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ETNA
Kent State University
etna@mcs.kent.edu

FINE STRUCTURE OF THE ZEROS OF ORTHOGONAL POLYNOMIALS, I. A TALE OF TWO PICTURES*

BARRY SIMON†

Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Mhaskar-Saff found a kind of universal behavior for the bulk structure of the zeros of orthogonal polynomials for large n. Motivated by two plots, we look at the finer structure for the case of random Verblunsky coefficients and for what we call the BLS condition: $\alpha_n = Cb^n + O((b\Delta)^n)$. In the former case, we describe results of Stoiciu. In the latter case, we prove asymptotically equal spacing for the bulk of zeros.

Key words. OPUC, clock behavior, Poisson zeros, orthogonal polynomials

AMS subject classifications. 42C05, 30C15, 60G55

1. Prologue: A Theorem of Mhaskar and Saff.

A recurrent theme of Ed Saff's work has been the study of zeros of orthogonal polynomials defined by measures in the complex plane. So I was happy that some thoughts I've had about zeros of orthogonal polynomials on the unit circle (OPUC) came to fruition just in time to present them as a birthday bouquet. To add to the appropriateness, the background for my questions was a theorem of Mhaskar-Saff [25] and the idea of drawing pictures of the zeros was something I learned from some of Ed's papers [32, 21]. Moreover, ideas of Barrios-López-Saff [4] played a role in the further analysis.

Throughout, $d\mu$ will denote a probability measure on $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ which is nontrivial in that it is not supported on a finite set. $\Phi_n(z)$ (resp. $\varphi_n(z)$) will denote the monic orthogonal polynomials (resp. orthonormal polynomials $\varphi_n = \Phi_n/\|\Phi_n\|$). I will follow my book [36, 37] for notation and urge the reader to look there for further background.

A measure is described by its Verblunsky coefficients

$$\alpha_n = -\overline{\Phi_{n+1}(0)} \tag{1.1}$$

which enter in the Szegő recursion

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z) \tag{1.2}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \tag{1.3}$$

where

$$P_n^*(z) = z^n \, \overline{P_n(1/\bar{z})} \tag{1.4}$$

Φ_n has all its zeros in $\mathbb{D}$ [36, Theorem 1.7.1]. We let $d\nu_n$ be the pure point measure on $\mathbb{D}$ which gives weight k/n to each zero of Φ_n of multiplicity k. For simplicity, we will suppose there is a $b \in [0,1]$ so that

$$\lim |\alpha_n|^{1/n} = b \tag{1.5}$$

(root asymptotics). If $b = 1$, we also need

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\alpha_j| = 0, \tag{1.6}$$

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†Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125 (bsimon@caltech.edu).
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which automatically holds if $b < 1$. Here is the theorem of Mhaskar-Saff [25]:

THEOREM 1.1 (Mhaskar-Saff [25]). *If* (1.5) *and* (1.6) *hold, then* $d\nu_n$ *converges weakly to the uniform measure on the circle of radius* b.

We note that both this result and the one of Nevai-Totik [28] I will mention in a moment define b by $\limsup|\alpha_n|^{1/n}$ and the Mhaskar-Saff result holds for $d\nu_{n(j)}$ where $n(j)$ is a subsequence with $\lim_{j\to\infty}|\alpha_{n(j)}|^{1/n(j)} = b$.

I want to say a bit about the proof of Theorem 1.1 in part because I will need a slight refinement of the first part (which is from Nevai-Totik [28]) and in part because I want to make propaganda for a new proof [36] of the second part.

The proof starts with ideas from Nevai-Totik [28] that hold when $b < 1$:

(1) By (1.2) and $|\Phi_n(e^{i\theta})| = |\Phi_n^*(e^{i\theta})|$, one sees inductively that

$$\sup_{n,z\in\partial\mathbb{D}} |\Phi_n^*(z)| \leq \prod_{j=0}^{\infty} (1 + |\alpha_j|) < \infty$$

and so, by the maximum principle and (1.4),

$$C \equiv \sup_{n,z\notin\mathbb{D}} |z|^{-n}|\Phi_n(z)| < \infty. \tag{1.7}$$

(2) By (1.3),

$$\sum_{n=0}^{\infty} |\Phi_{n+1}^*(z) - \Phi_n^*(z)| \leq C \sum_{n=0}^{\infty} |\alpha_n|\,|z|^{n+1} < \infty$$

if $|z|b < 1$ by (1.5). Thus, in the disk $\{z \mid |z| < b^{-1}\}$, Φ_n^* has a limit. Since $b < 1$, the Szegő condition (see [36, Chapter 2]) holds, so

$$\varphi_n^*(z) \to D(z)^{-1} \tag{1.8}$$

on $\mathbb{D}$ (see [36, Theorem 2.4.1]), we conclude that $D(z)^{-1}$ has an analytic continuation to the disk of radius b^{-1} and (1.8) holds there. (Nevai-Totik also prove a converse: If the Szegő condition holds, $d\mu_s = 0$ and $D(z)^{-1}$ has an analytic continuation to the disk of radius b^{-1}, then $\limsup|\alpha_n|^{1/n} \leq b$.)

(3) When (1.5) holds, $D(z)$ is analytic in $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$ so $D(z)^{-1}$ has no zeros in $\bar{\mathbb{D}}$. We define the Nevai-Totik points $\{z_j\}_{j=1}^N$ (N in $\{0, 1, 2, \ldots, \} \cup \infty$) with $1 > |z_1| \geq |z_2| \geq \cdots > b$ to be all the solutions of $D(1/\bar{z})^{-1} = 0$ in $\mathbb{A}_b = \{z \mid b < |z| < 1\}$. Since (1.8) holds and $\varphi_n^*(z) = 0 \Leftrightarrow \varphi_n(1/\bar{z}) = 0$, Hurwitz's theorem implies that the z_j are precisely the limit points of zeros of φ_n in the region $\mathbb{A}_b$. If $N = \infty$, $|z_j| \to b$ so we conclude that for each $\varepsilon > 0$, the number of zeros of φ_n in $\{z \mid |z| > b + \varepsilon\}$ is bounded as $n \to \infty$.

That concludes our summary of some of the results from Nevai-Totik. The next step is from Mhaskar-Saff.

(4) By (1.1), if $\{z_{jn}\}_{j=1}^n$ are the zeros of $\varphi_n(z)$ counting multiplicity, then

$$|\alpha_n| = \prod_{j=1}^{n} |z_{jn}|,$$

so, by (1.5),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log|z_{jn}| = \log b.$$

This together with (3) implies that the bulk of zeros must asymptotically lie on the circle of radius b and, in particular, any limit point of $d\nu_n$ must be a measure on $\{z \mid |z| = b\}$.

Mhaskar-Saff complete the proof by using potential theory to analyze the limit points of the ν_n. Instead, I will sketch a different idea from [36, Section 8.2] that exploits the CMV matrix (see [5] and [36, Sections 4.2–4.5]).

(5) If $\mathcal{C}^{(N)}$ is the $N \times N$ matrix in the upper left of $\mathcal{C}$, then

$$\Phi_N(z) = \det(z - \mathcal{C}^{(N)}),$$

and, in particular,

$$\int z^k \, d\nu_N(z) = \frac{1}{N} \operatorname{Tr}([\mathcal{C}^{(N)}]^k). \tag{1.9}$$

It is not hard to see that (1.6) implies that on account of the structure of $\mathcal{C}$, the right side of (1.9) goes to 0 as $N \to \infty$ for each $k > 0$. Thus any limit point, $d\nu$, of $d\nu_N$ has $\int z^k \, d\nu(z) = 0$ for $k = 1, 2, \dots$. That determines the measure $d\nu$ uniquely since the only measure on $b[\partial\mathbb{D}]$ with zero moments is the uniform measure.

That completes the sketch of the proof of the Mhaskar-Saff theorem. Before going on, I have two remarks to make. It is easy to see ([36, Theorem 8.2.6]) that $\langle \delta_\ell, \mathcal{C}^k \delta_\ell \rangle = \int_0^{2\pi} e^{ik\theta} |\varphi_\ell(e^{i\theta})|^2 \, d\mu(\theta)$. Thus the moments of the limit points of the Cesàro average

$$d\eta_N = \frac{1}{N} \sum_{j=0}^{N-1} |\varphi_j(e^{i\theta})|^2 \, d\mu(\theta)$$

are the same as the moments of the limits of $d\nu_N$ (so if $d\nu_N$ has a limit that lives on $\partial\mathbb{D}$, $d\eta_N$ has the same limit).

Second, a theorem like Mhaskar-Saff holds in many other situations. For example, if β_n is periodic and $\alpha_n - \beta_n \to 0$, then the $d\nu_n$ converge to the equilibrium measure for the essential support of $d\mu$, which is a finite number of intervals. (See Ed Saff's book with Totik [33].)

One critical feature of the Mhaskar-Saff theorem is its universality. So long as we look at cases where (1.5) and (1.6) hold, the angular distribution is the same. Our main goal here is to go beyond the universal setup where the results will depend on more detailed assumptions on asymptotics. In particular, we will want to consider two stronger conditions than root asymptotics, (1.5), namely, ratio asymptotics

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = b \tag{1.10}$$

for some $b \in (0, 1)$ and what we will call BLS asymptotics (or the BLS condition):

$$\alpha_n = Cb^n + O((b\Delta)^n), \tag{1.11}$$

where $b \in (0, 1)$, $C \in \mathbb{C}$, and $\Delta \in (0, 1)$.

The name BLS is for Barrios-López-Saff [4] who proved the following:

THEOREM 1.2 (Barrios-López-Saff [4]). *A set of Verblunsky coefficients obeys the BLS condition if and only if* $d\mu_s = 0$ *and the numeric inverse of the Szegő function* $D(z)^{-1}$*, defined initially for* $z \in \mathbb{D}$*, has a meromorphic continuation to a disk of radius* $(b\Delta')^{-1}$ *for some* $\Delta' \in (0, 1)$ *which is analytic except for a single pole at* $z = b^{-1}$.

ETNA
Kent State University
etna@mcs.kent.edu

This is Theorem 2 in [4]. They allow $b \in \mathbb{D}\backslash\{0\}$ but, by the rotation invariance of OPUC (see [36, Eqns. (1.6.62)–(1.6.67)]), any $b = |b|e^{i\theta}$ can be reduced to $|b|$. Another proof of Theorem 1.2 can be found in [36, Section 7.2] where it is also proven that

$$C = -\Big[\lim_{z\to b^{-1}} (z^{-1} - b)D(z)^{-1}\Big]\,\overline{D(b)} \tag{1.12}$$

and that if Δ or Δ' is in $(\frac{1}{2}, 1)$, then $\Delta = \Delta'$.

We summarize the contents of the paper after the next and second preliminary section. I'd like to thank Mourad Ismail, Rowan Killip, Paul Nevai, and Mihai Stoiciu for useful conversations.

2. Two Pictures and an Overview of the Fine Structure. Take a look at two figures (Figure 2.1 and Figure 2.2) from my book [36, Section 8.4]. The first shows the zeros of $\Phi_n(z)$ for $\alpha_n = (\frac{1}{2})^{n+1}$ and the second for α_n independent random variables uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$. (Of course, the second is for a particular choice of the random variables made by Mathematica using its random number generator.) They are shown for $n = 5, 10, 20, 50, 100$, and 200.

Figure 2.1 beautifully illustrates Theorem 1.1 and the Nevai-Totik theorem. All the zeros but one clearly approach the circle $|z| = \frac{1}{2}$. There is one zero that approaches approximately $0.84 + 0i$. It is the single Nevai-Totik zero in this case. That there are no limiting zeros inside $|z| = \frac{1}{2}$ is no accident; see Corollary 1 of [4] and Theorem 8.4 below. And the zeros certainly seem uniformly distributed — indeed, when I first ran the program that generated Figure 2.1, I was impressed by how uniform the distribution seemed to be, even for $N = 10$.

The conclusion of the Mhaskar-Saff theorem is not true for the example in Figure 2.2 (nor, of course, the hypotheses since (1.6) fails), although it would be if uniform density on $[-\frac{1}{2}, \frac{1}{2}]$ were replaced by any rotation invariant density, $d\gamma$, with $\int |\log(z)|\, d\gamma(z) < \infty$ (see [37, Theorem 10.5.19]). But, by Theorems 10.5.19 and 10.5.21 of [37], $d\nu_n$ has a limit $d\nu$ in the case of Figure 2.2, and this limit can be seen to be of the form $f(\theta)\frac{d\theta}{2\pi}$ with $f \in C^\infty$ and $f \not\equiv 1$ but not too far from 1 (e.g., all odd moments vanish and $\int z^2\, d\nu = (\frac{1}{24})^2$). My initial thought was that the roughness was trying to emulate the pure point spectrum.

I now think I was wrong in both initial reactions.

