, vol. 2, Chelsea, New York, 1959.

[20] F. GANTMACHER, *Theory of Matrices*, vol. 1, Chelsea, New York, 1959.

[21] G. GOLUB AND C. VAN LOAN, *Matrix Computations*, Johns Hopkins University Press, Baltimore, third ed., 1996.

[22] J. HENCH AND A. LAUB, *Numerical solution of the discrete-time periodic Riccati equation*, IEEE Trans. Automat. Control, 39 (1994), pp. 1197–1210.
[23] D. KRESSNER, *Numerical Methods and Software for General and Structured Eigenvalue Problems*, PhD thesis, TU Berlin, Institut für Mathematik, Berlin, Germany, 2004.
[24] L. KRONECKER, *Algebraische Reduction der Scharen bilinearer Formen*, S.B. Akad. Berlin, (1890), pp. 1225–1237.
[25] P. KUNKEL AND V. MEHRMANN, *Canonical forms for linear differential-algebraic equations with variable coefficients*, J. Comput. Appl. Math., 56 (1994), pp. 225–251.
[26] ———, *A new look at pencils of matrix valued functions*, Linear Algebra Appl., 212/213 (1994), pp. 215–248.
[27] P. KUNKEL AND R. STÖVER, *Symmetric collocation methods for linear differential-algebraic boundary value problems*, Numer. Math., 91 (2002), pp. 475–501.
[28] P. LANCASTER AND L. RODMAN, *The Algebraic Riccati Equation*, Oxford University Press, Oxford, 1995.
[29] W.-W. LIN, V. MEHRMANN, AND H. XU, *Canonical forms for Hamiltonian and symplectic matrices and pencils*, Linear Algebra Appl., 301–303 (1999), pp. 469–533.
[30] C. LUCAS, *Algorithms for Cholesky and QR Factorizations, and the Semidefinite Generalized Eigenvalue Problem*, PhD thesis, University of Manchester, Manchester, England, 2004.
[31] D. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Palindromic polynomial eigenvalue problems: Good vibrations from good linearizations*, technical report, DFG Research Center MATHEON, *Mathematics for key technologies* in Berlin, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany, 2005. url: http://www.matheon.de/ to appear in *SIAM J. Matrix Anal. Appl.*
[32] THE MATHWORKS, INC., COCHITUATE PLACE, 24 PRIME PARK WAY, NATICK, MASS, 01760, *MATLAB Version 6.0.0.88*, 2001.
[33] C. MEHL, *Condensed forms for skew-Hamiltonian/Hamiltonian pencils*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 454–476.
[34] V. MEHRMANN, *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution*, no. 163 in Lecture Notes in Control and Information Sciences, Springer-Verlag, Heidelberg, July 1991.
[35] V. MEHRMANN AND D. WATKINS, *Polynomial eigenvalue problems with hamiltonian structure*, Electr. Trans. Numer. Anal., 13 (2002), pp. 106–118. http://etna.math.kent.edu/vol.13.2002/pp106-118.dir/pp106-118.html.
[36] V. MEHRMANN AND H. XU, *Structured Jordan canonical forms for structured matrices that are Hermitian, skew Hermitian or unitary with respect to indefinite inner products*, Electr. J. Linear Algebra, 5 (1999), pp. 67–103.
[37] ———, *Numerical methods in control*, J. Comput. Appl. Math., 123 (2000), pp. 371–394.
[38] P. PETKOV, N. CHRISTOV, AND M. KONSTANTINOV, *Computational Methods for Linear Control Systems*, Prentice-Hall, Hertfordshire, UK, 1991.
[39] V. SIMA, *Algorithms for Linear-Quadratic Optimization*, vol. 200 of Pure and Applied Mathematics, Marcel Dekker, Inc., New York, NY, 1996.
[40] G. STEWART, *The efficient generation of random orthogonal matrices with an application to condition estimators*, SIAM J. Numer. Anal., 17 (1980), pp. 403–409.
[41] G. STEWART AND J.-G. SUN, *Matrix Perturbation Theory*, Academic Press, New York, 1990.
[42] R. C. THOMPSON, *The characteristic polynomial of a principal submatrix of a Hermitian pencil*, Linear Algebra Appl., 14 (1976), pp. 135–177.
[43] ———, *Pencils of complex and real symmetric and skew matrices*, Linear Algebra Appl., 147 (1991), pp. 323–371.
[44] P. VAN DOOREN, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl., 27 (1979), pp. 103–121.
[45] ———, *A generalized eigenvalue approach for solving Riccati equations*, SIAM J. Sci. Statist. Comput., 2 (1981), pp. 121–135.
[46] K. ZHOU, J. DOYLE, AND K. GLOVER, *Robust and Optimal Control*, Prentice-Hall, Upper Saddle River, NJ, 1995.

Appendix A. Proof of Theorem 3.1: Algorithm 1. We prove Theorem 3.1 constructively. The proof is provided by the following algorithm. Note that in the algorithm, some blocks in the partitioned matrices may be void, i.e., they may have zero rows or zero columns or both.

ALGORITHM 1. *Staircase algorithm for even pencils.*
For $N = -N^T, H = H^T \in \mathbb{R}^{n,n}$ this algorithm computes an orthogonal matrix $U \in \mathbb{R}^{n,n}$ such that $U^T NU$, $U^T HU$ are in the form of (3.1). In addition, the algorithm produces a sequence of inertias $(\pi_j, \nu_j, 0)$ of nonsingular, symmetric submatrices that will be used in Theorem 3.3.

Set flag $= 0$, $m = n_0 = q_0 = r_0 = 0$, $l = n$,

ETNA
Kent State University
etna@mcs.kent.edu

$$\mathcal{N} = \mathcal{N}_{22} = N, \quad \mathcal{H} = H, \quad U = I.$$

```
DO WHILE  flag = 0
```

    Perform a rank revealing factorization of $\mathcal{N}_{22} \in \mathbb{R}^{l-r_m, l-r_m}$,

$$\mathcal{N}_{22} = U_1 \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} U_1^T,$$

    with $\Delta \in \mathbb{R}^{2p,2p}$. Set

$$\mathcal{N}_1 = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix}^T \mathcal{N} \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_1 = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix}^T \mathcal{H} \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ \hat{\mathcal{H}}_{12}^T & \hat{\mathcal{H}}_{22} \end{bmatrix},$$

    partitioned analogously. (Here $\hat{\mathcal{H}}_{22} \in \mathbb{R}^{l-2p,l-2p}$).

    IF $2p = l$ THEN

        Set flag $= 1$ and

$$\mathcal{U} = \begin{bmatrix} I_{n_1+\ldots+n_m} & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & I_{q_1+\ldots+q_m} \end{bmatrix}.$$

    ELSE

        Set $m = m + 1$.

        Perform the Schur decomposition of $\hat{\mathcal{H}}_{22}$,

$$\hat{\mathcal{H}}_{22} = U_2 \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U_2^T,$$

        where $\Sigma \in \mathbb{R}^{\mu,\mu}$ is nonsingular with inertia index $(\pi_m, \nu_m, 0)$ and rank $r_m = \mu = \pi_m + \nu_m$.

        Set

$$\mathcal{N}_2 = \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix}^T \mathcal{N}_1 \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_2 = \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix}^T \mathcal{H}_1 \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} & \tilde{\mathcal{H}}_{13} \\ \tilde{\mathcal{H}}_{12}^T & \Sigma & 0 \\ \tilde{\mathcal{H}}_{13}^T & 0 & 0 \end{bmatrix},$$

        partitioned analogously.

        IF $\mu = l - 2p$ THEN

            Set flag $= 1$ and

$$\hat{\mathcal{U}} = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_{m-1}} \end{bmatrix} \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix},$$

$$\mathcal{U} = \begin{bmatrix} I_{n_1+\ldots+n_{m-1}} & 0 & 0 \\ 0 & \hat{\mathcal{U}} & 0 \\ 0 & 0 & I_{q_1+\ldots+q_{m-1}} \end{bmatrix}$$

ETNA
Kent State University
etna@mcs.kent.edu


```
ELSE
    Perform a rank revealing factorization or SVD
```

$$\tilde{\mathcal{H}}_{13} = U_3 \begin{bmatrix} \Gamma_m & 0 \\ 0 & 0 \end{bmatrix} V_3^T,$$

where $\Gamma_m \in \mathbb{R}^{\tau,\tau}$ is nonsingular.

