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# ON ONE QUESTION OF ED SAFF*

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*Dedicated to Ed Saff on the occasion of his 60th birthday*

**Abstract.** In relation to Fourier-Padé approximation, Ed Saff observed that Taylor and Lagrange interpolation projections satisfy the following property:

$$P(f) \cdot P(g) \in \Pi_n \Longrightarrow P(f \cdot g) = P(f) \cdot P(g).$$

We classify all projections that satisfy this property, thus answering a question of Saff. Some error formulas for approximation with the above-mentioned projections are also produced.

**Key words.** ideal projection, Hermite interpolation, error formula

**AMS subject classifications.** 41A05, 41A10, 41A45, 41A80

## 1. Preface.

Ages ago, Ed Saff [8] asked me the following question:

PROBLEM 1.1. *What are projections $P$ onto the space of polynomials $\Pi_n$ of degree at most $n$, that have the property that*

$$P(f) \cdot P(g) \in \Pi_n \Longrightarrow P(f \cdot g) = P(f) \cdot P(g)? \tag{1.1}$$

He observed that this property holds for both, Lagrange interpolating projections and Taylor projections, and is used in the study of Padé-Fourier approximation. I promised Ed an answer. Now, years later, it is time to fulfill that pledge.

Actually, the property (1.1) in greater generality is equivalent to an "ideal property" of projection $P$ as defined by G. Birkhoff [2]. In the next section we will prove that equivalence and give a complete self-contained proof that projections in $\mathbb{C}[z]$ that satisfy (1.1) are precisely the Hermite interpolation projections. In the last section we present some "Error formulas" for these projections.

Here are some notations. Let $\mathbb{F}$ stands for a field of real or complex numbers. Let $\mathbb{F}[z]$ denotes the space of polynomials in the indeterminate $z$ with coefficients in $\mathbb{F}$. We will use the word projection to mean a linear idempotent mapping on $\mathbb{F}[z]$.

DEFINITION 1.2. *A projection $P := H_{\Delta,\mathfrak{N}}$ is called a Hermite projection if there exists a finite set of distinct points $\Delta = \{z_1, \dots, z_m\} \subset \mathbb{F}$ and a set of integers $\mathfrak{N} = \{n_1, \dots, n_m\} \subset \mathbb{N}$ such that for every $f \in \mathbb{F}[z]$ and every $j = 1, \dots, m$:*

$$\left(H_{\Delta,\mathfrak{N}}f\right)^{(k)}(z_j) = f^{(k)}(z_j)\,; k = 0, \dots, n_j - 1, \tag{1.2}$$

*and*

$$\dim \operatorname{Im} H_{\Delta,\mathfrak{N}} = \sum_{j=1}^{m} (n_j - 1)\,. \tag{1.3}$$

REMARK 1.3. *Observe that in the case $n_j = 1$ for all $j$, the Hermite projector $H_{\Delta,\mathfrak{N}}$ is an interpolating projector. In the case $m = 1$, the Hermite projection $H_{\Delta,\mathfrak{N}}$ is the Taylor projection.*

Notice that here as well as in the rest of the paper no assumptions on the range of the projection is made.

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**2. Algebra of Ideal Projections.**

DEFINITION 2.1 (cf. [2]). *A projection $P$ on $\mathbb{F}[z]$ is called an ideal projection if* $\ker P$ *is an ideal in* $\mathbb{F}[z]$.

In the next proposition we collect a few properties of ideal projections:

PROPOSITION 2.2. *The following are equivalent*

1) *A projection $P$ on $\mathbb{F}[z]$ is ideal.*
2) *A projection $P$ on $\mathbb{F}[z]$ satisfies*

$$P(f \cdot g) = P(f \cdot P(g)), \forall f, g \in \mathbb{F}[z]. \tag{2.1}$$

3) *A projection $P$ on $\mathbb{F}[z]$ satisfies*

$$P(f \cdot g) = P(P(f) \cdot P(g)), \forall f, g \in \mathbb{F}[z]. \tag{2.2}$$

4) *A projection $P$ has the property*

$$P(f) \cdot P(g) \in \operatorname{Im} P \Longrightarrow P(f \cdot g) = P(f) \cdot P(g). \tag{2.3}$$

*Proof*. 1)$\Longrightarrow$2): Suppose that $\ker P$ is an ideal. Since $g - P(g) \in \ker P$, we have $f \cdot (g - P(g)) \in \ker P$. Hence $P(f \cdot g - f \cdot P(g)) = 0$, which implies (2.1).