Proof. [Expectation 1: Poisson Behavior] For Figure 2.2, I should have made the connection with work of Molchanov [27] and Minami [26] who proved in the case of random Schrödinger operators that, locally, eigenvalues in a large box had Poisson distribution. This leads to a conjecture. First some notation:

We say a collection of intervals $\Delta_1^{(n)}, \dots, \Delta_k^{(n)}$ in $\partial\mathbb{D}$ is canonical if $\Delta_j^{(n)} = \{e^{i\theta} \mid \theta \in [\frac{2\pi a_j}{n} + \theta_j, \frac{2\pi b_j}{n} + \theta_j]\}$ where $0 \le \theta_1 \le \dots \le \theta_k \le 2\pi$, and if $\theta_j = \theta_{j+1}$, then $b_j < a_{j+1}$. ☐

CONJECTURE 2.1. *Let $\{\alpha_\ell\}_{\ell=0}^\infty$ be independent identically distributed random variables, each uniformly distributed in $\{z \mid |z| \le \rho\}$ for some $\rho < 1$ and let $z_1^{(n)}, \dots, z_n^{(n)}$ be the random variable describing the zeros of the Φ_n associated to α. Then for some C_1, C_2,*

(1)

$$\mathbb{E}\big(\#\{z_j^{(n)} \mid |z_j^{(n)}| < 1 - e^{-C_1 n}\}\big) \le C_2(\log n)^2.$$

(2) *For any collection $\Delta_1^{(n)}, \dots, \Delta_k^{(n)}$ of canonical intervals and any $\ell_1, \dots, \ell_k$ in $\{0, 1, 2, \dots\}$,*

$$\mathbb{P}(\#\{j \mid \arg(z_j^{(n)}) \in \Delta_m\} = \ell_m \textit{ for } m = 1, \dots, k)$$

ETNA
Kent State University
etna@mcs.kent.edu

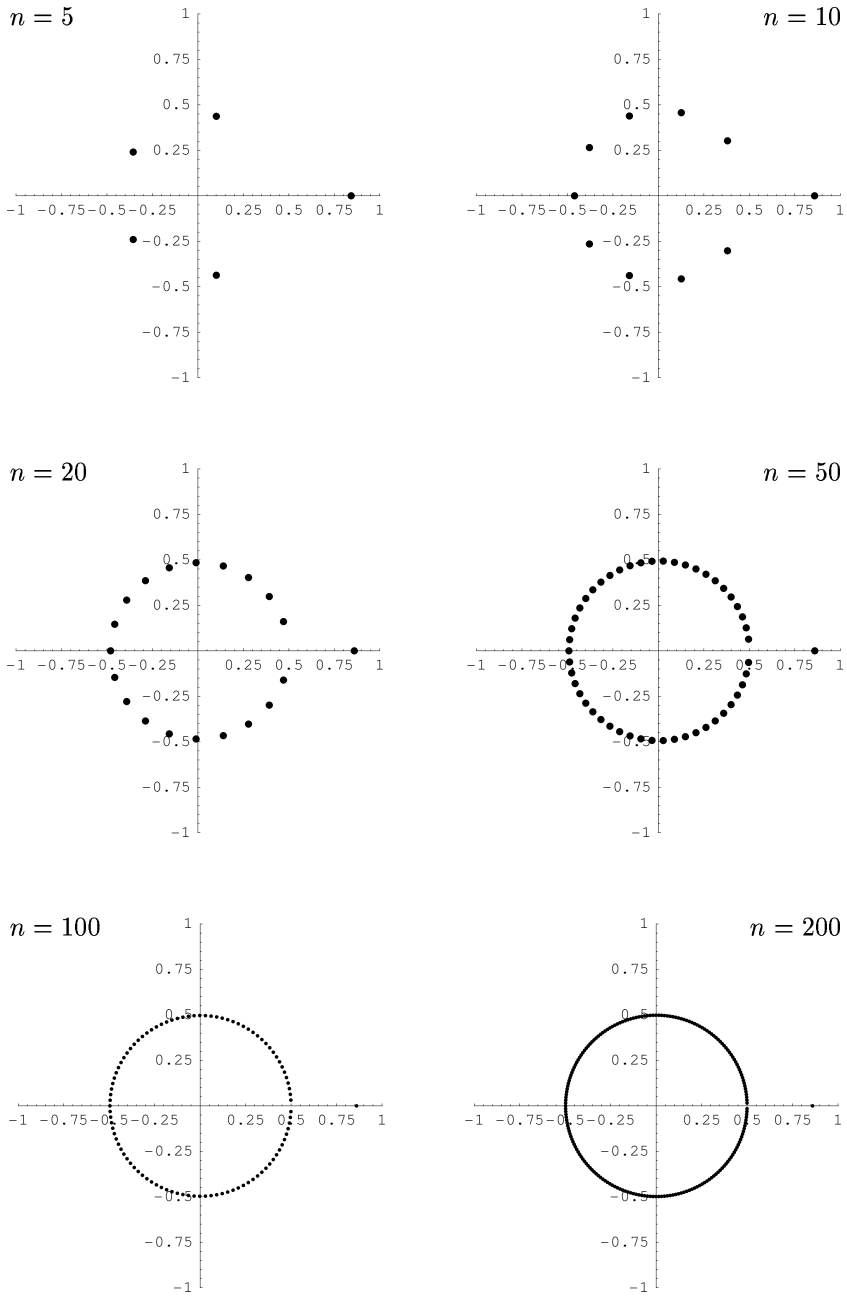

FIG. 2.1.

$$\rightarrow \prod_{m=1}^{k} \left[\frac{(b_m - a_m)^{\ell_m}}{\ell_m!} \, e^{-(b_m - a_m)} \right]. \tag{2.1}$$

REMARK 2.2. *1. This says that, asymptotically, the distribution of z's is the same as* n *points picked independently, each uniformly distributed.*

2. See the next section for a result towards this conjecture.

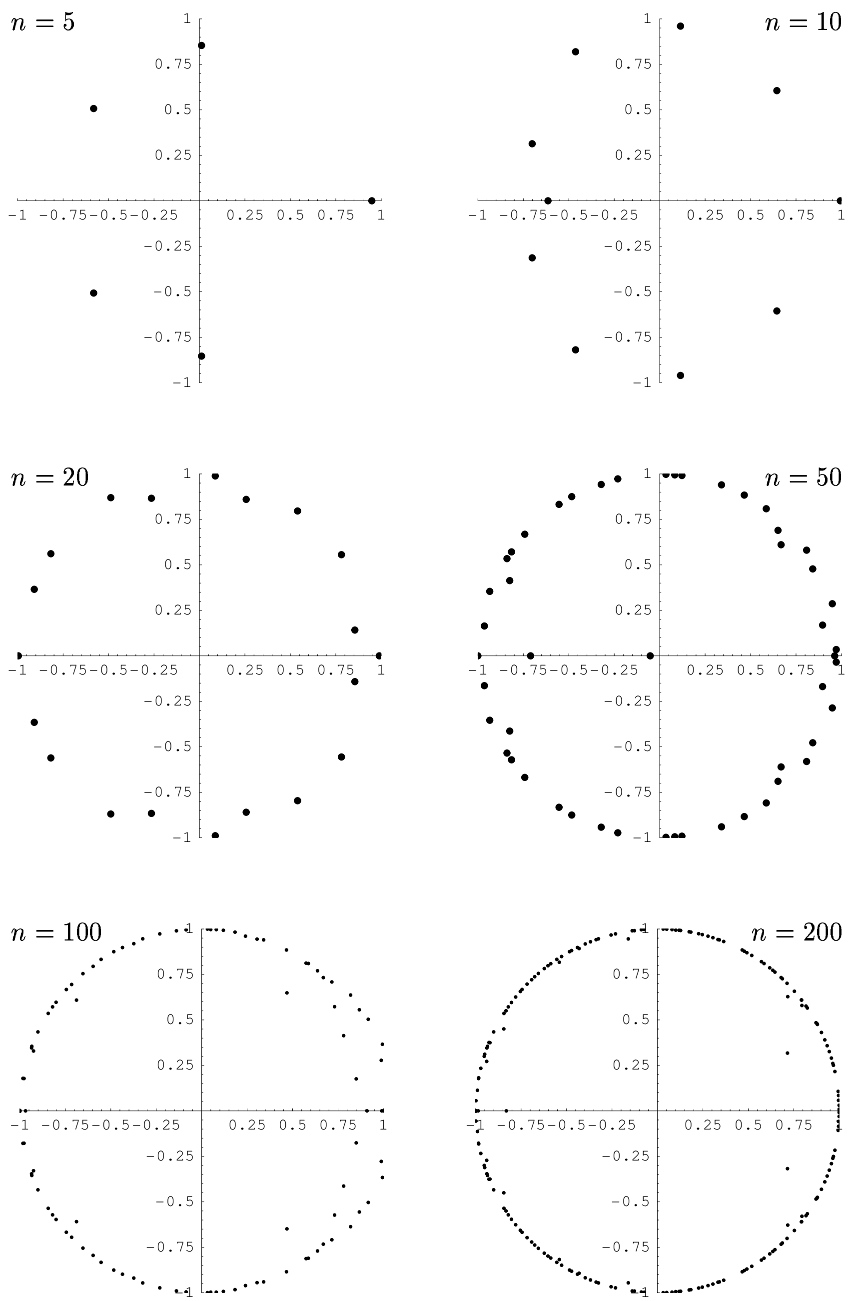

FIG. 2.2.

3. $\mathbb{E}$ *means expectation and* $\mathbb{P}$ *probability.*

4. I base the precise $e^{-C_1 n}$ *and* $(\log n)^2$ *on the results of Stoiciu [39], but I would regard as very interesting any result that showed, except for a small fraction (even if not as small as* $(\log n)^2/n$*), all zeros are very close (even if not as close as* $e^{-C_1 n}$*) for* $\partial\mathbb{D}$*.*

5. There is one aspect of this conjecture that is stronger than what is proven for the

ETNA
Kent State University
etna@mcs.kent.edu

Schrödinger case. The results of Molchanov [27] and Minami [26] are the analog of Conjecture 2.1 if $\theta_1 = \theta_2 = \cdots = \theta_k$, which I would call a local Poisson structure. That there is independence of distant intervals is conjectured here but not proven in the Schrödinger case. That this is really an extra result can be seen by the fact that Figure 2.2 is likely showing local Poisson structure about any $\theta_0 \neq 0, \pi$, but because the α's are real, the set of zeros is invariant under complex conjugation, so intervals about, say, $\pi/2$ and $3\pi/2$ are not independent.

As far as Figure 2.1 is concerned, it is remarkably regular so there is an extra phenomenon leading to

Proof. [Expectation 2: Clock Behavior] If for $b \in (0,1)$ and $C \in \mathbb{C}$, α_n/b^n converges to C sufficiently fast, then the non-Nevai-Totik zeros approach $|z| = b$ and the angular distance between nearby zeros in $2\pi/n$. □

REMARK 2.3. *1. Proving this expectation when "sufficiently fast" means BLS convergence is the main new result of this paper; see Section 4.*

2. This is only claimed for local behavior. We will see that, typically, errors in the distance between the zeros are $O(1/n^2)$ and will add up to shift zeros that are a finite distance from each other relative to a strict clock.

3. Clock behavior has been discussed for OPRL. Szegő [40] has C/n upper and lower bounds (different C's) in many cases and Erdös-Turan [11] prove local clock behavior under hypotheses on the measure, but their results do not cover all Jacobi polynomials. In Section 6, we will prove a clock result for a class of OPRL in terms of their Jacobi matrix parameters ($\sum_{n=1}^{\infty} n(|b_n| + |a_n - 1|) < \infty$), and in Section 7, a simple analysis that proves local clock behavior for Jacobi polynomials. I suppose this is not new, but I have not located a reference.

4. A closer look at Figure 2.1 suggests that this conjecture might not be true near $z = b$. In fact, the angular gap there is $2(2\pi/n) + o(1/n)$, as we will see.

I should emphasize that the two structures we suspect here are very different from what is found in the theory of random matrices. This is most easily seen by looking at the distribution function for distance between nearest zeros scaled to the local density. For the Poisson case, there is a constant density, while for clock, it is a point mass at a point $\theta_0 \neq 0$ depending on normalization. For the standard random matrix (GUE, GOE, CUE), the distribution is continuous but vanishing at 0 (see [24]).

Since any unitary with distinguished cyclic vector can be represented by a CMV matrix, CUE has a realization connected with OPUC, just not either the totally random or BLS case. Indeed, Killip-Nenciu [19] have shown that CUE is given by independent α_j's but not identically distributed.

In Section 3, we describe a new result of Stoiciu [39] on the random case. In Section 4, we overview our various clock results: paraorthogonal OPUC in Section 5, OPRL proven in Sections 6 and 7, and BLS in Sections 8–13. We mention some examples in Section 13.