Set $n_m = \tau$, $q_m = l - 2p - \mu$ and

$$\begin{aligned} \mathcal{N}_3 &= \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & V_3 \end{bmatrix}^T \mathcal{N}_2 \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & V_3 \end{bmatrix} \\ &= \left[\begin{array}{c|cc|cc} \mathcal{N}_{11} & \mathcal{N}_{12} & 0 & 0 & 0 \\ \hline -\mathcal{N}_{12}^T & \mathcal{N}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \end{aligned}$$

$$\begin{aligned} \mathcal{H}_3 &= \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & V_3 \end{bmatrix}^T \mathcal{H}_2 \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & V_3 \end{bmatrix} \\ &= \left[\begin{array}{c|cc|cc} \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{H}_{13} & \Gamma_m & 0 \\ \hline \mathcal{H}_{12}^T & \mathcal{H}_{22} & \mathcal{H}_{23} & 0 & 0 \\ \mathcal{H}_{13}^T & \mathcal{H}_{23}^T & \Sigma & 0 & 0 \\ \hline \Gamma_m^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \\ \hat{\mathcal{U}} &= \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_{m-1}} \end{bmatrix} \begin{bmatrix} I_{2p} & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & V_3 \end{bmatrix}, \\ \mathcal{U} &= \begin{bmatrix} I_{n_1+\ldots+n_{m-1}} & 0 & 0 \\ 0 & \hat{\mathcal{U}} & 0 \\ 0 & 0 & I_{q_1+\ldots+q_{m-1}} \end{bmatrix}. \end{aligned}$$

Set

$$\mathcal{N} = \begin{matrix} \\ 2p-\tau \\ \mu \end{matrix} \overset{\begin{matrix} 2p-\tau & \mu \end{matrix}}{\begin{bmatrix} \mathcal{N}_{22} & 0 \\ 0 & 0 \end{bmatrix}}, \quad \mathcal{H} = \begin{matrix} \\ 2p-\tau \\ \mu \end{matrix} \overset{\begin{matrix} 2p-\tau & \mu \end{matrix}}{\begin{bmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{23}^T & \Sigma \end{bmatrix}} \in \mathbb{R}^{l,l},$$

and $l = 2p - \tau + \mu$.

```
    END IF
  END IF
```

Form $H = \mathcal{U}^T H \mathcal{U}$, $N = \mathcal{U}^T N \mathcal{U}$, and $U = U\mathcal{U}$.

```
END WHILE
```

Algorithm 1 will stop after finitely many steps, because at each recursive call, the order of the even pencil decreases. At some stage $\hat{H}_{22}$ must be either nonsingular or void.

Our experimental MATLAB implementation of the even staircase algorithm 1 makes rank decisions using a singular value drop tolerance $\tau > 0$, i.e. singular values of magnitude less than an absolute threshold $\tau > 0$ are set to be zero. Ordinarily, τ should be slightly larger than the magnitude of errors or uncertainties in the data. For example, if the data are perturbed only by rounding errors and μ is the unit round, then it would not be unreasonable to use $\tau = \mu \|N\|_F$ for rank decisions on submatrices of N and $\tau = \mu \|H\|_F$ for rank decisions on submatrices of H.

Appendix B. Proof of Theorem 3.3. We prove Theorem 3.3 constructively using another staircase algorithm to obtain a more condensed even staircase form followed by a further reduction closer to the even Kronecker-like canonical form of Theorem 2.3. In contrast to Algorithm 1 these reductions use extra non-orthogonal transformations, so they are theoretical in nature and may not be well suited to finite precision computation. The extra work displays a relationship between successive values of the inertias $(\pi_j, \nu_j, 0)$.

ALGORITHM 2. Given $N = -N^T, H = H^T \in \mathbb{R}^{n,n}$, this algorithm computes a real nonsingular matrix $Y \in \mathbb{R}^{n,n}$ such that $Y^T NY, Y^T HY$ are in the even staircase form (3.1).

Set flag $=0$, $m = n_0 = q_0 = r_0 = 0$, $l = n$,

$$\mathcal{N}_{22} = N, \quad \mathcal{N} = \begin{bmatrix} \mathcal{N}_{22} & 0 \\ 0 & 0_{r_0} \end{bmatrix}, \quad \mathcal{H}_{22} = H, \quad \mathcal{H} = \begin{bmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{23}^T & \Sigma \end{bmatrix}, \quad U = I,$$

where $\Sigma = \Sigma^T \in \mathbb{R}^{r_0,r_0}$, and $\mathcal{H}_{23} \in \mathbb{R}^{l \times r_0}$.

Since $r_0 = 0$, the initial last row and column of $\mathcal{N}$ and $\mathcal{H}$ are void.

DO WHILE flag $=0$

Perform a rank revealing factorization of $\mathcal{N}_{22} \in \mathbb{R}^{l-r_m,l-r_m}$,

$$\mathcal{N}_{22} = U_1 \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} U_1^T,$$

with $\Delta \in \mathbb{R}^{2p,2p}$. Set

$$\mathcal{N}_1 = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix}^T \mathcal{N} \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_1 = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix}^T \mathcal{H} \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_m} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} & \hat{H}_{13} \\ \hat{\mathcal{H}}_{12}^T & \hat{\mathcal{H}}_{22} & \hat{\mathcal{H}}_{23} \\ \hat{\mathcal{H}}_{13}^T & \hat{\mathcal{H}}_{23}^T & \Sigma \end{bmatrix},$$

partitioned analogously. (Here $\hat{\mathcal{H}}_{22} \in \mathbb{R}^{l-2p-r_m,l-2p-r_m}$).

IF $2p = l - r_m$ THEN

Set flag $=1$ and

$$\mathcal{Y} = \begin{bmatrix} I_{n_1+\ldots+n_m} & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & I_{q_1+\ldots+q_m} \end{bmatrix}.$$

ELSE

Set $m = m + 1$.

Perform a congruence transformation with

$$X = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Sigma^{-1}\hat{\mathcal{H}}_{23}^T & I \end{bmatrix}$$

to annihilate the blocks $\hat{\mathcal{H}}_{23}$ and $\hat{\mathcal{H}}_{23}^T$ in $\mathcal{H}_1$. Set

$$\mathcal{N}_{1a} = X^T \mathcal{N}_1 X = \mathcal{N}_1, \quad \mathcal{H}_{1a} = X^T \mathcal{H}_1 X = \begin{bmatrix} \hat{\mathcal{H}}_{11} & \bar{\mathcal{H}}_{12} & \hat{H}_{13} \\ \bar{\mathcal{H}}_{12}^T & \bar{\mathcal{H}}_{22} & 0 \\ \hat{\mathcal{H}}_{13}^T & 0 & \Sigma \end{bmatrix}.$$

ETNA
Kent State University
etna@mcs.kent.edu

Perform the Schur decomposition

$$\bar{\mathcal{H}}_{22} = U_2 \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U_2^T,$$

where $\Sigma_m \in \mathbb{R}^{\tilde{\mu},\tilde{\mu}}$ is nonsingular.
Let $(\tilde{\pi}_m, \tilde{\nu}_m, 0)$ be the inertia index of Σ_m.
Set

$$\mathcal{N}_{1b} = \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix}^T \mathcal{N}_{1a} \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_{1b} = \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix}^T \mathcal{H}_{1a} \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{H}}_{11} & * & * & \hat{\mathcal{H}}_{13} \\ * & \Sigma_m & 0 & 0 \\ * & 0 & 0 & 0 \\ \hat{\mathcal{H}}_{13}^T & 0 & 0 & \Sigma \end{bmatrix},$$

partitioned analogously.
Let P be the permutation that interchanges the last two columns and rows of $\mathcal{H}_3$. Set

$$\mathcal{N}_2 = P^T \mathcal{N}_{1b} P = \left[\begin{array}{c|cc|c} \Delta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] =: \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_2 = P^T \mathcal{H}_{1b} P = \left[\begin{array}{c|cc|c} \tilde{\mathcal{H}}_{11} & * & \hat{\mathcal{H}}_{13} & * \\ \hline * & \Sigma_m & 0 & 0 \\ \hat{\mathcal{H}}_{13}^T & 0 & \Sigma & 0 \\ \hline * & 0 & 0 & 0 \end{array}\right]$$

$$=: \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} & \tilde{\mathcal{H}}_{13} \\ \tilde{\mathcal{H}}_{12}^T & \Sigma & 0 \\ \tilde{\mathcal{H}}_{13}^T & 0 & 0 \end{bmatrix}. \tag{B.1}$$