2)$\Longrightarrow$3). Is easily obtained by using (2.1) twice.

3)$\Longrightarrow$4). Follows from the idempotence of $P$:

$$P(f) \cdot P(g) \in \operatorname{Im} P \Longrightarrow P(P(f) \cdot P(g)) = P(f) \cdot P(g).$$

4)$\Longrightarrow$1). Suppose $P$ satisfies (2.3) and $f \in \ker P$. Then

$$0 = P(f) = P(f) \cdot P(g) \in \operatorname{Im} P$$

and by (2.3):

$$P(f \cdot g) = 0 \Longrightarrow f \cdot g \in \ker P.$$

Hence $\ker P$ is an ideal. □

A nice characterization (2.1) of ideal projections is due to Carl de Boor [4].

For the record, we now give a complete characterization of ideal projections. This is certainly not new, since the various generalizations to several variables are discussed in [6].

THEOREM 2.3. *A projection $P$ on $\mathbb{C}[z]$ is ideal if and only if $P = H_{\Delta,\mathfrak{N}}$ is a Hermite projection for some $\Delta$ and $\mathfrak{N}$.*

*In particular every ideal projection has a finite-dimensional range.*

*Proof.* Let $P$ be an ideal projection. Since $\ker P$ is an ideal in $\mathbb{C}[z]$ and since every ideal in $\mathbb{C}[z]$ is a principal ideal (cf. [1]), it implies that there exists a polynomial

$$p(z) = \prod_{j=0}^{m} (z - z_j)^{(n_j - 1)} \tag{2.4}$$

such that

$$\ker P = \langle p \rangle := \{g \cdot p : g \in \mathbb{C}[z]\}.$$

Since $f - P(f) \in \ker P$ for every $f \in \mathbb{C}[z]$, we conclude that

$$f(z) - P(f)(z) = g(z) \cdot \prod_{j=0}^{m} (z - z_j)^{(n_j - 1)} \tag{2.5}$$

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for some $g \in \mathbb{C}[z]$. It is easy to deduce from (2.5) that

$$(Pf)^{(k)}(z_j) = f^{(k)}(z_j)$$

just as (1.2) requires. Next observe that $\deg p = \sum_{j=1}^{m}(n_j - 1)$ and denote this degree as $(n+1)$. Then every non-zero polynomial $g \in \ker P = \{g \cdot p : g \in \mathbb{C}[z]\}$ is a polynomial of degree $(n+1)$ or higher and hence

$$\Pi_n \cap \ker P = \{0\}.$$

Finally since the ideal $\ker P$ contains a polynomial $p$ of exact degree $(n+1)$, we conclude that

$$\Pi_n \oplus \ker P = \mathbb{C}[z] \tag{2.6}$$

and therefore

$$\dim \Pi_n = \mathit{codim} \ker P = \dim \operatorname{Im} P = (n+1) = \sum_{j=1}^{m}(n_j - 1)$$

which proves (1.3).

Conversely suppose that $P = H_{\Delta,\mathfrak{N}}$ as defined by (1.2) and (1.3). Then it follows easily from (1.2) and the Leibniz rule for derivatives that

$$(g \cdot (H_{\Delta,\mathfrak{N}} f))^{(k)}(z_j) = (f \cdot g)^{(k)}(z_j); \quad k = 0, \dots, n_j - 1; \quad j = 0, \dots, n-1.$$

Hence

$$H_{\Delta,\mathfrak{N}}(f \cdot H_{\Delta,\mathfrak{N}}(g)) = H_{\Delta,\mathfrak{N}}(f \cdot g)$$

implies (2.1), which proves the theorem. □

REMARK 2.4. *It is interesting to observe that (2.6) and the Theorem 2.3 immediately implies that the Hermite interpolation problem always has unique solution in the space of polynomials $\Pi_n$. In fact (cf. [10]) the space $\Pi_n$ is a unique subspace in $\mathbb{C}[z]$ that has this property. Moreover, the space $\Pi_n$ is a unique Chebyshev subspace in $\mathbb{C}[z]$.*

The real version of the Theorem 2.3 no longer holds as stated.