3. Stoiciu's Results on the Random Case. Recall that given $\beta \in \partial\mathbb{D}$ and $\{\Phi_n\}_{n=0}^{\infty}$, a set of orthogonal polynomials, the paraorthogonal polynomials (POPs) [18, 16] are defined by

$$\Phi_n(z;\beta) = z\Phi_{n-1}(z) - \bar{\beta}\Phi_{n-1}^*(z).$$

They have all their zeros on $\partial\mathbb{D}$ (see, e.g., [36, Theorem 2.2.12]). Stoiciu [39] has proven the following result:

THEOREM 3.1 (Stoiciu [39]). *Let $\{\alpha_j\}_{j=0}^{\infty}$ be independent identically distributed random variables with common distribution uniform in $\{z \mid |z| \leq \sigma\}$ for some $\sigma < 1$. Let*

ETNA
Kent State University
etna@mcs.kent.edu

$\{\beta_j\}_{j=0}^{\infty}$ *be independent identically distributed random variables uniformly distributed on* $\partial\mathbb{D}$*. Let* $z_j^{(n)}$ *be the zeros of* $\Phi_n(z;\beta_{n-1})$*. Let* $\Delta_1^{(n)},\dots,\Delta_k^{(n)}$ *be canonical intervals with the same* θ*, that is,* $\theta_1=\theta_2=\cdots=\theta_k$*. Then* (2.1) *holds.*

This differs from Conjecture 2.1 in two ways: The zeros are of the POPs, not the OPUC, and the result is only local (i.e., all θ's are equal). While the proof has some elements in common with the earlier work on OPRL of Molchanov [27] and Minami [26], there are many differences forced, for example, by the fact that rank one perturbations of selfadjoint operators differ in many ways from rank two perturbations of unitaries. Since the proof is involved and the earlier papers have a reputation of being difficult, it seems useful to summarize here the strategy of Stoiciu's proof.

Following Minami, a key step is the proof of what are sometimes called fractional moment bounds and which I like to call Aizenman-Molchanov bounds after their first appearance in [2]. A key object in these bounds is the Green's function which has two natural analogs for OPUC:

$$G_{nm}(z)=\langle\delta_n,(\mathcal{C}-z)^{-1}\delta_m\rangle, \tag{3.1}$$

$$F_{nm}(z)=\langle\delta_n,(\mathcal{C}+z)(\mathcal{C}-z)^{-1}\delta_m\rangle. \tag{3.2}$$

These are related by

$$F_{nm}(z)=\delta_{nm}+2zG_{nm}(z), \tag{3.3}$$

so controlling one on $\partial\mathbb{D}$ is the same as controlling the other.

As we will see below, F and G lie in the Hardy space H^p for any $p<1$, so we can define

$$G_{nm}(e^{i\theta})=\lim_{r\uparrow 1}G_{nm}(re^{i\theta}) \tag{3.4}$$

for a.e. $e^{i\theta}$. In the random case, rotation invariance will then imply that for any $e^{i\theta}\in\partial\mathbb{D}$, (3.4) holds for a.e. α. In treatments of Aizenman-Molchanov bounds for Schrödinger operators, it is traditional to prove bounds on the analog of $G_{ij}(z)$ for $z=re^{i\theta}$ with $r<1$ uniform in r in $(\frac{1}{2},1)$. Instead, the Stoiciu proof deals directly with $r=1$, requiring some results on boundary values of H^p functions to replace a uniform estimate.

Given N, we define $\widehat{\mathcal{C}}^{(N)}$ to be the random CMV matrix ([5] and [36, Chapter 4]) obtained by setting α_N to the random value $\beta_N\in\partial\mathbb{D}$. $\widehat{\mathcal{C}}^{(N)}$ decouples into a direct sum of an $N\times N$ matrix, $\mathcal{C}^{(N)}$, and an infinite matrix which is identically distributed to the random $\mathcal{C}$ if N is even and $\tilde{\mathcal{C}}$, the random alternate CMV matrix, if N is odd. (This is a slight oversimplification. Only if $\beta_N=-1$ is the infinite piece of $\widehat{\mathcal{C}}^{(N)}$ a CMV matrix since the 1×1 piece in the $\mathcal{M}$ half of the $\mathcal{LM}$ factorization has $-\beta_N$ in place of 1. As explained in [36, Theorem 4.2.9], there is a diagonal unitary equivalence taking such a matrix to a CMV matrix with Verblunsky coefficients $-\beta_N^{-1}\alpha_{j+N+1}$ and the distribution of these is identical to the distribution of the α_{j+N+1}. We will ignore this subtlety in this sketch.)

We define $F_{nm}^{(N)}(z)$ and $G_{nm}^{(N)}(z)$ for $n,m\in(\{0,1,\dots,N-1\}$ by replacing $\mathcal{C}$ in (3.1)/(3.2) by $\mathcal{C}^{(N)}$.

$\mathcal{C}-\widehat{\mathcal{C}}^{(N)}$ is a rank two matrix with $(\mathcal{C}-\widehat{\mathcal{C}}^{(N)})_{nm}\neq 0$ only if $|n-N|\le 2$, $|m-N|\le 2$. Moreover, any matrix element of the difference is bounded in absolute value by 2. If $n\in\{0,\dots,N-1\}$, $m\ge N$, then $(\widehat{\mathcal{C}}^{(N)}-z)^{-1}_{nm}=0$, so the second resolvent formula implies

$$\begin{aligned} &n\le N-1,\ m\ge N\Rightarrow |G_{nm}(z)|\\ &\le 2\Bigg(\sum_{k=N-1,N-2,N-3}|G_{nk}^{(N)}(z)|\Bigg)\Bigg(\sum_{|k-N+\frac{1}{2}|\le\frac{5}{2}}|G_{km}(z)|\Bigg), \end{aligned} \tag{3.5}$$

which we will call the decoupling formula.

Similarly, we have

$$n,m \le N \Rightarrow |G_{nm}(z) - G_{nm}^{(N)}(z)| \le 2\Big(\sum_{k=N-1,N-2,N-3} |G_{nk}^{(N)}(z)|\Big)\Big(\sum_{|k-N+\frac{1}{2}|\le\frac{5}{2}} |G_{km}(z)|\Big) \tag{3.6}$$

and

$$n,m \ge N \Rightarrow |G_{nm}(z) - \widehat{G}_{nm}^{(N)}(z)| \le 2\Big(\sum_{k=N,N+1,N+2} |\widehat{G}_{nk}^{(N)}(z)|\Big)\Big(\sum_{|k-N+\frac{1}{2}|\le\frac{5}{2}} |G_{km}(z)|\Big), \tag{3.7}$$

where we recall that if $n, m \ge N$ and N is even, then $\widehat{G}_{nm}^{(N)}(z) = G_{n-N,m-N}(z, \{\alpha_{j+N+1}\}_{j=0}^{\infty})$ and if N is odd, then $\widehat{G}_{nm}^{(N)}(z) = G_{m-N,n-N}(z, \{\alpha_{j+N+1}\}_{j=0}^{\infty})$.

Stoiciu's argument has five parts, each with substeps:

Part 1: Some preliminaries concerning H^p properties of F_{ij}, positivity of the Lyapunov exponent, and exponential decay of F_{ij} for a.e. α.

Part 2: Proof of the Aizenman-Molchanov estimates.

Part 3: Using Aizenman-Molchanov estimates to prove that eigenvalues of $\mathcal{C}^{(N)}$ are, except for $O(\log N)$ of them, very close to eigenvalues of $\log N$ independent copies of $\mathcal{C}^{(N/\log N)}$.

Part 4: A proof that the probability of $\mathcal{C}^{(N)}$ having two eigenvalues in an interval of size $2\pi x/N$ is $O(x^2)$.

Part 5: Putting everything together to get the Poisson behavior.

Part 2 uses Simon's formula for G_{ij} (see [36, Section 4.4]) and ideas of Aizenman, Schenker, Friedrich, and Hundertmark [3], but the details are specific to OPUC and exploit the rotation invariance of the distribution in an essential way. Part 3 uses the strategy of Molchanov-Minami with some ideas of Aizenman [1], del Rio et al. [9], and Simon [38]. But again, there are OPUC-specific details that actually make the argument simpler than for OPRL. Part 4 is a new and, I feel, more intuitive argument than that used by Molchanov [27] or Minami [26]. It depends on rotation invariance. Part 5, following Molchanov and Minami, is fairly standard probability theory. Here are some of the details.

In the arguments below, we will act as if $\log N$ and $N/\log N$ are integers rather than doing what a true proof does: use integral parts and wiggle blocks of size $[N/\log N]$ by 1 to get $[\log N]$ of them that add to N.

Step 1.1 (H^p properties of Carathéodory functions). A Carathéodory function is an analytic function on $\mathbb{D}$ with $F(0) = 1$ and $\operatorname{Re} F(z) > 0$. By Kolmogorov's theorem (see [10, Section 4.2]), such an F is in H^p, $0 < p < 1$ with an a priori bound $(0 < p < 1)$,

$$\sup_{0<r<1} \int |F(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \le \cos\Big(\frac{p\pi}{2}\Big)^{-1}.$$

For any unit vector η, $\langle \eta, \frac{\mathcal{C}+z}{\mathcal{C}-z}\, \eta\rangle$ is a Carathéodory function so, by polarization, we have the a priori bounds

$$\sup_{0<r<1} \int |F_{nm}(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \le 2^{2-p} \cos\Big(\frac{p\pi}{2}\Big), \tag{3.8}$$

$0 < p < 1$, all m, n. The same bound holds for $F^{(N)}$.

Step 1.2 (Pointwise estimates on expectations). H^p functions have boundary values and the mean converges (see [10, Theorem 2.6]), so (3.8) holds for $r = 1$ and the sup dropped. If one averages over the random α (or random α and β for $F^{(N)}$) and uses the rotation invariance to see that the expectation is θ-independent, we find

$$\mathbb{E}(|F_{nm}(e^{i\theta})|^p) \leq 2^{2-p} \cos\left(\frac{p\pi}{2}\right) \tag{3.9}$$

for all n, m, all $\theta \in [0, 2\pi)$, and for all $F^{(N)}$.

Step 1.3 (Positive Lyapunov exponent). By the rotation invariance and the Thouless formula (see [37, Theorems 10.5.8 and 10.5.26]), the density of zeros is $d\theta/2\pi$, the Lyapunov exponent exists for all z and is given by (see [37, Theorem 12.6.2])

$$\gamma(z) = -\tfrac{1}{2} \int_{|z| \leq \sigma} \log(1 - |z|^2) \, \frac{d^2 z}{\pi \sigma^2} + \log(\max(1, |z|)),$$

and, in particular, $\gamma(e^{i\theta}) > 0$.

Step 1.4 (Pointwise decay of G). Let $z_0 \in \partial\mathbb{D}$ and let α be a random sequence of α's for which $\frac{1}{n} \log \|T_n(z; \alpha)\| \to \gamma$ and $|F_{00}(z_0)| < \infty$. Since $\gamma(z_0) > 0$, the Ruelle-Osceledec theorem (see [37, Theorem 10.5.29]) implies there is a $\lambda \neq 1$ for which the OPUC with boundary condition λ (see [36, Section 3.2]) obeys $|\varphi_n^{\lambda}(z_0)| \to 0$. It follows from Theorem 10.9.3 of [37] that $|G_{0\,n}(z_0)| \to 0$. Thus, for a.e. α,

$$\lim_{n \to \infty} |G_{0\,n}(z_0)| \to 0. \tag{3.10}$$

By Theorem 10.9.2 of [37], $F_{0\,0} \in i\mathbb{R}$ and this implies that the solutions π and ρ of Section 4.4 of [36] obey $|\pi_k(z_0)| = |\rho_k(z_0)|$ so $|(\mathcal{C} - z)^{-1}_{mn}|$ is symmetric in m and n. Thus (3.10) implies

$$\lim_{n \to \infty} |G_{n\,0}(z_0)| \to 0.$$

Step 1.5 (Decay of $\mathbb{E}(|G_{0\,n}(z_0)|^p)$). The proof of (3.10) shows, for fixed α, $|G_{0n}(z_0)|$ decays exponentially, but since the estimates are not uniform in α, one cannot use this alone to conclude exponential decay of the expectation. But a simple functional analytic argument shows that if h_n are functions on a probability measure space, $\sup_n \mathbb{E}(|h_n|^p) < \infty$ and $|h_n(\omega)| \to 0$ for a.e. ω, then $\mathbb{E}(|h_n|^r) \to 0$ for any $r < p$. It follows from (3.9) and (3.10) that for any $z_0 \in \partial\mathbb{D}$ and $0 < p < 1$,

$$\lim_{n \to \infty} \mathbb{E}(|G_{0\,n}(z_0)|^p) = 0.$$

Step 1.6 (General decay of $\mathbb{E}(|G|^p)$). By the Schwartz inequality and repeated use of (3.5), (3.6), and (3.7), one sees first for $p < \frac{1}{2}$ and then by Hölder's inequality that

$$\lim_{n \to \infty} \sup_{|m-k| \geq n} \mathbb{E}(|G_{mk}(z_0)|^p) = 0$$

and

$$\lim_{n \to \infty} \sup_{\substack{|m-k| \geq n \\ 0 \leq m,k \leq N-1 \\ \text{all } N}} \mathbb{E}(|G^{(N)}_{mk}(z_0)|^p) = 0. \tag{3.11}$$

That completes Part 1.

Step 2.1 (Conditional expectation bounds on diagonal matrix elements). Let

$$H(\alpha,\gamma)=\frac{\alpha+\gamma}{1+\bar{\alpha}\gamma}.$$

Then a simple argument shows that for $0<p<1$,

$$\sup_{\beta,\gamma}\int_{|\alpha|\leq 1}\left|\frac{1}{1-\beta H(\alpha,\gamma)}\right|^p d^2\alpha<\infty,$$

because, up to a constant, the worst case is $\beta=\gamma=1$, and in that case, the denominator vanishes only on $\mathrm{Re}\,\alpha=0$. Applying this to Khrushchev's formula (see [37, Theorem 9.2.4]) provides an a priori bound on the conditional expectation

$$\mathbb{E}(|F_{kk}(z)|^p\mid\{\alpha_j\}_{j\neq k})\leq C \tag{3.12}$$

where C is a universal constant depending only on σ (the radius of the support of the distribution of α) and a similar result for conditioning on $\{\alpha_j\}_{j\neq k-1}$, that is, averaging over α_{k-1}.