Set $r_m = r_{m-1} + \tilde{\mu}$.
IF $r_m = l - 2p$ THEN
    Set flag $= 1$ and

$$\hat{\mathcal{Y}} = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_{m-1}} \end{bmatrix} X \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix} P,$$

$$\mathcal{Y} = \begin{bmatrix} I_{n_1+\ldots+n_{m-1}} & 0 & 0 \\ 0 & \hat{\mathcal{Y}} & 0 \\ 0 & 0 & I_{q_1+\ldots+q_{m-1}} \end{bmatrix}.$$

ELSE
    Perform a rank revealing factorization or SVD

$$\tilde{\mathcal{H}}_{13} = U_3 \begin{bmatrix} \Gamma_m & 0 \\ 0 & 0 \end{bmatrix} V_3^T,$$

    where $\Gamma_m \in \mathbb{R}^{\tau,\tau}$ is nonsingular.

```
Set
```
$n_m = \tau$, $q_m = l - 2p - r_m$, and

$$\mathcal{N}_3 = \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & V_3 \end{bmatrix}^T \mathcal{N}_2 \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & V_3 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc|cc} \mathcal{N}_{11} & \mathcal{N}_{12} & 0 & 0 & 0 \\ \hline -\mathcal{N}_{12}^T & \mathcal{N}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\mathcal{H}_3 = \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & V_3 \end{bmatrix}^T \mathcal{H}_2 \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & V_3 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc|cc} \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{H}_{13} & \Gamma_m & 0 \\ \hline \mathcal{H}_{12}^T & \mathcal{H}_{22} & \mathcal{H}_{23} & 0 & 0 \\ \mathcal{H}_{13}^T & \mathcal{H}_{23}^T & \Sigma & 0 & 0 \\ \hline \Gamma_m^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\hat{\mathcal{Y}} = \begin{bmatrix} U_1 & 0 \\ 0 & I_{r_{m-1}} \end{bmatrix} X \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I_{r_{m-1}} \end{bmatrix} P \begin{bmatrix} U_3 & 0 & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & V_3 \end{bmatrix}.$$

```
Set
```

$$\mathcal{U} = \begin{bmatrix} I_{n_1+\ldots+n_{m-1}} & 0 & 0 \\ 0 & \hat{\mathcal{Y}} & 0 \\ 0 & 0 & I_{q_1+\ldots+q_{m-1}} \end{bmatrix}.$$

```
Set
```

$$\mathcal{N} = \begin{array}{c} \\ 2p-\tau \\ r_m \end{array} \begin{array}{c} \begin{array}{cc} 2p-\tau & r_m \end{array} \\ \begin{bmatrix} \mathcal{N}_{22} & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \mathcal{H} = \begin{array}{c} \\ 2p-\tau \\ r_m \end{array} \begin{array}{c} \begin{array}{cc} 2p-\tau & r_m \end{array} \\ \begin{bmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{23}^T & \Sigma \end{bmatrix} \end{array} \in \mathbb{R}^{l,l},$$

and $l = 2p - \tau + r_m$.

```
    END IF
  END IF
  Form H = 𝒴^T H 𝒴, N = 𝒴^T N 𝒴, Y = Y 𝒴.
END WHILE
```

We now show that the subpencils generated by two algorithms are equivalent. For this we need the following lemma.

LEMMA B.1. *Suppose that* $A \in \mathbb{R}^{m,n}$ *and* rank $A = r$. *If*

$$X_1^T A Y_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_2^T A Y_2 = \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma_1, \Sigma_2 \in \mathbb{R}^{r,r}$, $X_1, X_2 \in \mathbb{R}^{m,m}$, *and* $Y_1, Y_2 \in \mathbb{R}^{n,n}$ *are nonsingular, then there exist nonsingular matrices*

$$S = \begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix} \in \mathbb{R}^{m,m}, \quad Z = \begin{bmatrix} Z_1 & 0 \\ Z_2 & Z_3 \end{bmatrix} \in \mathbb{R}^{n,n}$$

where $S_1, Z_1 \in \mathbb{R}^{r,r}$, *such that*

$$X_1 = X_2 S, \qquad Y_1 = Y_2 Z, \qquad \Sigma_1 = S_1^T \Sigma_2 Z_1.$$

ETNA
Kent State University
etna@mcs.kent.edu

In particular, if $A = A^T$ or $A = -A^T$ and $X_1 = Y_1$, $X_2 = Y_2$, then $S = Z$ and $\Sigma_1 = Z_1^T \Sigma_2 Z_1$.

Proof. Let $S = X_2^{-1} X_1$ and $Z = Y_2^{-1} Y_1$. Then

$$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = S^T \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} Z.$$

The result follows directly by comparing the blocks on both sides. □

To show the relationship between Algorithms 2 and 1 we denote the blocks in Algorithm 2 by a ˜ and in Algorithm 1 by a ˆ.

Assume that at the beginning of the mth reduction

$$\alpha \tilde{\mathcal{N}} - \beta \tilde{\mathcal{H}} = K^T (\alpha \hat{\mathcal{N}} - \beta \hat{\mathcal{H}}) K \tag{B.2}$$

for some nonsingular matrix $K = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}$, where $K_{11} \in \mathbb{R}^{l-r_m, l-r_m}$, $K_{22} \in \mathbb{R}^{r_m, r_m}$. Then for $\tilde{\mathcal{N}}_{22}$ and $\hat{\mathcal{N}}_{22}$ in $\tilde{\mathcal{N}}$ and $\hat{\mathcal{N}}$, respectively, we have

$$\tilde{\mathcal{N}}_{22} = K_{11}^T \hat{\mathcal{N}}_{22} K_{11}.$$

Let

$$\tilde{U}_1^T \tilde{\mathcal{N}}_{22} \tilde{U}_1 = \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{U}_1^T \hat{\mathcal{N}}_{22} \hat{U}_1 = \begin{bmatrix} \hat{\Delta} & 0 \\ 0 & 0 \end{bmatrix}.$$

By Lemma B.1,

$$\tilde{U}_1 = K_{11}^{-1} \hat{U}_1 M,$$

where $M = \begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}$ is nonsingular and $M_{11} \in \mathbb{R}^{2p,2p}$. Then a simple calculation yields

$$\alpha \tilde{\mathcal{N}}_1 - \beta \tilde{\mathcal{H}}_1 = \tilde{M}^T (\alpha \hat{\mathcal{N}}_1 - \beta \hat{\mathcal{H}}_1) \tilde{M},$$

where

$$\tilde{M} = \begin{bmatrix} \hat{U}_1 & 0 \\ 0 & I_{r_m} \end{bmatrix}^T K \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & I_{r_m} \end{bmatrix} = \left[\begin{array}{cc|c} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]$$

with $M_{33} = K_{22}$. Clearly, then

$$\begin{bmatrix} \tilde{\hat{\mathcal{H}}}_{22} & \tilde{\hat{\mathcal{H}}}_{23} \\ \tilde{\hat{\mathcal{H}}}_{23}^T & \tilde{\Sigma} \end{bmatrix} = \begin{bmatrix} M_{22} & 0 \\ M_{32} & M_{33} \end{bmatrix}^T \begin{bmatrix} \hat{\hat{\mathcal{H}}}_{22} & \hat{\hat{\mathcal{H}}}_{23} \\ \hat{\hat{\mathcal{H}}}_{23}^T & \hat{\Sigma} \end{bmatrix} \begin{bmatrix} M_{22} & 0 \\ M_{32} & M_{33} \end{bmatrix}. \tag{B.3}$$

In Algorithm 2 we then determine a nonsingular matrix $\tilde{Z}$ such that

$$\tilde{Z}^T \begin{bmatrix} \tilde{\hat{\mathcal{H}}}_{22} & \tilde{\hat{\mathcal{H}}}_{23} \\ \tilde{\hat{\mathcal{H}}}_{23}^T & \tilde{\Sigma} \end{bmatrix} \tilde{Z} = \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix},$$

ETNA
Kent State University
etna@mcs.kent.edu

where $\tilde{\Sigma} := \operatorname{diag}(\Sigma_m, \tilde{\Sigma})$ and in Algorithm 1 we determine an orthogonal matrix $\hat{Z}$ such that

$$\hat{Z}^T \begin{bmatrix} \hat{\hat{\mathcal{H}}}_{22} & \hat{\mathcal{H}}_{23} \\ \hat{\mathcal{H}}_{23}^T & \hat{\Sigma} \end{bmatrix} \hat{Z} = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}.$$

By (B.3), the new $\tilde{\Sigma}$ and $\hat{\Sigma}$ must have the same inertia and the same size $r_{m+1} \times r_{m+1}$. Moreover, by Lemma B.1,

$$\tilde{Z} = \begin{bmatrix} M_{22} & 0 \\ M_{32} & M_{33} \end{bmatrix}^{-1} \hat{Z} L,$$

where $L = \begin{bmatrix} L_{22} & 0 \\ L_{32} & L_{33} \end{bmatrix}$ is nonsingular and $L_{22} \in \mathbb{R}^{r_{m+1}, r_{m+1}}$.