EXAMPLE 2.5. *Define a mapping $P : \mathbb{R}[x] \to \Pi_1 = span\{1, x\}$ as follows:*

$$P\left(x^j\right) = \begin{cases} 1 & \text{if } j = 0(\text{mod}4) \\ x & \text{if } j = 1(\text{mod}4) \\ -1 & \text{if } j = 2(\text{mod}4) \\ -x & \text{if } j = 3(\text{mod}4) \end{cases}$$

*for all $j = 0, 1, \dots$ and extend it by linearity to $\mathbb{R}[x]$. Clearly $P$ is a linear mapping onto $\Pi_1$ and*

$$P(1) = 1, \quad P(x) = x.$$

*Therefore $P$ is a projection. Next observe that*

$$x^2 - P\left(x^2\right) = x^2 + 1 \neq 0.$$

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*Consequently, the projection $P$ does not interpolate $f(x) = x^2$ and hence is not a Hermite projection.*

*However for every finite sum $f = \sum a_k x^k$ with real coefficients, the expression*

$$f(\pm i) := \sum a_k (\pm i)^k$$

*is a well-defined (complex) scalar, and it is easy to see that*

$$Pf(\pm i) = f(\pm i)$$

*since it is so for every monomial. In particular $P$ satisfies (2.3).*

This example suggests an easy (although a bit awkward) modification of the Theorem 2.3 for the real case:

THEOREM 2.6. *A projection $P$ on $\mathbb{R}[z]$ is ideal if and only if there exists a finite sets of distinct points $\Delta = \{x_1, \dots, x_m\} \subset \mathbb{R}$, $\Delta' = \{z_1, \dots, z_s\} \subset \mathbb{C}\backslash\mathbb{R}$ and sets of integers $\mathfrak{N} = \{n_1, \dots, n_m\} \subset \mathbb{N}$, $\mathfrak{N}' = \{n'_1, \dots, n'_s\} \subset \mathbb{N}$ such that for every $f \in \mathbb{F}[z]$ and every $j = 1, \dots, m$:*

$$(Pf)^{(k)}(x_j) = f^{(k)}(x_j); \quad k = 0, \dots, n_j - 1, \tag{2.7}$$

*for every $l = 1, \dots, s$*

$$(Pf)^{(k)}(z_l) = f^{(k)}(z_l); \quad k = 0, \dots, n'_l - 1, \tag{2.8}$$

*and*

$$\dim \operatorname{Im} P = \sum_{j=1}^{m} (n_j - 1) + 2 \sum_{l=1}^{s} (n'_l - 1). \tag{2.9}$$

*Proof.* The proof is the same as that of the previous theorem, with one obvious modification. This time the ideal

$$\ker P = \langle p \rangle := \{g \cdot p : g \in \mathbb{C}[z]\}$$

is generated by the polynomial $p$ of the form

$$p(z) = \left(\sqcap_{j=0}^{m} (x - x_j)^{(n_j - 1)}\right) \left(\sqcap_{l=0}^{s} ((x - z_l)(x - \bar{z}_l))^{(n'_l - 1)}\right). \tag{2.10}$$

Observe that (2.8) implies

$$(Pf)^{(k)}(\bar{z}_l) = f^{(k)}(\bar{z}_l); \quad k = 0, \dots, n'_l - 1. \tag{2.11}$$

The rest of the argument is the same as in the proof of the Theorem 2.3. □

The Theorems 2.3 and 2.6 explain the special role that Taylor, Lagrange and, in full generality, Hermite interpolation plays in Approximation Theory. The ideal property of these interpolants allow us to view approximation as the process of division. For instance the kernel of the Taylor projection $T_n$ onto $\Pi_n$ is an ideal generated by polynomial $z^{n+1}$. The process of division of $f$ by $z^n$:

$$f = z^{n+1} q(f) + T_n(f). \tag{2.12}$$

Ironically it is the Taylor polynomial that is (in the language of algebra) the remainder of the division, while the “remainder” in Taylor Theorem is the main part.