Step 2.2 (Conditional expectation bounds on near diagonal matrix elements). Since $\rho^{-1}\leq(1-\sigma^2)^{-1/2}\equiv Q$ for all α with $|\alpha|\leq\sigma$, we have, by equation (1.5.30) of [36], that

$$\left|\frac{\varphi_k}{\varphi_m}\right|\leq(2Q)^{|k-m|}$$

on $\partial\mathbb{D}$. A similar estimate for the solutions π and ρ of Section 4.4 of [36] (using $|\pi_k|=|\rho_k|$; see the end of Step 1.4) proves

$$\left|\frac{\pi_k}{\pi_m}\right|\leq(2Q)^{|k-m|}.$$

This implies, by Theorem 4.4.1 of [36], that

$$\left|\frac{G_{kl}(z)}{G_{mn}(z)}\right|\leq(2Q)^{|k-m|+|\ell-n|}, \tag{3.13}$$

something clearly special to OPUC. This together with (3.12) and (3.3) implies

$$\mathbb{E}(|G_{mn}(z)|^p\mid\{\alpha_j\}_{j\neq k})\leq(2Q)^{|m-k|+|n-k|}(1+2C).$$

Step 2.3 (Double decoupling). This step uses an idea of Aizenman, Schenker, Friedrich, and Hundertmark [3]. Given n, we look at $N<n-3$ and decouple first at N and then at $N+3$ to get, using (3.5) and (3.7), that

$$|G_{0\,n}(z)|\leq 4\Bigg(\sum_{k=N-1,N-2,N-3}|G_{0\,k}^{(N)}(z)|\Bigg)\Bigg(\sum_{\substack{|k-N+\frac{1}{2}|\leq\frac{5}{2}\\|\ell-N+\frac{7}{2}|\leq\frac{5}{2}}}|G_{k\ell}|\Bigg)\Bigg(\sum_{\ell=N+3,N+4,N+5}|\widehat{G}_{\ell n}^{(N+3)}(z)|\Bigg).$$

ETNA
Kent State University
etna@mcs.kent.edu

Using (3.13) and generously overestimating the number of terms, we find

$$|G_{0\,n}(z)| \leq 4\cdot 3\cdot 6\cdot 6\cdot 3(2Q)^{10}|G^{(N)}_{0\,N-1}(z)|\,|G_{N+1\,N+1}(z)|\,|\widehat{G}^{(N+3)}_{N+3\,n}(z)|.$$

Raise this to the p-th power and average over α_{N+1} with $\{\alpha_k\}_{k\neq N+1}$ fixed. Since $G^{(N)}_{0\,N-1}$ and $\widehat{G}^{(N+3)}_{N+3\,N}$ are independent of α_{N+1}, they come out of the conditional expectation which can be bounded by (3.12).

After that replacement has been made, the other two factors are independent. Thus, if we integrate over the remaining α's and use the structure of $\widehat{G}$, we get

$$\mathbb{E}(|G_{0\,n}(z)|^p) \leq C_p \mathbb{E}(|G^{(N)}_{0\,N-1}(z)|^p)\mathbb{E}(|G^{(N)}_{0\,n-N-3}|^p), \tag{3.14}$$

where C_p is p-dependent but N-independent.

Step 2.4 (Aizenman-Molchanov bounds). By (3.11), for p fixed, we can pick N so large that in (3.14), we have $C_p\mathbb{E}(|G^{(N)}_{0\,N-1}(z)|^p) < 1$. If we iterate, we then get exponential decay, that is, we get the Aizenman-Molchanov bound; for any $p\in(0,1)$, there is D_p and κ_p so that

$$\mathbb{E}(|G_{nm}(z)|^p) \leq D_p\, e^{-\kappa_p|n-m|} \tag{3.15}$$

and $n,m\in[0,N-1]$,

$$\mathbb{E}(|G^{(N)}_{nm}(z)|^p) \leq D_p\, e^{-\kappa_p|n-m|}. \tag{3.16}$$

One gets (3.16) from (3.15) by repeating Step 1.6.

That completes Part 2.

Step 3.1 (Dynamic localization). In the Schrödinger case, Aizenman [1] shows (3.16) bounds imply bounds on $\sup_t|(e^{-ith})_{nm}|$. The analog of this has been proven by Simon [38]. Thus, (3.16) implies

$$\mathbb{E}\big(\sup_\ell|(\mathcal{C}^\ell)_{nm}|^p\big) \leq D_p\, e^{-\kappa_p|n-m|}, \tag{3.17}$$

and similarly with $\mathcal{C}$ replaced by $\mathcal{C}^{(N)}$.

Step 3.2 (Pointwise a.e. bounds). For a.e. α, there is $D(\alpha)$ so that

$$[(\mathcal{C}^{(N)})^\ell]_{nm} \leq D(\alpha)(N+1)^8\, e^{-\kappa|n-m|} \tag{3.18}$$

for, by (3.17) and its N analog with $p=\frac{1}{2}$,

$$\mathbb{E}\bigg(\sum_{\substack{n,m,N\\ n,m\leq N}} \sup_\ell |(C^\ell)_{nm}|^{1/2}(N+1)^{-4}e^{+\frac{1}{2}\kappa_{1/2}|n-m|}\bigg) < \infty$$

Step 3.3 (SULE for OPUC). Following del Rio, Jitomirskaya, Last, and Simon [9], we can now prove SULE in the following form. For each eigenvalue ω_k of $\mathcal{C}^{(N)}$, define m_k to maximize the component of the corresponding eigenvector u_k (the eigenvalues are simple), that is,

$$|u_{k,m_k}| = \max_{\ell=1,\ldots,N}|u_{k,\ell}|$$

ETNA
Kent State University
etna@mcs.kent.edu

Since

$$\frac{1}{L}\sum_{\ell=0}^{L-1}\bar{\omega}_k^\ell[(\mathcal{C}^{(N)})^\ell]_{nm}\to u_{k,n}\bar{u}_{k,m}$$

and $\max|u_{k,l}|\geq N^{-1/2}$ (since $\sum_{\ell=0}^{N-1}|u_{k,\ell}|^2=1$), (3.18) implies

$$|u_{k,n}|\leq D(\alpha)(N+1)^{8.5}\,e^{-\kappa|n-m_k|},$$

which is what del Rio et al. call semi-uniform localization (SULE).

Step 3.4 (Bound on the distribution of u_k). If $|n-m_k|\geq\kappa^{-1}[(9.5)\log(N+1)+\log D(\alpha)]$, $|u_{k,n}|\leq(N+1)^{-1}$ so $\sum_{\text{such }n}|u_{k,n}|^2\leq\frac{1}{2}$, and thus,

$$|u_{k,m_k}|^2\geq\frac{1}{C[\log(N)+\log D(\alpha)]}. \tag{3.19}$$

Since $\sum_{k=1}^N|u_{k,\ell}|^2=1$ for each ℓ, (3.19) implies for each m,

$$\#\{k\mid m_k=m\}\leq C[\log(N)+\log D(\alpha)].$$

Step 3.5 (Decoupling except for bad eigenvalues). Let $(C^\sharp)^{(N)}$ be the matrix obtained from $\mathcal{C}^{(N)}$ by decoupling in $\log(N)$ blocks of size $N/\log(N)$ where decoupling is done with random values of $\beta_{jN/\log(N)}$ in $\partial\mathbb{D}$. Call an eigenvalue of $\mathcal{C}^{(N)}$ bad if its m_k lies within $C_1[\log(N)+\log D(\alpha)]$ and good if not. A good eigenvector is centered at an m_k well within a single block and, by taking C_1 large, is of order at most $O(N^{-2})$ at the decoupling points. It follows, by using trial functions, that good eigenvalues move by at most C_2N^{-2} if $\mathcal{C}^{(N)}$ is replaced by $(\mathcal{C}^\sharp)^{(N)}$.

Step 3.6 (Decoupling of probabilities). Fix the k intervals of Theorem 3.1. We claim if $z_j^{(N)}$ are the eigenvalues of $\mathcal{C}^{(N)}$ and $z_j^{\sharp(N)}$ of $\mathcal{C}^{\sharp(N)}$, then

$$\begin{aligned}&\mathbb{P}(\#(\{j\mid z_j^{(N)}\in\Delta_m^{(N)}\}=\ell_m,\,m=1,\dots,k)\\&\qquad-\mathbb{P}(\#(j\mid z_j^{\sharp(N)}\in\Delta_m^{(N)})=\ell_m,\,m=1,\dots,k)\to 0\end{aligned} \tag{3.20}$$

This follows if we also condition on the set where $D(\alpha)\leq D$ because the distribution of bad eigenvalues conditioned on $D(\alpha)\leq D$ is rotation invariant, and so the conditional probability is rotation invariant. Thus, with probability approaching 1, no bad eigenvalues lie in the $\Delta_m^{(N)}$. Also, since the conditional distribution of good eigenvalues is $d\theta/2\pi$, they will lie within $O(N^{-2})$ of the edge with probability N^{-1}. Thus (3.20) holds with the conditioning. Since $\lim_{D\to\infty}(D(\alpha)>D)\to 0$, (3.20) holds.

That concludes Part 3 of the proof. For the fourth part, we note that the POP

$$\Phi_n=z\Phi_{n-1}-\bar{\beta}_n\Phi_{n-1}^*=0\Leftrightarrow\frac{\beta_nz\Phi_{n-1}}{\Phi_n^*}=1.$$

Step 4.1 (Definition and properties of η_n). Define $\eta_n(\theta;\alpha_0,\dots,\alpha_{n-1},\beta_n)$ for $\theta\in[0,2\pi]$

$$\frac{\beta_nz\Phi_{n-1}}{\Phi_{n-1}^*}=\exp(i\eta_n)|_{z=e^{i\theta}}\,.$$

The ambiguity in η_n is settled by usually thinking of it as only defined mod 2π, that is, in $\mathbb{R}/2\pi\mathbb{Z}$. η_n is then real analytic and has a derivative $d\eta_n/d\theta$ lying in $\mathbb{R}$. We first claim

$$\frac{d\eta_n}{d\theta} > 0, \tag{3.21}$$

for if $\tilde{\eta}(\theta)$ is defined by $e^{i\tilde{\eta}} = (z - z_0)/(1 - z\bar{z}_0)$ for $z_0 \in \mathbb{R}$, and $z = e^{i\theta}$, then

$$\frac{d\tilde{\eta}}{d\theta} = \frac{1 - |z_0|^2}{|e^{i\theta} - z_0|^2} > 0, \tag{3.22}$$

from which (3.21) follows by writing Φ_{n-1} as the product of its zeros, all of which lie in $\mathbb{D}$. We also have

$$\int_0^{2\pi} \frac{d\eta_n}{d\theta}\, d\theta = 2\pi n. \tag{3.23}$$

This follows from the argument principle if we note $\beta_n z\Phi_{n-1}/\Phi^*_{n-1}$ is analytic in $\mathbb{D}$ with n zeros there. Alternatively, since the Poisson kernel maps 1 to 1, (3.22) implies $\int \frac{d\tilde{\eta}}{2\theta}\frac{d\theta}{2\pi} = 1$, which also yields (3.23).

Step 4.2 (Independence of $\eta_n(e^{i\theta})$ and $\frac{d\eta_n}{d\theta}(e^{i\theta_1})$). β_n drops out of $d\eta_n/d\theta$ at all points. On the other hand, β_n is independent of $z\Phi_{n-1}/\Phi^*_{n-1}$ and uniformly distributed. It follows that $\eta_n(e^{i\theta_0})$ and $\frac{d\eta_n}{d\theta}(e^{i\theta_1})$ at any θ_0 and θ_1 are independent. Moreover, $\eta_n(e^{i\theta})$ is uniformly distributed.