We then have

$$\alpha\tilde{\mathcal{N}}_2 - \beta\tilde{\mathcal{H}}_2 = \tilde{L}^T(\alpha\hat{\mathcal{N}}_2 - \beta\hat{\mathcal{H}}_2)\tilde{L}, \tag{B.4}$$

where

$$\tilde{L} = \begin{bmatrix} I_{2p} & 0 \\ 0 & \hat{Z} \end{bmatrix}^{-1} \tilde{M} \begin{bmatrix} I_{2p} & 0 \\ 0 & \tilde{Z} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix},$$

with $L_{11} = M_{11}$.

Let $\tilde{\mathcal{H}}_{13}$ be the block of $\tilde{\mathcal{H}}_2$ in (B.1) and $\hat{\mathcal{H}}_{13}$ be the corresponding block in $\hat{\mathcal{H}}_2$. By comparing the blocks in (B.4) we have

$$\tilde{\mathcal{H}}_{13} = L_{11}^T \hat{\mathcal{H}}_{13} L_{33}.$$

Let

$$\tilde{U}^T \tilde{\mathcal{H}}_{13} \tilde{V} = \begin{bmatrix} \tilde{\Gamma}_m & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{U}^T \hat{\mathcal{H}}_{13} \hat{V} = \begin{bmatrix} \hat{\Gamma}_m & 0 \\ 0 & 0 \end{bmatrix}$$

be the computed rank revealing factorizations. Then $\tilde{\Gamma}_m$ and $\hat{\Gamma}_m$ must have the same size $\tau \times \tau$. Again by Lemma B.1

$$\tilde{U} = L_{11}^{-1} \hat{U} S, \quad \tilde{V} = L_{33}^{-1} \hat{V} T,$$

where

$$S = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix},$$

and $S_{11}, T_{11} \in \mathbb{R}^{\tau,\tau}$. Then

$$\alpha\tilde{\mathcal{N}}_3 - \beta\tilde{\mathcal{H}}_3 = \tilde{S}^T(\alpha\hat{\mathcal{N}}_3 - \beta\hat{\mathcal{H}}_3)\tilde{S},$$

where

$$\tilde{S} = \begin{bmatrix} \hat{U} & 0 & 0 \\ 0 & I_{r_{m+1}} & 0 \\ 0 & 0 & \hat{V} \end{bmatrix}^{-1} \tilde{L} \begin{bmatrix} \tilde{U} & 0 & 0 \\ 0 & I_{r_{m+1}} & 0 \\ 0 & 0 & \tilde{V} \end{bmatrix}$$

$$= \begin{bmatrix} S_{11} & 0 & 0 & 0 & 0 \\ S_{21} & S_{22} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 \\ S_{41} & S_{42} & S_{43} & S_{44} & 0 \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} \end{bmatrix},$$

ETNA
Kent State University
etna@mcs.kent.edu

with $S_{33} = L_{22}$ and $S_{44} = T_{11}$, $S_{54} = T_{21}$, and $S_{55} = T_{22}$. It is then evident that the newly generated subpencils $\alpha\tilde{\mathcal{N}} - \beta\tilde{\mathcal{H}}$ and $\alpha\hat{\mathcal{N}} - \beta\hat{\mathcal{H}}$ satisfy

$$\alpha\tilde{\mathcal{N}} - \beta\tilde{\mathcal{H}} = \begin{bmatrix} S_{22} & 0 \\ S_{32} & S_{33} \end{bmatrix}^T (\alpha\hat{\mathcal{N}} - \beta\hat{\mathcal{H}}) \begin{bmatrix} S_{22} & 0 \\ S_{32} & S_{33} \end{bmatrix},$$

which the same as (B.2). Since both algorithms start with

$$\tilde{\mathcal{N}} = \hat{\mathcal{N}} = N, \quad \tilde{\mathcal{H}} = \hat{\mathcal{H}} = H,$$

it follows by induction that they generate the same integers n_j, q_j, r_j. The inertia indices satisfy $\tilde{\pi}_j = \pi_j - \pi_{j-1}$ and $\tilde{\nu}_j = \nu_j - \nu_{j-1}$. □

In the following we will show that by carrying out some further block Gauß elimination steps, the staircase form computed by Algorithm 2 can be reduced close to the even Kronecker-like form. In Algorithm 2 it is not necessary to move all the blocks Σ toward the center. So the permutation with P in (B.1) does not necessarily have to be carried out. The staircase form has the following block structure.

$$\hat{N} = Y^T N Y = \left[\begin{array}{cccc|c|cccc} N_{11} & \cdots & \cdots & N_{1,m} & N_{1,m+1} & N_{1,m+2} & \cdots & N_{1,2m} & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\ \vdots & & \ddots & \vdots & \vdots & N_{m-1,m+2} & \iddots & & \\ -N_{1,m}^T & \cdots & \cdots & N_{m,m} & N_{m,m+1} & 0 & & & \\ \hline -N_{1,m+1}^T & \cdots & \cdots & -N_{m,m+1}^T & N_{m+1,m+1} & & & & \\ \hline -N_{1,m}^T & \cdots & -N_{m-1,m+2}^T & 0 & & 0 & & & \\ \vdots & \iddots & \iddots & & & & \ddots & & \\ -N_{1,2m}^T & \iddots & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{array}\right] \begin{array}{c} n_1 \\ \vdots \\ \vdots \\ n_m \\ \hline l \\ \hline \tilde{q}_m \\ \vdots \\ \vdots \\ \tilde{q}_1 \end{array},$$

$$\hat{H} = Y^T H Y = \left[\begin{array}{cccc|c|cccc} H_{11} & \cdots & \cdots & H_{1,m} & H_{1,m+1} & H_{1,m+2} & \cdots & \cdots & H_{1,2m+1} \\ \vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\ H_{1,m}^T & \cdots & \cdots & H_{m,m} & H_{m,m+1} & H_{m,m+2} & & & \\ \hline H_{1,m+1}^T & \cdots & \cdots & H_{m,m+1}^T & H_{m+1,m+1} & & & & \\ \hline H_{1,m+2}^T & \cdots & \cdots & H_{m,m+2}^T & & H_{m+2,m+2} & & & \\ \vdots & & \iddots & & & & \ddots & & \\ \vdots & \iddots & & & & & & \ddots & \\ H_{1,2m+1}^T & & & & & & & & H_{2m+1,2m+1} \end{array}\right] \begin{array}{c} n_1 \\ \vdots \\ \vdots \\ n_m \\ \hline l \\ \hline \tilde{q}_m \\ \vdots \\ \vdots \\ \tilde{q}_1 \end{array}, \tag{B.5}$$

where for $\tilde{r}_j = r_j - r_{j-1}$, $\tilde{q}_j = q_j + \tilde{r}_j$, $(1 \le j \le m)$, and $l = 2p + \tilde{r}_{m+1}$. The blocks have the following properties.

$$N_{j,2m+1-j} \in \mathbb{R}^{n_j,\tilde{q}_{j+1}}, \qquad \operatorname{rank} N_{j,2m+1-j} = \tilde{q}_{j+1}, \quad 1 \le j \le m-1,$$

$$N_{m+1,m+1} = \begin{bmatrix} \Delta & 0 \\ 0 & 0_{\tilde{r}_{m+1}} \end{bmatrix}, \qquad \Delta \in \mathbb{R}^{2p,2p},$$

$$H_{j,2m+2-j} = \begin{array}{c} \\ n_j \end{array} \!\!\begin{array}{ccc} \tilde{r}_j & n_j & q_j - n_j \\ \left[\Pi_j\right. & \Gamma_j & \left. 0 \right] \end{array}, \qquad 1 \le j \le m,$$