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Lagrange interpolation (that in full generality becomes Hermite interpolation and includes Taylor) also have this property. The kernel of Lagrange interpolating projector is also an ideal of functions (polynomials) that vanish on a given set of points, hence it is also the remainder of the division of $f$ by a polynomial $\omega(x) = \prod_{j=1}^{n}(x - x_j)$.

Now the equivalent properties (2.2) and (2.3), can be understood as a purely algebraic fact: The remainder of the product is equal to the remainder of the product of the remainders.

**3. Error Formulas.** The plethora of forms for the error in Taylor, Lagrange and Hermite interpolation can also be understood from the ideal point-of-view. Here is the general perspective.

Let $P$ be an ideal projection, and let the ideal $\ker P$ be generated by a polynomial $h \in \mathbb{F}[z]$:

$$\ker P = \{h \cdot g : g \in \mathbb{F}[z]\}.$$

Since $f - Pf \in \ker P$, hence

$$f - Pf = h \cdot A(f). \tag{3.1}$$

It is easy to see that (3.1) defines a linear operator:

$$A : \mathbb{F}[z] \to \mathbb{F}[z] \tag{3.2}$$

and

$$\ker A = \ker(I - P) = \mathrm{Im}\, P. \tag{3.3}$$

We claim that various factorizations of this operator $A$ give rise to the error formulas. We start with the following general lemma:

LEMMA 3.1. *Let $A : X \to Y$ and $B : X \to Z$ be two linear operators. Then*

$$A = CB \tag{3.4}$$

*for some linear operator $C$ if and only if*

$$\ker B \subset \ker A. \tag{3.5}$$

*Proof.* The necessity is obvious. For sufficiency, define a linear operator

$$J : X/\ker B \to X/\ker A$$

that maps an equivalence class $[f]_B \in X/\ker B$ into the equivalence class $[f]_A \in X/\ker A$.

Assume that $g \in [f]_B$. Then $(f - g) \in \ker B$ and by (3.5), we conclude that $(f - g) \in \ker A$. Hence the operator $J$ is well-defined and the diagram:

$$\begin{array}{ccc} & X & \\ \alpha \swarrow & & \searrow \beta \\ X/\ker A & \underset{J}{\longleftarrow} & X/\ker B \end{array}$$

commutes. Here $\alpha$ and $\beta$ are the canonical embeddings.

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Consider the following diagram:

$$\begin{array}{ccccc} Y & \overset{A}{\longleftarrow} & X & \overset{B}{\longrightarrow} & \mathrm{Im}\, B \subset Z \\ \nwarrow \delta & \alpha \swarrow & \searrow \beta & \gamma \nearrow \swarrow & \gamma^{-1} \\ X \,/\ker A & \underset{J}{\longleftarrow} & X/\ker B & & \end{array}$$

with $\delta$ and $\gamma$ defined as natural injections. Since $\gamma$ is also onto, it has an inverse mapping $\gamma^{-1}$.

This diagram is commutative since every triangular diagram in it is commutative. Thus

$$A = \left(\delta J \gamma^{-1}\right) B$$

which proves the lemma. □

In particular, if the interpolation projection has $\Pi_n$ as its range then for all $k \leq n+1$, the operators

$$B_k := \frac{d^k}{dz^k} : \mathbb{F}[z] \to \mathbb{F}[z] \tag{3.6}$$

have $\ker B_k \subset \Pi_n$. Thus, by (3.3)

$$\ker A = \mathrm{Im}\, P = \Pi_n \supset \ker B$$

and by Lemma 3.1 we have:

$$f - Pf = h \cdot C_k\left(f^{(k)}\right). \tag{3.7}$$

Usually the operator $C$ in (3.7) has an integral form. For instance in the real case (cf. [3]) of Lagrange interpolation $H_\Delta$ with $\Delta = \{z_1, \ldots, z_m\} \subset \mathbb{R}$, the kernel of $H_\Delta$ is an ideal generated by polynomial $h(z) = \sqcap_{j=1}^{m}(z - z_j)$

$$f(z) - H_\Delta f(z) = \sqcap_{j=1}^{m}(z - z_j) \cdot \int K_\Delta(t, z) f^{(m)}(t)dt,$$

where $K_\Delta(t, z)$ is a B-spline at the nodes $\Delta' = \{z_1, \ldots, z_m, z\}$. In the complex case various integral representations are described in [5], [9] and [7]. We will now extend these results to the ideal projections with an arbitrary range.