Step 4.3 (Calculation of $\mathbb{E}(d\eta_n/d\theta)$). As noted, $d\eta_n/d\theta$ is only dependent on $\{\alpha_j\}_{j=0}^{n-2}$ and, by rotation covariance of the α's (see [36, Eqns. (1.6.62)–(1.6.68)]),

$$\frac{d\eta_n}{d\theta}(\theta_0; e^{-(j+1)\varphi}\alpha_j) = \frac{d\eta_n}{d\theta}(\theta_0 - \varphi; \alpha_j).$$

It follows that since the α_j's are rotation invariant that $\mathbb{E}(\frac{d\eta_n}{d\theta}(\theta_0))$ is independent of θ_0 and so, by applying $\mathbb{E}$ to (3.23),

$$\mathbb{E}\left(\frac{d\eta}{d\theta}(\theta_0)\right) = n. \tag{3.24}$$

Step 4.4 (Bound on the conditional expectation). Let I_n be an interval on $\partial\mathbb{D}$ of size $2\pi y/n$. Let $\lambda_0 \in I_n$ and consider the conditional probability

$$\mathbb{P}(I_n \text{ has 2 or more eigenvalues} \mid \lambda_0 \text{ is an eigenvalue}), \tag{3.25}$$

(where we use "eigenvalue" to refer to zeros of the POP since they are eigenvalues of a $\mathcal{C}^{(N)}$). If that holds, $\eta_n(\lambda_0) = 1$ and, for some λ_1 in I, $\eta(\lambda_1) = 1$, so $\int_{I_n} \frac{d\eta_n}{d\theta} d\theta \geq 2\pi$. Thus the conditional probability (3.25) is bounded by the conditional probability

$$\mathbb{P}\left(\int_{I_n} \frac{d\eta_n}{d\theta}\, d\theta \geq 2\pi \;\middle|\; \eta(\lambda_0) = 1\right). \tag{3.26}$$

While (3.25) is highly dependent on the value of $\eta(\lambda_0)$, (3.26) is not since $d\eta_n/d\theta$ is independent of $\eta(\lambda_0)$. Thus, by Chebyshev's inequality,

$$(3.25) \leq (3.26) \leq \mathbb{P}\left(\int_{I_n} \frac{d\eta_n}{d\theta}\, d\theta \geq 2\pi\right)$$

ETNA
Kent State University
etna@mcs.kent.edu

$$\leq (2\pi)^{-1}\mathbb{E}\biggl(\int_{I_n} \frac{d\eta_n}{d\theta}\,d\theta\biggr) = \biggl(\frac{2\pi y}{n}\biggr)(2\pi)^{-1}n = y.$$

by (3.24).

<u>*Step 4.5*</u> (Two eigenvalue estimate). By a counting argument,

$$\mathbb{P}(I_n \text{ has exactly } m \text{ eigenvalues}) = \frac{1}{m}\int_{\theta\in I_n} \mathbb{P}(I_n \text{ has exactly } m \text{ eigenvalues} \mid \eta_n(\theta)=1)\,d\kappa(\theta),$$

where $d\kappa(\theta)$ is the density of eigenvalues which is $\frac{n}{2\pi}d\theta$ by rotation invariance. Since $m \geq 2 \Rightarrow \frac{1}{m} \leq \frac{1}{2}$, we see

$$\mathbb{P}(I_n \text{ has 2 or more eigenvalues}) \leq \tfrac{1}{2}\int_{I_n} (3.25)\,d\kappa(\theta) \leq \frac{y}{2}\biggl(\frac{n}{2\pi}\,\frac{2\pi y}{n}\biggr) = \frac{y^2}{2}. \tag{3.27}$$

The key is that for y small, (3.27) is small compared to the probability that I_n has at least one eigenvalue which is order y. This completes Part 4.

<u>*Step 5.1*</u> (Completion of the proof). It is essentially standard theory of Poisson processes that an estimate like (3.27) for a sum of a large number of independent point processes implies the limit is Poisson. The argument specialized to this case goes as follows. Use Step 3.6 to consider $\log N$ independent of POPs of degree $N/\log N$. The union of the $\Delta_m^{(N)}$ lies in a single interval, $\tilde{\Delta}^{(N)}$, of size C/N (here is where the $\theta_0 = \cdots = \theta_k$ condition is used) which is $y_N \frac{2\pi}{(N/\log N)}$ with $y_N = C/\log N$. Thus the probability of any single POP having two zeros in $\tilde{\Delta}^{(N)}$ is $O((\log N)^{-2})$. The probability of any of the $\log N$ POPs having two zeros is $O((\log N)^{-1}) \to 0$.

The probability of any single eigenvalue in a $\Delta_m^{(N)}$ is $O(1/\log N)$, so each interval is described by precisely the kind of limit where the Poisson distribution results. Since, except for a vanishing probability, no interval has eigenvalues from a POP with an eigenvalue in another, and the POPs are independent, we get independence of intervals. This completes our sketch of Stoiciu's proof of his result.

4. Overview of Clock Theorems. The rest of this paper is devoted to proving various theorems about equal spacings of zeros under suitable hypotheses. In this section, we will state the main results and discuss the main themes in the proofs. It is easiest to begin with the case of POPs for OPUC:

THEOREM 4.1. *Let $\{\alpha_j\}_{j=0}^\infty$ be a sequence of Verblunsky coefficients so that*

$$\sum_{j=0}^\infty |\alpha_j| < \infty \tag{4.1}$$

and let $\{\beta_j\}_{j=1}^\infty$ be a sequence of points on $\partial\mathbb{D}$. Let $\{\theta_j^{(n)}\}_{j=1}^n$ be defined so $0 \leq \theta_1^{(n)} \leq \cdots \leq \theta_n^{(n)} < 2\pi$ and so that $e^{i\theta_j^{(n)}}$ are the zeros of the POPs

$$\Phi_n^{(\beta)}(z) = z\Phi_{n-1}(z) - \bar{\beta}_n\Phi_{n-1}^*(z).$$

Then (*with* $\theta_{n+1}^{(n)} \equiv \theta_1^{(n)} + 2\pi$)

$$\sup_{j=1,\dots,n} n\left|\theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{2\pi}{n}\right| \to 0 \tag{4.2}$$

as $n \to \infty$.

The intuition behind the theorem is very simple. Szegő's theorem and Baxter's theorem imply on $\partial\mathbb{D}$ that (with $\varphi_n^{(\beta)} = \Phi_n^{(\beta)}/\|\Phi_{n-1}\|$)

$$\varphi_n^{(\beta)}(e^{i\theta}) \sim e^{in\theta}\,\overline{D(e^{i\theta})^{-1}} - \bar{\beta}_n D(e^{i\theta})^{-1} \tag{4.3}$$

and the zeros of the right side of (4.3) obey (4.2)! (4.3) holds only on $\partial\mathbb{D}$ and does not extend to complex θ without much stronger hypotheses on α. That works since we know by other means that $\varphi_n^{(\beta)}$ has all its zeros on $\partial\mathbb{D}$. But when one looks at true OPUC, we will not have this a priori information and will need stronger hypotheses on the α's.

There is a second issue connected with the $\sim$ in (4.3). It means uniform convergence of the difference to zero. If f_n and g are uniformly close, f_n must have zeros close to the zeros of g, and we will have enough control on the right side of (4.3) to be sure that $\varphi_n^{(\beta)}$ has zeros near those of the right side of (4.3). So uniform convergence will be existence of zeros.

A function like $f_n(x) = \sin(x) - \frac{2}{n}\sin(nx)$, which has three zeros near $x = 0$, shows uniqueness is a more difficult problem.

There are essentially two ways to get uniqueness. One involves control over derivatives and/or complex analyticity which will allow uniqueness via an appeal to an intermediate value theorem or a use of Rouché's theorem. These will each require extra hypotheses on the Verblunsky coefficients or Jacobi parameters. In the case of genuine OPUC where we already have to make strong hypotheses for existence, we will use a Rouché argument.

There is a second way to get uniqueness, namely, by counting zeros. Existence will imply an odd number of zeros near certain points. If we have n such points and n zeros, we will get uniqueness. This will be how we will prove Theorem 4.1. Counting will be much more subtle for OPRL because the close zeros will lie in $[-2,2]$ (if $a_n \to 1$ and $b_n \to 0$) and there can be zeros outside. For counting to work, we will need only finitely many mass points outside $[-2,2]$. This will be obtained via a Bargmann bound, which explains why our hypothesis in the next theorem is what it is:

THEOREM 4.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be Jacobi parameters that obey*

$$\sum_{n=1}^{\infty} n(|a_n - 1| + |b_n|) < \infty. \tag{4.4}$$

Let $\{P_n\}_{n=0}^{\infty}$ *be the monic orthogonal polynomials and let* $\{E_j\}_{j=1}^{J}$ ($J < \infty$) *be the mass points of the associated measure which lie outside* $[-2,2]$. *Then for* n *sufficiently large,* $P_n(x)$ *has precisely* J *zeros outside* $[-2,2]$ *and the other* $n-J$ *in* $[-2,2]$. *Define* $0 < \theta_1^{(n)} < \cdots < \theta_{n-J}^{(n)} < \pi$ *so that the zeros of* $P_n(x)$ *on* $[-2,2]$ *are exactly at* $\{2\cos(\theta_\ell^{(n)})\}_{\ell=1}^{n-J}$. *Then*

$$\sup_{j=1,\dots,n-J-1} n\left|\theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{2\pi}{2n}\right| \to 0. \tag{4.5}$$

REMARK 4.3. *1. The Jacobi parameters are defined by the recursion relation*

$$xP_n(x) = P_{n-1}(x) + b_{n+1}P_n(x) + a_n^2 P_{n-1}(x)$$

(with $P_0(x) = 1$ and $P_{-1}(x) = 0$).

2. It is known for all Jacobi polynomials that the Jacobi parameters have $|b_n| + |a_n - 1| = O(n^{-2})$. So (4.4) fails. In Section 7, we will provide a different argument that proves clock behavior for Jacobi polynomials.

3. We will also say something about $|\theta_1|$ and $|\pi - \theta_{n-J}|$, but the result is a little involved so we put the details in Section 6.

Finally, we quote the result for OPUC obeying the BLS condition:

THEOREM 4.4. *Suppose a set of Verblunsky coefficients obeys (1.11) for $C \in \mathbb{C}$, $C \neq 0$, $b \in (0,1)$, and $\Delta \in (0,1)$. Then the number, J, of Nevai-Totik points is finite, and for n large, $\Phi_n(z)$ has J zeros near these points. The other $n - J$ zeros can be written $\{z_j^{(n)}\}_{j=1}^{n-J}$ where $z_j^{(n)} \equiv |z_j^{(n)}| e^{i\theta_j^{(n)}}$ with (for n large) $\theta_0^{(n)} \equiv 0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_{n-J}^{(n)} < 2\pi = \theta_{n-J+1}^{(n)}$. We have that*

$$\sup_j \big|\, |z_j^{(n)}| - b \,\big| = O\left(\frac{\log(n)}{n}\right), \tag{4.6}$$

$$\sup_{j=0,\dots,n-J} n \left| \theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{2\pi}{n} \right| \to 0, \tag{4.7}$$

and

$$\frac{|z_{j+1}^{(n)}|}{|z_j^{(n)}|} = 1 + O\left(\frac{1}{n \log n}\right). \tag{4.8}$$

REMARK 4.5. *1. We will see that "usually," the right side of (4.6) can be replaced by $O(1/n)$ and the right side of (4.8) by $1 + O(1/n^2)$.*

2. Since $\theta_0 = 0$ and $\theta_{n-J+1} = 2\pi$ are not zeros, the angular gap between $z_1^{(n)}$ and $z_{n-J}^{(n)}$ is $2(2\pi/n)$.

3. (4.7) and (4.8) imply that $|z_{j+1}^{(n)} - z_j^{(n)}| \to \frac{2\pi}{n} b$.

The key to the proofs of Theorems 4.2 and 4.4 will be careful asymptotics for P_n and Φ_n. For P_n, we will use well-known Jost-Szegő asymptotics. For Φ_n, our analysis seems to be new.

We will also prove two refined results on the Nevai-Totik zeros, one of which has a clock!

THEOREM 4.6. *Suppose that (1.5) holds with $b < 1$. Let z_0 obey $|z_0| > b$ and $D(1/\bar{z}_0)^{-1} = 0$ (i.e., z_0 is a Nevai-Totik zero). Let z_n be zeros of $\varphi_n(z)$ near z_0 for n large. Then for some $\varepsilon > 0$ and n large,*

$$|z_n - z_0| \leq e^{-\varepsilon n}. \tag{4.9}$$

REMARK 4.7. *In general, if z_0 is a zero of order k of $\overline{D(1/\bar{z})}^{-1}$, then there are k choices of z_n and all obey (4.9).*

THEOREM 4.8. *Suppose that (1.11) holds for $C \in \mathbb{C}$, $C \neq 0$, $b \in (0,1)$, and $\Delta \in (0,1)$. There exists $\Delta_2, \Delta_3 \in (0,1)$ so that if $b < z_0 < b\Delta_2$ is a zero of order k of $\overline{D(1/\bar{z})}^{-1}$, then for large n, the k zeros of $\varphi_n(z)$ near z_0 have a clock pattern:*

$$z_n^{(j)} = z_0 + C_1 \left(\frac{b}{|z_0|}\right)^{n/k} \exp\left(-2\pi i \,\frac{n}{k}\, \arg(z_0)\right) e^{2\pi i j/k} + O\left(\left(\frac{b\Delta_3}{|z_0|}\right)^{n/k}\right), \tag{4.10}$$

ETNA
Kent State University
etna@mcs.kent.edu

for $j = 0, 1, \ldots, k-1$ (*so the* k *zeros form a* k*-fold clock*).

This completes the description of clock theorems we will prove in this article, but I want to mention three other situations where the pictures in [36] suggest there are clock theorems plus a fourth situation:

(A) **Periodic Verblunsky Coefficients.** As Figures 8.8 and 8.9 of [36] suggest, if the Verblunsky coefficients are periodic (or converge sufficiently rapidly to the periodic case), the zeros are locally equally spaced, but are spaced inversely proportional to a local density of states. We will prove this in a future paper [35]. For earlier related results, see [22, 29, 30].