ETNA
Kent State University
etna@mcs.kent.edu

$$H_{2m+2-j,2m+2-j} = \begin{array}{c} \\ \tilde{r}_j \\ q_j \end{array} \begin{array}{c} \begin{array}{cc} \tilde{r}_j & q_j \end{array} \\ \left[\begin{array}{cc} \Sigma_j & 0 \\ 0 & 0 \end{array}\right] \end{array}, \qquad 1 \le j \le m,$$

$$H_{m+1,m+1} = \left[\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{m+1} \end{array}\right], \quad \Sigma_{11} \in \mathbb{R}^{2p,2p}, \quad \Sigma_{m+1} \in \mathbb{R}^{\tilde{r}_{m+1},\tilde{r}_{m+1}},$$

where all the blocks $\Delta, \Sigma_j, \Gamma_j$ are nonsingular. Without loss of generality, we may assume that $\Delta = J_{2p}$ and $\Sigma_j = \mathrm{diag}(I_{\tilde{\pi}_j}, -I_{\tilde{\nu}_j})$.

The property that $N_{j,2m+1-j}$ has full column rank can be shown as follows.

After the first step of reduction we have

$$N = \left[\begin{array}{c|c|ccc} \mathcal{N}_{11} & \mathcal{N}_{12} & 0 & 0 & 0 \\ \hline -\mathcal{N}_{12}^T & \mathcal{N}_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

In the next step, after having compressed $\mathcal{N}_{22}$, N is changed to

$$\left[\begin{array}{c|cc|ccc} \hat{\mathcal{N}}_{11} & \hat{\mathcal{N}}_{12} & \hat{\mathcal{N}}_{13} & 0 & 0 & 0 \\ \hline -\hat{\mathcal{N}}_{12}^T & \Delta & 0 & 0 & 0 & 0 \\ -\hat{\mathcal{N}}_{13}^T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since $\left[\begin{array}{cc} \mathcal{N}_{11} & \mathcal{N}_{12} \\ -\mathcal{N}_{12}^T & \mathcal{N}_{22} \end{array}\right]$ and Δ are nonsingular, $\hat{\mathcal{N}}_{13}$ has to be of full column rank. It is easily seen that $\hat{\mathcal{N}}_{13}$ is equivalent to $N_{1,2m}$. So $N_{1,2m}$ has full column rank. By induction, it follows that the other blocks $N_{j,2m+1-j}$ have full column rank as well.

We now begin further reductions on the pencil (B.5). The reduction process is described in the following algorithm.

ALGORITHM 3. Let $N := Y^T N Y$ and $H := Y^T H Y$ be given as in (B.5).

Annihilate the blocks Σ_{12} and Σ_{12}^T with pivot block Σ_{m+1} in $H_{m+1,m+1}$.
Annihilate the blocks in $N_{m,m+1}$ $(-N_{m,m+1}^T)$ above and to the left of Δ in
$N_{m+1,m+1}$ with the pivot block Δ. Then

$$\left[\begin{array}{cc} N_{m,m} & N_{m,m+1} \\ -N_{m,m+1}^T & N_{m+1,m+1} \end{array}\right] = \left[\begin{array}{c|cc} \tilde{N}_{m,m} & 0 & \tilde{\Phi}_m \\ \hline 0 & \Delta & 0 \\ -\tilde{\Phi}_m^T & 0 & 0 \end{array}\right].$$

Because by the reduction procedure $\left[\begin{array}{cc} N_{m,m} & N_{m,m+1} \\ -N_{m,m+1}^T & N_{m+1,m+1} \end{array}\right]$ is nonsingular, $\tilde{\Phi}_m$ has to be of full column rank. So we can determine a nonsingular matrix X such that $Z^T \tilde{\Phi}_j = \left[\begin{array}{c} 0 \\ I_{\tilde{r}_{m+1}} \end{array}\right]$. Then

$$\left[\begin{array}{cc} Z & 0 \\ 0 & I \end{array}\right]^T \left[\begin{array}{cc} N_{m,m} & N_{m,m+1} \\ -N_{m,m+1}^T & N_{m+1,m+1} \end{array}\right] \left[\begin{array}{cc} Z & 0 \\ 0 & I \end{array}\right] = \left[\begin{array}{cc|cc} \Psi_{11} & \Psi_{12} & 0 & 0 \\ -\Psi_{12}^T & \Psi_{22} & 0 & \tilde{I}_{r_{m+1}} \\ \hline 0 & 0 & \Delta & 0 \\ 0 & -\tilde{I}_{r_{m+1}} & 0 & 0 \end{array}\right].$$

ETNA
Kent State University
etna@mcs.kent.edu

We then annihilate Ψ_{22}, Ψ_{12}, and $-\Psi_{12}^T$ by performing another block Gauß congruence transformation with pivot blocks $I_{\tilde r_{m+1}}$ and $-I_{\tilde r_{m+1}}$. Again by the nonsingularity of $\begin{bmatrix} N_{m,m} & N_{m,m+1} \\ -N_{m,m+1}^T & N_{m+1,m+1} \end{bmatrix}$, Ψ_{11} has to be nonsingular.
So it can be compressed further to $\Psi_m := J_{(n_m - \tilde r_{m+1})/2}$ by performing one more congruence transformation. Eventually,

$$\begin{bmatrix} N_{m,m} & N_{m,m+1} \\ -N_{m,m+1}^T & N_{m+1,m+1} \end{bmatrix} \longrightarrow \left[\begin{array}{cc|cc} \Psi_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\tilde r_{m+1}} \\ \hline 0 & 0 & \Delta & 0 \\ 0 & -I_{\tilde r_{m+1}} & 0 & 0 \end{array}\right].$$

Applying the same sequence of congruence transformations to H, it is easy to check that the block structures and ranks will not change.

We now proceed by working on H. First we simplify Γ_j and Γ_j^T to I_{n_m} in $H_{m,m+2}$ and $H_{m,m+2}^T$ by post-multiplying $\mathrm{diag}(I, \Gamma_j^{-1}, I)$ to $H_{m,m+2}$ and pre-multiplying its transpose to $H_{m,m+2}^T$. Note that this transformation does not affect $H_{m+2,m+2}$ and the blocks in N.

Second, we annihilate Π_m, $-\Pi_m^T$ in $H_{m,m+2}$, $H_{m,m+2}^T$ and $H_{m,m+1}, H_{m,m+1}^T, H_{m,m}$ with pivot block I_{n_m} in $H_{m,m+2}$ and $H_{m,m+2}^T$.

FOR $j = m-1, \dots, 1$

a) Annihilate the blocks $N_{j,j+1}, \dots, N_{j,2m-j}$ as well as $-N_{j,j+1}^T, \dots, -N_{j,2m-j}^T$ in N with the nonsingular blocks in $-N_{j+1,2m-j}^T, \dots, -N_{m,m+1}^T$, $N_{m+1,m+1}$, $N_{m,m+1}, \dots, N_{j+1,2m-j}$ as pivots.

Simplify $N_{j,2m+1-j}$ and $-N_{j,2m+1-j}^T$ to $\begin{bmatrix} 0 \\ I_{\tilde q_{j+1}} \end{bmatrix}$ and $-\begin{bmatrix} 0 \\ I_{\tilde q_{j+1}} \end{bmatrix}^T$, respectively.

Annihilate the blocks in $N_{j,j}$ with pivot blocks $I_{\tilde q_{j+1}}$ and $-I_{\tilde q_{j+1}}$ from $N_{j,2m+1-j}$ and $-N_{j,2m+1-j}^T$ to get

$$N_{j,j} = \begin{bmatrix} \Psi_j & 0 \\ 0 & 0 \end{bmatrix}.$$

With the same argument as before, Ψ_j must be nonsingular and thus we reduce Ψ_j to $J_{(n_j - \tilde q_{j+1})/2}$.

b) Reduce the blocks Γ_j, Γ_j^T in $H_{j,2m+2-j}$ and $H_{j,2m+2-j}^T$ to I_{n_j}.
Annihilate the blocks $H_{j,j}$, $H_{j,j+1}, \dots, H_{j,2m+1-j}$, $H_{j,j+1}^T, \dots, H_{j,2m+1-j}^T$, as well as Π_j, π_j^T in $H_{j,2m+2-j}$ and $H_{j,2m+2-j}^T$ with block pivot I_{n_j} from $H_{j,2m+2-j}$ and $H_{j,2m+2-j}^T$.