DEFINITION 3.2. *Define $\mathbb{C}[[z]]$ to be the ring of all formal power series in $z$ with coefficients in $\mathbb{C}$.*

THEOREM 3.3. *Let $P$ be an ideal projection with* $\ker P =< h >$. *Let $B$ be an operator on $\mathbb{C}[z]$ such that*

$$\ker B \subset \mathrm{Im}\, P. \tag{3.8}$$

*Then there exists an operator $C$ defined on $\mathbb{C}[z]$ such that*

$$f - Pf = h \cdot C(Bf). \tag{3.9}$$

*More over the operator $C$ can be written as an integral operator*

$$Cg(z) = \int\limits_{|\zeta|=|z|+1} K(z, \zeta) g(\zeta) d\zeta \tag{3.10}$$

*with $K(z, \zeta) \in \mathbb{C}[[\zeta^{-1}]]$ for every $z$.*

*Proof.* The existence of operator $C$ follows directly from (3.8) and Lemma 3.1, since

$$\ker B \subset \operatorname{Im} P = \ker(I - P).$$

It remains to prove (3.10). Let $p_k$ be a polynomial defined by

$$p_k = P\left(z^k\right),$$

then

$$C\left(\sum_{k=0}^{N} a_k z^k\right) = \sum_{k=0}^{N} a_k p_k.$$

Letting $f(z) = \sum_{k=0}^{N} a_k z^k$, we have

$$a_k = \frac{1}{2\pi i} \int\limits_{|\zeta|=|z|+1} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta$$

and hence

$$Cf(z) = \int\limits_{|\zeta|=|z|+1} f(\zeta)\left[\frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{p_k(z)}{\zeta^{k+1}}\right] d\zeta. \tag{3.11}$$

Setting

$$K(z,\zeta) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{p_k(z)}{\zeta^{k+1}}$$

we obtain the desired conclusion. Notice that while $K(z,\zeta)$ is only a formal power series in $\zeta^{-1}$, for every polynomial $f$ only finitely many terms in (3.11) are non-zero. Thus (3.11) indeed defines a linear mapping on $\mathbb{C}[z]$. ◻

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REFERENCES

[1] M. F. ATIYAH AND I. G. MACDONALD, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[2] G. BIRKHOFF, *The algebra of multivariate interpolation*, in Constructive Approaches to Mathematical Models, C. V. Coffman and G. J. Fix, eds., Academic Press, New York, 1979, pp. 345–363.

[3] C. DE BOOR, *Divided differences*, Surv. Approx. Theory, 1 (2005), pp. 49–63.

[4] ———, *Ideal interpolation*, in Approximation Theory XI: Gatlinburg, 2004, C. K. Chui, M. Neamtu, and L. L. Schumaker, eds., Nasboro Press, 2005, pp. 59–91.

[5] I. IBRAGIMOV, *Interpolation Methods and Some of Their Applications*, Nauka, Moscow, 1971.

[6] H. M. MOELLER, *Hermite interpolation in several variables using ideal-theoretic methods*, in Constructive Theory of Functions of Several Variables, Oberwolfach 1976, W. Schempp and K. Zeller, eds., Lecture Notes in Mathematics, 571, Springer-Verlag, Berlin, 1977, pp. 155-163.

[7] E. B. SAFF, *Polynomial and rational approximation in the complex domain*, in Approximation Theory (New Orleans, La., 1986), Proc. Sympos. Appl. Math., vol. 36, Amer. Math. Soc., Providence, RI, 1986, pp. 21–49.

[8] ———, 1998. Personal Communication.

[9] E. B. SAFF AND J. C. SNADER, *The error for quadrature methods: a complex variables approach*, Amer. Math. Monthly, 94 (1987), pp. 175–180.

[10] B. SHEKHTMAN, *Uniqueness of chebyshev spaces and ideal complements*, in Frontiers in Interpolation and Approximation, N. K. Govil, H. N. Mhaskar, R. N. Mohapatra, Z. Nashed, and J. Szabados, eds., pp. 407–425, Chapman & Hall/CRC, Boca Raton, FL, 2007.