(B) **Barrios-López-Saff [4]** consider α_n's which are decaying as b^n with a periodic modulation. A strong version of their consideration is that there is a period p sequence, $c_1, c_2, \ldots, c_p$, all nonzero with

$$\alpha_n = b^n c_n + O((b\Delta)^n),$$

with $0 < \Delta < 1$. In [34], we will prove clock behavior for such α_n's. In general, there are p missing points in the clock at $b\omega^j$ with $\omega = \exp(2\pi i/p)$ a p-th root of unity. Indeed, we will treat the more general

$$\alpha_n = \sum_{\ell=1}^{m} c_\ell e^{in\theta_\ell} b^n + O((b\Delta)^n),$$

for any $\theta_1, \ldots, \theta_m \in [0, 2\pi)$.

(C) **Power Regular Baxter Weights.** Figure 8.5 of [36] (which shows zeros for $\alpha_n = (n+2)^{-2}$) suggests that if $\beta > 1$ and $n^\beta \alpha_n \to C$ sufficiently fast, then one has a strictly clock result. By [4], all zeros approach $\partial\mathbb{D}$, and we believe that if their phases are $0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_n < 2\pi$ and $\theta_{n+1} = \theta_1 + 2\pi$, then $\sup_j |n[\theta_{j+1} - \theta_j] - 2\pi| \to C$.

(D) **Slow Power Decay.** Figures 8.6 and 8.7 of [36] (which show $\alpha_n = (n+2)^{-1/2}$ and $(n+2)^{-1/8}$) were shown by Ed Saff at a conference as a warning that pictures can be misleading because they suggest there is a gap in the spectrum while we know that the Mhaskar-Saff theorem applies! In fact, I take their prediction of the gap seriously and suggest if $(n+2)^\beta \alpha_n \to C$ fast enough for $\beta < 1$, then we have clock behavior away from $\theta = 0$, that is, if θ_0 is fixed and θ_j, θ_{j+1} are the nearest zeros to θ_0, then $n(\theta_{j+1} - \theta_j) \to 2\pi$, but that there is a single zero near $\theta = 0$ with the next nearby zero θ' obeying $n|\theta'| \to \infty$.

(C) and (D) present interesting open problems.

5. Clock Theorems for POPs in Baxter's Class. In this section, we will prove Theorem 4.1.

LEMMA 5.1. *If* (4.1) *holds, then*

$$\sup_{e^{i\theta} \in \mathbb{D}} \left| \frac{e^{i\theta} \varphi_{n-1}(e^{i\theta})}{\varphi_{n-1}^*(e^{i\theta})} - \frac{e^{in\theta}\, \overline{D(e^{i\theta})}^{-1}}{D(e^{i\theta})^{-1}} \right| \to 0 \tag{5.1}$$

as $n \to \infty$.

REMARK 5.2. *1. Recall* $\varphi_n^{(\beta)} = \Phi_n^{(\beta)} / \|\Phi_{n-1}\|$, *that is,*

$$\varphi_n^{(\beta)} = z\varphi_{n-1} - \bar\beta \varphi_{n-1}^*. \tag{5.2}$$

2. Implicit here is the fact that $D(z)$ *defined initially on* $\mathbb{D}$ *has a continuous extension to* $\bar{\mathbb{D}}$.

Proof. Baxter's theorem (see [36, Theorem 5.2.2]) says that $D(z)$ lies in the Wiener algebra and, in particular, has a (unique) continuous extension to $\bar{\mathbb{D}}$, and that $\varphi_{n-1}^*(z) \to D(z)^{-1}$ uniformly on $\bar{\mathbb{D}}$ and, in particular, uniformly on $\partial\mathbb{D}$. (5.2) plus $\varphi_{n-1}(e^{i\theta}) = e^{i(n-1)\theta}\,\overline{\varphi_{n-1}^*(e^{i\theta})}$ completes the proof. □

Since D is nonvanishing on $\bar{\mathbb{D}}$ (see [36, Theorem 5.2.2]), the argument principle implies $D(e^{i\theta}) = |D(e^{i\theta})|e^{i\psi(\theta)}$ with ψ continuous and $\psi(2\pi) = \psi(0)$. We will suppose $\psi(0) \in (-\pi, \pi]$.

LEMMA 5.3. *For each n and each $\eta \in [2\psi(0), 2\psi(0) + 2\pi)$, there are solutions, $\tilde{\theta}_{j,\tilde{\eta}}^{(n)}$, of*

$$n\tilde{\theta} + 2\psi(\tilde{\theta}) = 2\pi j + \tilde{\eta}, \tag{5.3}$$

for $j = 0, 1, \ldots, n-1$. We have that

$$\sup_{\tilde{\eta}, j=0,1,2,\ldots,n-1} n\left|\tilde{\theta}_{j+1,\tilde{\eta}}^{(n)} - \tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \frac{2\pi}{n}\right| \to 0, \tag{5.4}$$

where $\tilde{\theta}_{n,\tilde{\eta}}^{(n)} \equiv 2\pi + \tilde{\theta}_{0,\tilde{\eta}}^{(n)}$. Moreover, for any ε, there is an N so for $n > N$,

$$\sup_{|\eta-\eta'|<\frac{\varepsilon}{2}} n|\tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \tilde{\theta}_{j,\tilde{\eta}'}^{(n)}| \leq \varepsilon. \tag{5.5}$$

Proof. As $\tilde{\theta}$ runs from 0 to 2π, the LHS of (5.3) runs from $2\psi(0)$ to $2\pi n + 2\psi(0)$. By continuity, (5.3) has solutions. If there are multiple ones, pick the one with $\tilde{\theta}_{j,\tilde{\eta}}^{(n)}$ as small as possible.

Since ψ is bounded, there is C so

$$\left|\tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \frac{2\pi j}{n}\right| \leq \frac{C}{n},$$

so subtracting (5.3) for $j+1$ from (5.3) for j,

$$\left|\tilde{\theta}_{j+1,\tilde{\eta}}^{(n)} - \tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \frac{2\pi}{n}\right| \leq \frac{2}{n} \max_{|\theta-\theta'|\leq\frac{C+1}{n}} |\psi(\theta) - \psi(\theta')|. \tag{5.6}$$

Since ψ is continuous on $[0, 2\pi]$, it is uniformly continuous, and thus the max in (5.6) goes to zero, which implies (5.4).

To prove (5.5), we note that subtracting (5.3) for $\tilde{\eta}$ from (5.3) for $\tilde{\eta}'$,

$$n|\tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \tilde{\theta}_{j,\tilde{\eta}'}^{(n)}| \leq |\tilde{\eta} - \tilde{\eta}'| + 2|\psi(\tilde{\theta}_{j,\tilde{\eta}}^{(n)}) - \psi(\tilde{\theta}_{j,\tilde{\eta}'}^{(n)})|. \tag{5.7}$$

The inequality (5.7) first implies

$$|\tilde{\theta}_{j,\tilde{\eta}}^{(n)} - \tilde{\theta}_{j,\tilde{\eta}'}^{(n)}| \leq \frac{C}{n},$$

and then implies (5.5) picking N so

$$\sup_{|\theta-\theta'|\leq\frac{C}{n}} |\psi(\theta) - \psi(\theta')| < \frac{\varepsilon}{4}. \qquad \square$$

REMARK 5.4. *The proof shows that (5.5) continues to hold for any solutions of (5.3).*

Proof. [Proof of Theorem 4.1] The phase, $\zeta_n(\theta)$, of $z\varphi_{n-1}/\varphi_{n-1}^*\big|_{z=e^{i\theta}}$ is monotone increasing and runs from $\zeta_n(0)$ to $2\pi n = \zeta_n(0)$ at $\theta = 2\pi$ (see Step 4.1 in Section 3 for the monotonicity), so for any fixed $\beta_n = e^{-i\eta_n}$ with $\eta_n \in [\zeta_n(0), \zeta_n(0) + 2\pi)$, there are exactly n solutions, $\theta_j^{(n)}$, $j = 0, 1, \dots, n-1$ of

$$\zeta_n(\theta_j^{(n)}) = 2\pi j + \eta_n.$$

By (5.1), $\theta_j^{(n)}$ solves (5.3) with

$$\eta' - \eta_n \to 0$$

as $n \to \infty$. By the remark after Lemma 5.3, that shows that (4.2) holds. □

I believe this result has an extension to a borderline where $\alpha_n = C_0 n^{-1} +$ error, where the error goes to zero sufficiently fast (ℓ^1 error may suffice; for applications, $|\text{error}| \leq Cn^{-2}$ is all that is needed). The extension is on zeros away from $\theta = 0$. I believe in this case that D has an extension to $\partial\mathbb{D}\backslash\{1\}$ (see [37, Section 12.1]) and one has convergence there. Replacing uniformity should be some control of derivatives of φ_n^* (an $O(n)$ bound). By the Szegő mapping (see [37, Section 13.1]), this would provide another approach to Jacobi polynomials.

6. Clock Theorems for OPRL With Bargmann Bounds. Our goal here is to prove Theorem 4.2 which shows clock behavior for OPUC when (4.4) holds. It is illuminating to consider two simple examples first:

EXAMPLE 6.1. *Take* $a_n \equiv 1$, $b_n \equiv 0$. *Then*

$$P_n(2\cos\theta) = c_n \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

the Chebyshev polynomials of the second kind. The zeros are precisely at

$$\theta_j = \frac{\pi j}{n+1} \qquad j = 1, 2, \dots, n.$$

Note

$$\theta_{j+1} - \theta_j = \frac{\pi}{n+1} = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right), \tag{6.1}$$

for $j = 1, \dots, n-1$. *In addition,*

$$\theta_1 = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right) \qquad \pi - \theta_n = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right). \tag{6.2}$$

EXAMPLE 6.2. *Take* $a_1 = \sqrt{2}$, $a_n = 1$ *(*$n \geq 2$*),* $b_n = 0$. *Then*

$$P_n(2\cos\theta) = c_n \cos(n\theta),$$

the Chebyshev polynomials of the first kind. The zeros are precisely at

$$\theta_j = \frac{\pi(j - \frac{1}{2})}{n} \qquad j = 1, \dots, n.$$

The identities (6.1) *hold in this case also but instead of* (6.2)*, we have*

$$\theta_1 = \frac{\pi}{2n} + O\left(\frac{1}{n^2}\right) \qquad \pi - \theta_n = \frac{\pi}{2n} + O\left(\frac{1}{n^2}\right).$$

We will make heavy use of the construction of the Jost function in this case. For Jacobi matrices, Jost functions go back to a variety of papers of Case and collaborators; see, for example, [6, 15]. I will follow ideas of Killip-Simon [20] and Damanik-Simon [7, 8], as discussed in [37, Chapter 13].

First, we note some basic facts about the zeros of $P_n(x)$, some only true when (4.4) holds.

PROPOSITION 6.3. *Let $d\mu$ be a measure on $\mathbb{R}$ whose Jacobi parameters obey (4.4). Then*

(a) $\mathrm{supp}(d\mu) = [-2,2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$ *with* $E_1^- < \cdots < E_{N_-}^- < -2$ *and* $E_1^+ > E_2^+ > \cdots > E_{N_+}^+ > 2$.

(b) $N_+ < \infty$ *and* $N_- < \infty$.

(c) *For any n, $P_n(x)$ has at most one zero in each (E_j^-, E_{j+1}^-) $(j = 1, \dots, N_-)$, in each (E_{j+1}^+, E_j^+) $(j = 1, \dots, N_+)$, and in $(E_{N_-}^-, -2)$ and $(2, E_{N_+}^+)$.*

(d) *For some N_0 and $n > N_0$, $P_n(x)$ has exactly one zero in each of the above intervals and all other zeros lie in $(-2, 2)$.*

Proof. (a) holds because if J_0 is the free Jacobi matrix (the one with $a_n \equiv 1$, $b_n \equiv 0$), then $J - J_0$ is compact.

(b) This follows from the Bargmann bound for Jacobi matrices as proven by Geronimo [13, 14] and Hundertmark-Simon [17].

(c) That there is at most one zero in any interval disjoint from $\mathrm{supp}(d\mu)$ is a standard fact [12].

(d) By a simple variational argument, using the trial functions in (1.2.61) of [36], each $E_j^\pm$ is a limit point of zeros. This and (c) imply that each interval has a zero for large N. By a comparison argument, the E_j's cannot be zeros for such n and also shows that ± 2 are not zeros. Since all zeros lie in $[E_1^-, E_1^+]$ (see [36, Subsection 1.2.5]), the other zeros lie in $[-2, 2]$. □

REMARK 6.4. *It is possible N_+ and/or N_- are zero, in which case the above proof changes slightly, for example, E_1^- is replaced by -2.*

Next, we use the fact (see [37, Theorem 13.6.5]) that when (4.4) holds, there is a Jost function, $u(z)$, described most simply in the variable z with $E = z + z^{-1}$ ($z \in \mathbb{D}$ maps to $\mathbb{C}\backslash[-2,2]$ and $\partial\mathbb{D}$ is a twofold cover on $[-2,2]$ with $e^{i\theta} \to 2\cos\theta$).