END FOR j

With this further reduction, the matrices N and H are transformed as

$$X^T N X =$$

$$\left[\begin{array}{cccc|c|cccc}
N_{11} & & & & & & & N_{1,2\delta} & 0 \\
 & \ddots & & & & & \iddots & \iddots & \\
 & & \ddots & & & N_{m-1,m+1} & \iddots & & \\
 & & & N_{m,m} & N_{m,m+1} & 0 & & & \\
\hline
 & & & -N_{m,m+1}^T & N_{m+1,m+1} & & & & \\
\hline
 & & -N_{m-1,m+1}^T & 0 & & 0 & & & \\
 & \iddots & \iddots & & & & \ddots & & \\
-N_{1,2m}^T & \iddots & & & & & & \ddots & \\
0 & & & & & & & & 0
\end{array}\right]
\begin{array}{c}
n_1 \\ \vdots \\ \vdots \\ n_m \\ \hline l \\ \hline \tilde q_m \\ \vdots \\ \vdots \\ \tilde q_1
\end{array},$$

$$X^T H X =$$
$$\left[\begin{array}{ccc|c|ccc} 0 & & & & & & H_{1,2m+1} \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 0 & & H_{m,m+2} & & \\ \hline & & & H_{m+1,m+1} & & & \\ \hline & & H_{m,m+2}^T & & H_{m+2,m+2} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ H_{1,2m+1}^T & & & & & & H_{2m+1,2m+1} \end{array}\right] \begin{array}{c} n_1 \\ \vdots \\ n_m \\ \hline l \\ \hline \tilde{q}_m \\ \vdots \\ \tilde{q}_1 \end{array} \tag{B.6}$$

for some nonsingular matrix X, where (note that $q_j = \tilde{q}_j - \tilde{r}_j$)

$$N_{j,2m+1-j} = \begin{array}{c} \\ n_j - \tilde{q}_{j+1} \\ \tilde{r}_{j+1} \\ q_{j+1} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{r}_{j+1} & q_{j+1} \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ I & 0 \\ 0 & I \end{array}\right] \end{array}, \qquad 1 \le j \le m-1,$$

$$N_{j,j} = \begin{array}{c} \\ n_j - \tilde{q}_{j+1} \\ \tilde{q}_{j+1} \end{array} \begin{array}{c} \begin{array}{cc} n_j - \tilde{q}_{j+1} & \tilde{q}_{j+1} \end{array} \\ \left[\begin{array}{cc} J & 0 \\ 0 & 0 \end{array}\right] \end{array}, \quad \tilde{q}_{m+1} = \tilde{r}_{m+1}, \qquad 1 \le j \le m,$$

$$N_{m,m+1} = \begin{array}{c} \\ n_m - \tilde{r}_{m+1} \\ \tilde{r}_{m+1} \end{array} \begin{array}{c} \begin{array}{cc} 2p & \tilde{r}_{m+1} \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array}\right] \end{array},$$

$$N_{m+1,m+1} = \left[\begin{array}{cc} \Delta & 0 \\ 0 & 0_{\tilde{r}_{m+1}} \end{array}\right], \qquad \Delta = J_p,$$

$$H_{j,2m+2-j} = n_j \begin{array}{c} \begin{array}{ccc} \tilde{r}_j & n_j & q_j - n_j \end{array} \\ \left[\begin{array}{ccc} 0 & I & 0 \end{array}\right] \end{array}, \qquad 1 \le j \le m,$$

$$H_{2m+2-j,2m+2-j} = \begin{array}{c} \\ \tilde{r}_j \\ q_j \end{array} \begin{array}{c} \begin{array}{cc} \tilde{r}_j & q_j \end{array} \\ \left[\begin{array}{cc} \Sigma_j & 0 \\ 0 & 0 \end{array}\right] \end{array}, \qquad 1 \le j \le m,$$

$$H_{m+1,m+1} = \left[\begin{array}{cc} \Theta & 0 \\ 0 & \Sigma_{m+1} \end{array}\right], \qquad \Theta = \Theta^T \in \mathbb{R}^{2p,2p}.$$

Note that all $\Sigma_j = \mathrm{diag}(I_{\tilde{\pi}_j}, -I_{\tilde{\nu}_j})$, $1 \le j \le m+1$ are signature matrices.

By performing a congruence transformation to the pencil with $X^T N X$, $X^T H X$ in (B.6) with an appropriate permutation, we obtain the structured Kronecker form (2.2) of $\alpha N - \beta H$. This leads to the conclusion in Theorem 3.3.

Let us illustrate this complicated process by an example.

EXAMPLE B.2. Let

$$\alpha N - \beta H = \alpha \left[\begin{array}{cc|c|c|c|cc|cccc} 0 & 0 & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline & & 0 & & 1 & 0 & 0 & & & & \\ \hline & & & \Delta & & & & & & & \\ \hline & & -1 & & 0 & & & & & & \\ \hline -1 & 0 & 0 & & & 0 & 0 & & & & \\ 0 & -1 & 0 & & & 0 & 0 & & & & \\ \hline 0 & 0 & & & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & & 0 & 0 & 0 & 0 \end{array}\right]$$

ETNA
Kent State University
etna@mcs.kent.edu

$$
-\beta\left[\begin{array}{cc|c|c|c|cc|cccc}
0 & 0 & & & & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & & & & 0 & 0 & 0 & 0 & 1 & 0\\ \hline
 & & 0 & & 0 & 1 & 0 & & & & \\ \hline
 & & & \Theta & & & & & & & \\ \hline
 & & 0 & & 0 & & & & & & \\ \hline
 & 0 & 1 & & & 0 & 0 & & & & \\
0 & 0 & 0 & & & 0 & 0 & & & & \\ \hline
0 & 0 & & & & & & -1 & 0 & 0 & 0\\
1 & 0 & & & & & & 0 & 0 & 0 & 0\\
0 & 1 & & & & & & 0 & 0 & 0 & 0\\
0 & 0 & & & & & & 0 & 0 & 0 & 0
\end{array}\right],
$$

where Δ is nonsingular. We have $m = 3$, and

$$
\begin{array}{llllll}
n_1 = 2, & n_2 = 1, & n_3 = 0; & q_1 = 3, & q_2 = 2, & q_3 = 1,\\
\tilde{r}_1 = 1, & \tilde{r}_2 = 0, & \tilde{r}_3 = 0, & \tilde{r}_4 = 0. & &
\end{array}
$$

Then

$$
\begin{array}{l}
q_1 - n_1 = 1, \quad q_2 - n_2 = 1, \quad q_3 - n_3 = 1, \qquad (\text{for } \mathcal{S}_{j-1})\\
k_1 = (n_1 - q_2 - \tilde{r}_2)/2 = 0, \quad k_2 = (n_2 - q_3 - \tilde{r}_3)/2 = 0, \qquad (\text{for } \mathcal{I}_{2j}),\\
\tilde{r}_1 = \tilde{\nu}_1 = 1, \quad \tilde{r}_2 = 0, \quad \tilde{r}_3 = 0, \quad \tilde{r}_4 = 0, \qquad (\text{for } \mathcal{I}_{2j-1})
\end{array}
$$

By Theorem 3.3, we conclude that the pencil has one singular block $\mathcal{O}_1 = \alpha 0 - \beta 0$; one block $\mathcal{S}_1$; one block $\mathcal{S}_2$; and one block $\mathcal{I}_1$ with $s = -1$. The canonical structure of $\alpha N - \beta H$ associated with the finite eigenvalues is the same as that of $\alpha\Delta - \beta\Theta$.

In fact, if we rearrange the columns and rows of N and H in the order 4, 8, 1, 3, 5, 6, 9, 2, 7, 10, 11 and let P be the corresponding permutation matrix, then

$$
P^T(\alpha N - \beta H)P = \alpha\left[\begin{array}{c|c|ccccc|ccc|c}
\Delta & & & & & & & & & & \\ \hline
 & 0 & & & & & & & & & \\ \hline
 & & 0 & 0 & 0 & 1 & 0 & & & & \\
 & & 0 & 0 & 1 & 0 & 0 & & & & \\
 & & 0 & -1 & 0 & 0 & 0 & & & & \\
 & & -1 & 0 & 0 & 0 & 0 & & & & \\
 & & 0 & 0 & 0 & 0 & 0 & & & & \\ \hline
 & & & & & & & 0 & 1 & 0 & \\
 & & & & & & & -1 & 0 & 0 & \\
 & & & & & & & 0 & 0 & 0 & \\ \hline
 & & & & & & & & & & 0
\end{array}\right]
$$

$$
-\beta\left[\begin{array}{c|c|ccccc|ccc|c}
\Theta & & & & & & & & & & \\ \hline
 & -1 & & & & & & & & & \\ \hline
 & & 0 & 0 & 0 & 0 & 1 & & & & \\
 & & 0 & 0 & 0 & 1 & 0 & & & & \\
 & & 0 & 0 & 0 & 0 & 0 & & & & \\
 & & 0 & -1 & 0 & 0 & 0 & & & & \\
 & & -1 & 0 & 0 & 0 & 0 & & & & \\ \hline
 & & & & & & & 0 & 0 & 1 & \\
 & & & & & & & 0 & 0 & 0 & \\
 & & & & & & & -1 & 0 & 0 & \\ \hline
 & & & & & & & & & & 0
\end{array}\right],
$$

ETNA
Kent State University
etna@mcs.kent.edu

which is in the structured Kronecker form.

Appendix C. A structured algorithm for computing the structured Schur form of regular skew-symmetric/symmetric pencil of index at most 1**.** In this appendix we present an algorithm for computing the structured Schur form (4.2). We call a matrix $U \in \mathbb{R}^{2n,2n}$ *orthogonal-symplectic* if $U^TU = I_{2n}$ and $U^TJ_nU = J_n$. In this algorithm we will denote by $G(i,j)$ a Givens rotation operating in rows or columns i and j. If $G_{ij} \in \mathbb{R}^{n,n}$ then $G_s(i,j) := \mathrm{diag}(G_{ij}, G_{ij})$ is orthogonal-symplectic and also $G(n,2n) \in \mathbb{R}^{2n,2n}$ is orthogonal-symplectic. Finally in the algorithm we use the orthogonal-symplectic permutation matrix $P = [e_n, e_1, \ldots, e_{n-1}, e_{2n}, e_{n+1}, \ldots, e_{2n-1}]$.

ALGORITHM 4. For a regular skew-Hamiltonian/Hamiltonian pencil $\alpha Z - \beta M$ of index at most 1 with $Z = SJ_nS^TJ_n^T \in \mathbb{R}^{2n,2n}$ and $S = \mathrm{diag}(D, 0, D, 0)$, where $D \in \mathbb{R}^{p,p}$ is positive diagonal, this algorithm computes orthogonal matrices Q_1, Q_2 and orthogonal symplectic matrices U_1, U_2 such that $Q_1^TMQ_2$, $Q_1^TSU_1$, $U_2^T(J_nS^TJ_n^T)Q_2$ are in the form (4.2).

Let

$$M = MP, \quad Q_1 = U_1 = I_{2n}, \quad Q_2 = U_2 = P,$$
$$T = P^T(J_nS^TJ_n^T)P = \mathrm{diag}(0, D, 0_{n-p-1}, 0, D, 0_{n-p-1}).$$

Step 1. Reduce M, S, T to a form that is as (4.2) with the exception that M_{44} is lower Hessenberg.

FOR $k = 1, \ldots, n$

% *Annihilate* $m_{n+k,k}, \ldots, m_{2n-1,k}$

FOR $j = n+k, \ldots, 2n-1$

Determine $G(j, j+1)$ to annihilate m_{jk}. Set

$$M = G^T(j, j+1)M, \quad S = G^T(j, j+1)S, \quad Q_1 = Q_1G(j, j+1).$$

Determine $G_s(j-n, j-n+1)$ to annihilate $s_{j,j+1}$. Set

$$S = SG_s(j-n, j-n+1), \quad U_1 = U_1G_s(j-n, j-n+1).$$

Determine $G(j-n, j-n+1)$ to annihilate $s_{j-n+1,j-n}$. Set

$$M = G^T(j-n, j-n+1)M, \quad S = G^T(j-n, j-n+1)S,$$
$$Q_1 = Q_1G(j-n, j-n+1).$$

END FOR

% *Annihilate* $m_{2n,k}$

Determine $G(n, 2n)$ to annihilate $m_{2n,k}$. Set

$$M = G^T(n, 2n)M, \quad S = G^T(n, 2n)S, \quad Q_1 = Q_1G(n, 2n).$$

Determine another $G(n, 2n)$ to annihilate $s_{2n,n}$. Set

$$S = SG(n, 2n), \quad U_1 = U_1G(n, 2n).$$

% *Annihilate* $m_{n,k}, \ldots, m_{k+1,k}$

FOR $j = n, \ldots, k+1$

```
        Determine G(j-1,j) to annihilate m_{j,k}.  Set
```
$$M = G^T(j-1,j)M, \quad S = G^T(j-1,j)S, \quad Q_1 = Q_1 G(j-1,j).$$
```
        Determine G_s(j-1,j) to annihilate s_{j,j+1}.  Set
```
$$S = SG_s(j-1,j), \quad U_1 = U_1 G_s(j-1,j).$$
```
        Determine G(n+j-1,n+j) to annihilate s_{n+j-1,n+j}.  Set
```
$$M = G^T(n+j-1,n+j)M, \quad S = G^T(n+j-1,n+j)S,$$
$$Q_1 = Q_1 G(n+j-1,n+j).$$
```
    END FOR
% Annihilate m_{n+k,k+1}, ..., m_{n+k,n-1}
    FOR j = k+1,...,n-1
        Determine G(j,j+1) to annihilate m_{n+k,j}.  Set
```
$$M = MG(j,j+1), \quad T = TG(j,j+1), \quad Q_2 = Q_2 G(j,j+1).$$
```
        Determine G_s(j,j+1) to annihilate t_{j+1,j}.  Set
```
$$T = G_s^T(j,j+1)T, \quad U_2 = U_2 G_s(j,j+1).$$
```
        Determine G(n+j,n+j+1) to annihilate t_{n+j,n+j+1}.  Set
```
$$M = MG(n+j,n+j+1), \quad T = TG(n+j,n+j+1),$$
$$Q_2 = Q_2 G(n+j,n+j+1).$$
```
    END FOR
% Annihilate m_{n+k,n}
    Determine G(n,2n) to annihilate m_{n+k,n}.  Set
```
$$M = MG(n,2n), \quad T = TG(n,2n), \quad Q_2 = Q_2 G(n,2n).$$
```
    Determine another G(n,2n) to annihilate t_{2n,n}.  Set
```
$$T = G^T(n,2n)T, \quad U_2 = U_2 G(n,2n).$$
```
% Annihilate m_{n+k,2n}, ..., m_{n+k,n+k+2}
    FOR j = 2n,...,n+k+2
        Determine G(j-1,j) to annihilate m_{n+k,j}.  Set
```
$$M = MG(j-1,j), \quad T = TG(j-1,j), \quad Q_2 = Q_2 G(j-1,j).$$
```
        Determine G_s(j-n-1,j-n) to annihilate t_{j-1,j}.  Set
```
$$T = G_s^T(j-n-1,j-n)T, \quad U_2 = U_2 G_s(j-n-1,j-n).$$
```
        Determine G(j-n-1,j-n) to annihilate t_{j-n,j-n-1}.  Set
```
$$M = MG(j-n-1,j-n), \quad T = TG(j-n-1,j-n),$$
$$Q_2 = Q_2 G(j-n-1,j-n).$$

ETNA
Kent State University
etna@mcs.kent.edu


```
        END FOR
END FOR
(Note that the (3,3) block of T now is zero.)
% Annihilate m_{n+1,n+2}, ..., m_{n+p,n+p+1}
FOR k = n+1, ..., n+p
        Determine G(k,k+1) to annihilate m_{k,k+1}.  Set
```

$$M = MG(k,k+1), \quad T = TG(k,k+1), \quad Q_2 = Q_2G(k,k+1).$$

```
END FOR
```

Step 2. Reduce M_{44} to lower quasi-triangular form.