PROPOSITION 6.5. *Let $d\mu$ be a measure on $\mathbb{R}$ whose Jacobi parameters obey (4.4). There exists a function $u(z)$ on $\bar{\mathbb{D}}$, analytic on $\mathbb{D}$, continuous on $\bar{\mathbb{D}}$, and real on $\bar{\mathbb{D}} \cap \mathbb{R}$, so that*

(a) *Uniformly on* $[0, 2\pi]$,

$$(\sin\theta) p_{n-1}(2\cos\theta) - \mathrm{Im}(\overline{u(e^{i\theta})}\, e^{in\theta}) \to 0. \tag{6.3}$$

(b) *The only zeros u has in $\mathbb{D}$ are at those points $\beta_j^\pm \in \mathbb{D}$ with $\beta_j^\pm + (\beta_j^\pm)^{-1} = E_j^\pm$.*

(c) *The only possible zeros of u and $\partial\mathbb{D}$ are at $z = \pm 1$, and if $u(\pm 1) = 0$, then $\lim_{\theta\to 0} \theta^{-1} u(\pm e^{i\theta}) = c$ exists and is nonzero.*

Proof. This is part of Theorems 13.6.4 and 13.6.5 of [37]. □

Proof. [Proof of Theorem 4.2] Write

$$u(e^{i\theta}) = |u(e^{i\theta})| e^{i\eta(\theta)}$$

Mod 2π, η is uniquely defined on $(0, \pi)$ and $(\pi, 2\pi)$ since u is nonvanishing there.

Suppose first that $u(\pm 1) \neq 0$. Then η can be chosen continuously at ± 1 and so η can be chosen continuously on $[0, 2\pi]$ with

$$\eta(2\pi) - \eta(0) = 2\pi(N_+ + N_-),$$

by the argument principle and the fact that the number of zeros of u in $\mathbb{D}$ is $N_+ + N_-$.

It follows that if

(6.4) $$g_n(\theta) = n\theta - \eta(\theta)$$

then

$$g_n(2\pi) = 2\pi(n - N_+ - N_-),$$

and, in particular,

$$\operatorname{Im}(\overline{u(e^{i\theta})}\, e^{in\theta})$$

has at least $2(n - N_+ - N_-)$ zeros at points where $\tilde{\theta}_j$

(6.5) $$g_n(\tilde{\theta}_j) = \pi j \qquad j = 0, 1, \ldots, 2(n - N_+ - N_-) - 1.$$

Since u is real,

$$\tilde{\theta}_{2(n-N_+-N_-)-j} = 2\pi - \tilde{\theta}_j$$

and $\tilde{\theta}_0 = 0$, $\tilde{\theta}_{n-N_+-N_-} = \pi$.

By the boundedness of η, (6.4), and (6.5),

(6.6) $$\tilde{\theta}_{j+1} - \tilde{\theta}_j = \frac{\pi}{n} + o\left(\frac{1}{n}\right)$$

uniformly in j.

By (6.3), $\sin(\theta)p_{n-1}(2\cos\theta)$, $\theta \in [0, 2\pi]$ has at least $n - N_+ - N_- + 1$ zeros on $[0, \pi]$ at points θ_j with $\theta_j - \tilde{\theta}_j = o(1/n)$. Since 0 and π are zeros of $\sin\theta$ and for large n, $p_{n-1}(2\cos\theta)$ only has $n - N_+ - N_- - 1$ zeros on $(0, \pi)$, $\{\theta_j\}$ are the only zeros. (6.6) implies (4.5) and also

$$\theta_1 = \frac{\pi}{n} + O\left(\frac{1}{n}\right) \qquad \theta_{n-N_+-N_--1} = \pi - \frac{\pi}{n} + O\left(\frac{1}{n}\right).$$

Next, we turn to the case where u vanishes as $+1$ and/or -1. Suppose that $u(1) = 0$, $u(-1) \neq 0$. By (a) of Proposition 6.5 and the reality of u (i.e., $\overline{u(e^{i\theta})} = u(e^{-i\theta})$), we have that $u(e^{i\theta}) = ic\theta + o(\theta)$ where $c \neq 0$, so

$$\eta(0) = \pm\frac{\pi}{2} \qquad \eta(2\pi) = \mp\frac{\pi}{2} \qquad (\text{mod } 2\pi).$$

By the argument principle taking into account the zero at $z = 1$,

$$\eta(2\pi) - \eta(0) = 2\pi(N_+ + N_- + \tfrac{1}{2})$$

As in the regular case, (6.6) holds, but in place of (6.3),

$$\theta_1 = \frac{\pi}{2n} + O\left(\frac{1}{n}\right) \qquad \theta_{n-N_+-N_--1} = \pi - \frac{\pi}{n} + O\left(\frac{1}{n}\right).$$

The other cases are similar. □

To summarize, we use a definition.

DEFINITION 6.6. *We say J has a resonance at $+2$ if and only if $u(1) = 0$ and a resonance at -2 if $u(-1) = 0$.*

THEOREM 6.7. *Let (4.4) hold. Let $\theta_j^{(n)}$ be the points where $P_n(2\cos\theta) = 0$, $0 < \theta_1^{(n)} < \cdots < \theta_{n-N_+-N_-}^{(n)} < \pi$. Then*

$$\theta_1^{(n)} = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right)$$

if $+2$ is not a resonance and

$$\theta_1^{(n)} = \frac{\pi}{2n} + O\left(\frac{1}{n^2}\right)$$

if $+2$ is a resonance. Similar results hold for $\theta_{n-N_+-N_-}^{(n)}$ with regard to a resonance at -2.

7. Clock Theorems for Jacobi Polynomials. The Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, are defined [31, 40] to be orthogonal for the weight

$$w(x) = (1-x)^\alpha (1+x)^\beta$$

on $[-1,1]$ where $\alpha, \beta > -1$ (to insure integrability of the weight). The P_n's are normalized by

$$P_n^{(\alpha,\beta)}(1) = \frac{[\prod_{j=0}^{n-1}(1+\alpha-j)]}{n!}.$$

They are neither monic nor orthonormal, but it is known ([40, Eqns. (4.3.2) and (4.3.3)]) that $P_n^{(\alpha,\beta)}(x)$ differ from the normalized polynomials by $(n+1)^{1/2}c_n^{(\alpha,\beta)}$ with $0 < \inf_n c_n^{(\alpha,\beta)} \leq \sup_n c_n^{(\alpha,\beta)} < \infty$ (not uniform in α, β), and $2^n P_n^{(\alpha,\beta)}(x)$ have leading term $d_n^{(\alpha,\beta)}$ with a similar estimate to c_n.

Our goal in this section is to prove

THEOREM 7.1. *Fix α, β. Let $\theta_j^{(n)}$, $j = 1, 2, \dots, n$, be defined by*

$$0 < \theta_1^{(n)} < \cdots < \theta_n^{(n)} < \pi$$

and $x_j^{(n)} = \cos(\theta_j^{(n)})$ are all the zeros of $P_n^{(\alpha,\beta)}(x)$. Then for each $\varepsilon > 0$,

$$\sup_{j;\,\theta_j\in[\varepsilon,\pi-\varepsilon]} n\left|\theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{\pi}{n}\right| \to 0 \tag{7.1}$$

as $n \to \infty$.

REMARK 7.2. *1. It is not hard to see that $\theta_j \in [\varepsilon, \pi-\varepsilon]$ can be replaced by $\delta n < j < (1-\delta)n$ for each $\delta > 0$.*

2. For restricted values of α, β, this result is a special case of results of Erdös-Turan [11]. Szegő [40] has bounds of the form

$$Cn^{-1} < \theta_{j+1}^{(n)} - \theta_j^{(n)} < Dn^{-1}$$

with C, D ε-dependent. I have not found (7.1), but the proof depends on such well-known results in such a simple way that I'm sure it must be known!

ETNA
Kent State University
etna@mcs.kent.edu

Proof. We will depend on two classical results. The first is Darboux's formula ([40, Theorem 8.21.8]) for the large n asymptotics of $P_n^{(\alpha,\beta)}$:

$$P_n(\cos\theta) = n^{-1/2}k(\theta)^{-1/2}\cos(n\theta+\gamma(\theta)) + O(n^{-3/2}), \tag{7.2}$$

where

$$k(\theta) = \pi^{-\frac{1}{2}}\sin\left(\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\cos\left(\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}},$$
$$\gamma(\theta) = \tfrac{1}{2}(\alpha+\beta+1)\theta - \left(\alpha+\tfrac{1}{2}\right)\tfrac{\pi}{2},$$

and where the $O(n^{-3/2})$ is uniform in $\theta \in [\varepsilon, \pi-\varepsilon]$ for each fixed $\varepsilon > 0$ and fixed α, β. (7.2) is just pointwise Szegő-Jost asymptotics on $[-1,1]$ with explicit phase (k is determined by the requirement that $k(\theta)w(\cos\theta)d(\cos\theta)$ must be a multiple of $d\theta$).

The second formula we need ([31, Eqn. (13.8.4)]) is

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(1+\alpha+\beta+n)P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{7.3}$$

which is a simple consequence of the Rodrigues formula.

Fix θ_0. Define $\theta^{(n)}$ by

$$\theta^{(n)} = \frac{2\pi}{n}\left[\frac{n\theta_0}{2\pi}\right],$$

where $[y]$ = integral part of y. Then $\theta^{(n)} \le \theta_0 < \theta^{(n)} + \frac{2\pi}{n}$ and $n\theta^{(n)} \in 2\pi\mathbb{Z}$. Define

$$f_n(y) = n^{1/2}P_n^{(\alpha,\beta)}\left(\cos\left(\theta^{(n)} + \frac{y}{n}\right)\right).$$

Then (7.2) implies that uniformly in $\theta_0 \in [\varepsilon, \pi-\varepsilon]$ and $y \in [-Y, Y]$ (any $\varepsilon > 0$, $Y < \infty$), we have as $n \to \infty$,

$$f_n(y) \to k(\theta_0)^{-1/2}\cos(y+\gamma(\theta_0)).$$

Moreover, by (7.3), $f_n'(y)$ converges uniformly to a continuous limit. Standard functional analysis (essentially the fundamental theorem of calculus!) says that if $f_n \to f_\infty$ and $f_n' \to g_\infty$ uniformly for C^1 functions f_n, then $g_\infty = f_\infty'$. Thus

$$f_n'(y) \to -k(\theta_0)^{-1/2}\sin(y+\gamma(\theta_0))$$

In particular, since $k(\theta_0)$ is bounded above and below on $[-\varepsilon, \pi-\varepsilon]$, we see that for $\alpha, \beta, \varepsilon, Y$ fixed, there are $C_1, C_2 > 0$ and N, so for all $\theta_0 \in [\varepsilon, \pi-\varepsilon]$, $|y_j| < Y$ for $j = 1, 2$, and $n > N$, we have

$$|y_1 - y_2| < \pi, \quad |f_n(y_j)| < C_1 \Rightarrow |f_n(y_1) - f_n(y_2)| \ge C_2(y_1 - y_2). \tag{7.4}$$

In the usual way (7.4) implies that for n large, there is at most one solution of $f_n(y) = 0$ within π of another solution. Since (7.2) implies existence, we can pinpoint the zeros of P_n in $[\varepsilon, \pi-\varepsilon]$ as single points near $\{\frac{\pi}{2n} + \frac{j\pi}{n} - \gamma(\frac{\pi}{2n} + \frac{j\pi}{n}) + o(\frac{1}{n})\}$. From this, (7.1) follows. ∎

8. Asymptotics Away From the Critical Region. This is the first of several sections which focus on proving Theorem 4.4. The key will be asymptotics of $\varphi_n(z)$ in the region near $|z| = b$. In this section, for background, we discuss asymptotics away from $|z| = b$. We start with $|z| > b$. The first part of the following is a translation of results of Nevai-Totik [28] from asymptotics of φ_n^* to φ_n:

THEOREM 8.1. *If* (1.5) *holds for* $0 < b < 1$, *then* D *is analytic in* $\{z \mid |z| < b^{-1}\}$ *and for* $|z| > b$,

$$\lim_{n\to\infty} z^{-n}\varphi_n(z) = \overline{D(1/\bar{z})}^{-1}, \tag{8.1}$$

and (8.1) *holds uniformly in any region* $|z| \geq b + \varepsilon$ *with* $\varepsilon > 0$. *Indeed, on any region* $\{z \mid b + \varepsilon < |z| \leq 1\}$,

$$|\varphi_n(z) - \overline{D(1/\bar{z})}^{-1} z^n| \leq C_\varepsilon \left(b + \frac{\varepsilon}{2}\right)^n. \tag{8.2}$$

REMARK 8.2. *The point of* (8.2) *is that the error in* (8.1) *is approximately* $O(b^n/|z|^n)$, *which is exponentially small if* $|z| > b + \varepsilon$. *It is remarkable that we get exponentially small errors with only* (1.5).

Proof. By step (2) in the proof of Theorem 1.1,

$$|\varphi_{n+1}^*(z) - \varphi_n^*(z)| \leq \tilde{C}_\varepsilon [\max(1, |z|)]^n \left|b + \frac{\varepsilon}{2}\right|^n. \tag{8.3}$$

As noted there, this implies (1.8) which, using

$$\varphi_n(z) = z^n\, \overline{\varphi_n^*(1/\bar{z})} \tag{8.4}$$

implies (8.1). The bound (8.3) then implies

$$|\varphi_n^*(z) - D(z)^{-1}| \leq C_\varepsilon [\max(1, |z|)]^n \left|b + \frac{\varepsilon}{2}\right|^n$$

if $|z|(b + \frac{\varepsilon}{3}) < 1$, and this yields (8.2) after using (8.4). ∎

REMARK 8.3. *The restriction* $|z| \leq 1$ *for* (8.2) *comes from* $|z| \leq 1$ *in* (1.7). *But, by Theorem 8.1,* (1.7) *holds if* $|z| > b$, *and so we can conclude* (8.2) *in any region* $\{z \mid b + \varepsilon < |z| < b^{-1} - \varepsilon\}$. *By the maximum principle, we have that*

$$\sup_{|z| \leq b+\varepsilon} |b + \varepsilon|^{-n} |\Phi_n(z)| < \infty,$$

which, plugged into the machine in (8.2), *implies for* $|z| > b^{-1} - \varepsilon$, *we have*

$$|\varphi_n(z) - \overline{D(1/\bar{z})}^{-1} z^n| \leq C_\varepsilon |z|^n \left(b + \frac{\varepsilon}{2}\right)^n,$$

which is exponentially small compared to $|z|^n$.