Partition the matrices M, S, T as in (4.2). Apply the periodic QZ-algorithm, see e.g. ([7, 22, 23]) to the formal product $S_{44}^{-T}S_{22}^{-1}M_{22}T_{22}^{-1}T_{44}^{-T}M_{44}^T$ to determine orthogonal matrices W_j, $j = 1, \ldots, 6$ such that $W_1^TS_{44}W_2$ and $W_5^TT_{44}W_6$ are lower triangular, $W_3^TS_{22}W_2$ and $W_3^TM_{22}W_4$, $W_5^TT_{22}W_4$ are upper triangular, and $W_1^TM_{44}W_6$ is lower quasi-triangular.
Let

$$\mathcal{Q}_1 = \text{diag}(I_p, W_3, I_p, W_1), \quad \mathcal{Q}_2 = \text{diag}(I_p, W_4, I_p, W_6),$$
$$\mathcal{U}_1 = \text{diag}(I_p, W_2, I_p, W_2), \quad \mathcal{U}_2 = \text{diag}(I_p, W_5, I_p, W_5).$$

Set

$$M = \mathcal{Q}_1^TM\mathcal{Q}_2, \quad S = \mathcal{Q}_1^TS\mathcal{U}_1, \quad T = \mathcal{U}_2^TT\mathcal{Q}_2,$$
$$Q_1 = Q_1\mathcal{Q}_1, \quad Q_2 = Q_2\mathcal{Q}_2, \quad U_1 = U_1\mathcal{U}_1, \quad U_2 = U_2\mathcal{U}_2.$$

Once the form (4.2) has been obtained, we introduce

$$\hat{S} = Q_1^TSU_1, \quad \hat{T} = U_2^TTQ_2 \quad \text{and} \quad \hat{M} = Q_1^TMQ_2. \tag{C.1}$$

Because $T = J_nS^TJ_n^T$, $M = -J_nMJ_n^T$, $U_1J_n = J_nU_1$, and $U_2J_n = J_nU_2$, we have

$$\begin{aligned} U_1^TT(J_nQ_1J_n^T) &= J_n\hat{S}^TJ_n^T, \\ (J_nQ_2J_n^T)^TSU_2 &= J_n\hat{T}^TJ_n^T, \\ (J_nQ_2J_n^T)^TM(J_nQ_1^TJ_n)^T &= -J_n\hat{M}^TJ_n^T \end{aligned} \tag{C.2}$$

and from $Z = ST$, we have

$$\begin{aligned} Q_1^TZ(J_nQ_1J_n^T) &= Q_1^TSU_1U_1^TT(J_nQ_1J_n^T) = \hat{S}J_n\hat{S}^TJ_n^T, \\ (J_nQ_2J_n^T)^TZQ_2 &= (J_nQ_2J_n^T)SU_2U_2^TTQ_2 = J_n\hat{T}J_n^T\hat{T}. \end{aligned} \tag{C.3}$$

It was shown in [3] that the finite eigenvalues of $\alpha Z - \beta M$ are exactly the finite eigenvalues of

$$\alpha\mathcal{Z} - \beta\mathcal{M} = \alpha\begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} - \beta\begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix}$$

(with doubled algebraic multiplicity). Let $\mathcal{Q}_1 = \text{diag}(Q_1, J_nQ_2J_n^T,)$, $\mathcal{Q}_2 = \text{diag}(J_nQ_1J_n^T, Q_2)$. If follows from (C.1), (C.2), (C.3), and (4.2) that

$$\mathcal{Q}_1^T(\alpha\mathcal{Z} - \beta\mathcal{M})\mathcal{Q}_2 = \alpha\begin{bmatrix} \hat{S}J_n\hat{S}^TJ_n^T & 0 \\ 0 & J_n\hat{T}J_n^T\hat{T} \end{bmatrix} - \beta\begin{bmatrix} 0 & \hat{M} \\ -J_n\hat{M}^TJ_n^T & 0 \end{bmatrix}$$

$$= \alpha \left[\begin{array}{cccc|cccc} 0 & * & * & * & & & & \\ 0 & S_{22}S_{44}^T & * & * & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & * & S_{44}S_{22}^T & & & & \\ \hline & & & & 0 & * & * & * \\ & & & & 0 & T_{44}^T T_{22} & * & * \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & * & T_{22}^T T_{44} \end{array}\right]$$

$$-\beta \left[\begin{array}{cccc|cccc} & & & & M_{11} & * & * & * \\ & & & & 0 & M_{22} & * & * \\ & & & & 0 & 0 & M_{33} & 0 \\ & & & & 0 & 0 & * & M_{44} \\ \hline -M_{33}^T & * & * & * & & & & \\ 0 & -M_{44}^T & * & * & & & & \\ 0 & 0 & -M_{11}^T & 0 & & & & \\ 0 & 0 & * & -M_{22}^T & & & & \end{array}\right].$$

Rearranging the rows and columns by a block permutation in the order $1, 5, 2, 6, 3, 7, 4, 8$, the pencil is equivalent to the pencil

$$\alpha \left[\begin{array}{cc|cc} 0 & * & * & * \\ 0 & \mathcal{A} & * & * \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & * & \mathcal{A}^T \end{array}\right] - \beta \left[\begin{array}{cc|cc} \mathcal{C} & * & * & * \\ 0 & \mathcal{B} & * & * \\ \hline 0 & 0 & -\mathcal{C}^T & 0 \\ 0 & 0 & * & -\mathcal{B}^T \end{array}\right],$$

where the asterisks indicate (possibly) nonzero blocks and

$$\mathcal{A} = \begin{bmatrix} S_{22}S_{44}^T & 0 \\ 0 & T_{44}^T T_{22} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & M_{22} \\ -M_{44}^T & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & M_{11} \\ -M_{33}^T & 0 \end{bmatrix}.$$

The finite eigenvalues of $\alpha\mathcal{Z} - \beta\mathcal{M}$ are exactly those of $\alpha\mathcal{A} - \beta\mathcal{B}$ and $\alpha\mathcal{A}^T + \beta\mathcal{B}^T$. It is easily verified that

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} (\alpha\mathcal{A} + \beta\mathcal{B}) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \alpha\mathcal{A} - \beta\mathcal{B},$$

the eigenvalues of $\alpha\mathcal{A} - \beta\mathcal{B}$ and $\alpha\mathcal{A}^T + \beta\mathcal{B}^T$ are the same. Therefore, $\alpha Z - \beta M$ and $\alpha\mathcal{A} - \beta\mathcal{B}$ have the same finite eigenvalues with the same algebraic multiplicity.

Since $S_{22}, S_{44}, T_{22}, T_{44}$ are nonsingular, the pencil $\alpha\mathcal{A} - \beta\mathcal{B}$ is equivalent to

$$\mathcal{A}^{-1}\mathcal{B} = \begin{bmatrix} 0 & S_{44}^{-T} S_{22}^{-1} M_{22} \\ -T_{22}^{-1} T_{44}^{-T} M_{44}^T & 0 \end{bmatrix}.$$

Then, obviously, the eigenvalues of $\mathcal{A}^{-1}\mathcal{B}$ are the square roots of the eigenvalues of the matrix

$$(\mathcal{A}^{-1}\mathcal{B})^2 = -\begin{bmatrix} S_{44}^{-T} S_{22}^{-1} M_{22} T_{22}^{-1} T_{44}^{-T} M_{44}^T & 0 \\ 0 & T_{22}^{-1} T_{44}^{-T} M_{44}^T S_{44}^{-T} S_{22}^{-1} M_{22} \end{bmatrix}.$$

Note that the two diagonal blocks have the same eigenvalues. Using the triangular forms of these blocks, the eigenvalues of $(\mathcal{A}^{-1}\mathcal{B})^2$ can be computed from the diagonal blocks of the upper quasi-triangular matrix

$$-S_{44}^{-T} S_{22}^{-1} M_{22} T_{22}^{-1} T_{44}^{-T} M_{44}^T.$$

This then allows to compute the eigenvalues of $\alpha\mathcal{A} - \beta\mathcal{B}$.

Because $\alpha N - \beta H$ is equivalent to $\alpha Z - \beta M$, the finite eigenvalues of $\alpha N - \beta H$ can be obtained from $\alpha\mathcal{A} - \beta\mathcal{B}$, as well.