Barrios-López-Saff [4] proved that ratio asymptotics (1.10) implies that $\varphi_{n+1}(z)/\varphi_n(z) \to b$ for $|z| < b$, thereby also proving there are no zeros of φ_n in each disk $\{z \mid |z| < b - \varepsilon\}$ if n is large (see also [37, Section 9.1]). Here we will get a stronger result from a stronger hypothesis:

THEOREM 8.4. *Suppose that*

$$b^{-n}\alpha_n \to C \neq 0$$

as $n \to \infty$. *Then for any* $|z| < b$,

$$b^{-n}\varphi_n(z) \to \bar{C}(z-b)^{-1}D(z)^{-1}. \tag{8.5}$$

Moreover, if BLS asymptotics (1.11) *holds, then in each region* $\{z \mid |z| < b - \varepsilon\}$,

$$|b^{-n}\varphi_n(z) - \bar{C}(z-b)^{-1}D(z)^{-1}| \le C_1\tilde{\Delta}^n, \tag{8.6}$$

for some $\tilde{\Delta} < 1$.

REMARK 8.5. *In [34], we will prove a variant of this result that only needs ratio asymptotics as an assumption.*

Proof. Define

$$u_n(z) = \varphi_n(z)b^{-n} \qquad A_n = -\bar{\alpha}_n b^{-n-1}.$$

Then, Szegő recursion says

$$u_{n+1} = \left(\frac{z}{b}\right)u_n + A_n\varphi_n^*(z). \tag{8.7}$$

Iterating, we see that

$$u_n = \sum_{j=1}^{n} A_{n-j}\varphi_{n-j}^*(z)\left(\frac{z}{b}\right)^{j-1} + \left(\frac{z}{b}\right)^n u_0. \tag{8.8}$$

Since $A_m \to -\bar{C}b^{-1}$, $\varphi_m^* \to D^{-1}$, and $|z|/b < 1$, (8.8) implies u_n has a limit u_∞. (8.7) then implies

$$bu_\infty = zu_\infty - \bar{C}D(z)^{-1},$$

which implies (8.5).

If (1.11) holds, then

$$A_n - A_\infty = O(\Delta^n). \tag{8.9}$$

Moreover, Szegő recursion for Φ_n^* implies if $|z| < 1$,

$$|\Phi_{n+1}^* - \Phi_n^*| \le C_1 b^n |\rho_n^{-1} - 1|,$$

and then since $|\rho_n^{-1} - 1| \le C_1|\alpha_n|$ if $|\alpha_n| < \frac{1}{4}$, $|\varphi_{n+1}^* - \varphi_n^*| \le C_2 b^n$, and so

$$|z| \le 1 \Rightarrow |\varphi_n^*(z) - D(z)^{-1}| \le C_3 b^n, \tag{8.10}$$

(8.8), (8.9), and (8.10) imply (8.6) with $\tilde{\Delta} = \max(\Delta, b)$. ◻

9. Asymptotics in the Critical Region. The key result in controlling the zeros when the BLS condition holds is

THEOREM 9.1. *Let the BLS condition* (1.11) *hold for a sequence,* $\{\alpha_n\}_{n=0}^\infty$*, of Verblunsky coefficients and some* $b \in (0,1)$ *and* $C \in \mathbb{C}$. *Then there exist* D, Δ_1, *and* Δ_2 *with* $0 < \Delta_1 < \Delta_2 < 1$ *so that if*

$$b\Delta_2 < |z| < b\Delta_2^{-1}, \tag{9.1}$$

then

$$|\varphi_n(z) - \overline{D(1/\bar{z})}^{-1} z^n - \bar{C}(z-b)^{-1} D(z)^{-1} b^n| \leq D(b\Delta_1)^n. \tag{9.2}$$

REMARK 9.2. *1. Implicit in* (9.2) *is that* $\overline{D(1/\bar{z})}^{-1}$ *has an analytic continuation (except at* $z = b$*; see Remark 2 in the following) to the region* (9.1)*, that is,* $D(z)^{-1}$ *has an analytic continuation to* $\{z \mid |z| < b^{-1}\Delta_2^{-1}\}$ *except for* $z = b^{-1}$.

2. Since $\varphi_n(z)$ *is analytic at* $z = b$ *and* $D(b)^{-1} \neq 0$*, the poles in* $\overline{D(1/\bar{z})}^{-1}$ *and* $D(z)/(z-b)^{-1}$ *must cancel, that is,* (1.4) *must hold.*

3. In this way, Theorem 9.1 includes a new proof of one direction of Theorem 1.2, that is, that the BLS condition implies that $D(z)^{-1}$ *is meromorphic in* $\{z \mid |z| < b^{-1}\Delta_2^{-1}\}$ *with a pole only at* $z = b^{-1}$ *with* (1.4).

4. The condition $\Delta_1 < \Delta_2$ *implies that the error* $O((b\Delta_1)^n)$ *is exponentially smaller than both* z^n *and* b^n *in the region where* (9.1) *holds.*

We will prove (9.2) by considering the second-order equation obeyed by φ_n for n so large that $\alpha_{n-1} \neq 0$ (see [36, Eqn. (1.5.47)]):

$$\varphi_{n+1} = \rho_n^{-1}\left(z + \frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}}\right)\varphi_n - \frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}} \frac{\rho_{n-1}}{\rho_n} z\varphi_{n-1} \tag{9.3}$$

(the only other applications I know of this formula are in [4] and Mazel et al. [23]). By (1.11) which implies $\rho_n = 1 + O(b^{2n})$ and $\bar{\alpha}_n/\bar{\alpha}_{n-1} = b + O(\Delta^n)$, we have

$$\rho_n^{-1}\left(z + \frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}}\right) = z + b + O(b^{2n} + \Delta^n), \tag{9.4}$$

$$\left(\frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}}\right)\frac{\rho_{n-1}}{\rho_n} z = bz + O(b^{2n} + \Delta^n). \tag{9.5}$$

In [34], we will analyze this critical region by an alternate method that, instead of analyzing (9.3) as a second-order homogeneous difference equations, analyzes the more usual Szegő recursion as a first-order inhomogeneous equation.

We thus study the pair of difference equations:

$$u_{n+1} = \rho_n^{-1}\left(z + \frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}}\right)u_n - \frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}} \frac{\rho_{n-1}}{\rho_n} z u_{n-1} \tag{9.6}$$

and

$$u_{n+1}^{(0)} = (z+b)u_n^{(0)} - bz u_{n-1}^{(0)}. \tag{9.7}$$

expanding solutions of u_n in terms of solutions of $u_n^{(0)}$.

Two solutions of (9.7) are b^n and z^n (since $x^2 - (z+b)x + bz$ is solved by $x = b, z$). These are linearly independent if $b \neq z$ but not at $b = z$, so it is better to define

$$x_n = z^n \qquad y_n = \frac{z^n - b^n}{z - b}$$

with y_n interpreted as nb^{n-1} at $z = b$.

ETNA
Kent State University
etna@mcs.kent.edu

We rewrite (9.7) as

$$\begin{pmatrix} u_{n+1}^{(0)} \\ u_n^{(0)} \end{pmatrix} = M^{(0)} \begin{pmatrix} u_n^{(0)} \\ u_{n-1}^{(0)} \end{pmatrix},$$

with

$$M^{(0)} = \begin{pmatrix} z+b & -bz \\ 1 & 0 \end{pmatrix},$$

and (9.6) as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = (M^{(0)} + \delta M_n) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \tag{9.8}$$

where δM_n is affine in z and, by (9.4)/(9.5), obeys

$$\|\delta M_n\| \le C(1+|z|)(b^{2n} + \Delta^n). \tag{9.9}$$

Now we use variation of parameters, that is, we define c_n, d_n by

$$u_n = c_n x^n + d_n y^n, \tag{9.10}$$

$$u_{n-1} = c_n x^{n-1} + d_n y^{n-1}, \tag{9.11}$$

or $(u_n u_{n-1})^t = Q_n (c_n d_n)^t$ where

$$Q_n = \begin{pmatrix} x^n & y^n \\ x^{n-1} & y^{n-1} \end{pmatrix}. \tag{9.12}$$

Since $\det(Q_n) = -z^{n-1} b^{n-1}$, we have

$$Q_{n+1}^{-1} = -z^{-n} b^{-n} \begin{pmatrix} y^n & -y^{n+1} \\ -x^n & x^{n+1} \end{pmatrix}. \tag{9.13}$$

Since x_n and y_n solve (9.7),

$$Q_{n+1}^{-1} M^{(0)} Q_n = \mathbf{1}.$$

Moreover, since

$$|x_n| \le |z|^n \qquad |y_n| \le n \ \ \max(|z|, |b|)^n,$$

(9.9), (9.12), and (9.13) imply that

$$\delta \widetilde{M}_n \equiv Q_{n+1}^{-1} \delta M_n Q_n$$

obeys

$$\|\delta \widetilde{M}_n\| \le C_n \left[\frac{\max(|z|, |b|)}{\min(|z|, |b|)} \right]^n (1+|z|)(b^{2n} + \Delta^n).$$

In particular, in the region (9.1),

$$\|\delta \widetilde{M}_n\| \le C \Delta_1^{2n},$$

if we take $\Delta_1 = \Delta_2^2$ and $\Delta_2 < 1$ is picked so that $\max(b^2, \Delta) < \Delta_2$. Since, in the region (9.1),

$$|z^n| \leq |b|^n \Delta_2^{-n},$$

we have that

$$n[|z|^n + |b|^n] \| \delta \widetilde{M}_n \| \leq C(b\Delta_1)^n. \tag{9.14}$$

We thus have the tools to prove the main input needed for Theorem 9.1:

PROPOSITION 9.3. *There exist $0 < \Delta_1 < \Delta_2 < 1$ and N so that for z in the region (9.1), there are two solutions $u_n^+(z)$ and $u_n^-(z)$ of (9.6) for $n \geq N$ with*
(i)

$$|u_n^+ - x^n| - |u_n^- - y^n| \leq D_1(b\Delta_1)^n. \tag{9.15}$$

(ii) *$u_n^\pm(z)$ are analytic in the region (9.1).*
(iii) *$(u_{n+1}^\pm, u_n^\pm)^t$ are independent for $^+$ and $^-$ for $n \geq N$.*

Proof. If u_n is related to c_n, d_n by (9.10)/(9.11), then (9.6) is equivalent to (9.8) and then to

$$\begin{pmatrix} c_{n+1} \\ d_{n+1} \end{pmatrix} = (1 + \delta \widetilde{M}_n) \begin{pmatrix} c_n \\ d_n \end{pmatrix}.$$

By (9.14) and the fact that $\delta \widetilde{M}_n$ is analytic in the region (9.2), we see

$$L_n(z) = \prod_{j=n}^{\infty} (1 + \delta \widetilde{M}_n(z))$$

exists, is analytic in z (in the region (9.1)), and invertible for all z in the region and $n \geq N$ sufficiently large.

Define

$$\begin{pmatrix} c_n^+ \\ d_n^+ \end{pmatrix} = L_n(z)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and c_n^-, d_n^- by the same formula with $\binom{1}{0}$ replaced by $\binom{0}{1}$ and then $u_n^\pm$ by (9.10). Analyticity of u is immediate from the analyticity of $L_n(z)$ and (9.15) follows from (9.14).

Independence follows from the invertibility of $L_n(z)^{-1}$. □

Proof. [Proof of Theorem 9.1] By the independence, we can write

$$(\varphi_{N+1}(z), \varphi_N(z)) = f_1(z)(u_{N+1}^+(z), u_N^+(z)) + f_2(z)(u_{N+1}^-(z), u_N^-(z)),$$

where f_1, f_2 are analytic in the region (9.1) since φ and $u^\pm$ are. By the fact that $\varphi, u^\pm$ obey (9.6), we have for all $n \geq N$ that

$$\varphi_n(z) = f_1(z) u_n^+(z) + f_2(z) u_n^-(z). \tag{9.16}$$

Suppose first $|z| < b$ in the region (9.1). Then, since $b^{-n}|z|^n \to 0$, (9.15) implies that as $n \to \infty$,

$$b^{-n} u_n^+ \to 0 \qquad b^{-n} u_n^- \to -\frac{1}{z-b}.$$

ETNA
Kent State University
etna@mcs.kent.edu

We conclude, by (8.5), that in that region,

$$f_2(z) = -\bar{C} D(z)^{-1},$$

so, by analyticity, this holds in all of the region (9.1).

Next, suppose $|z| > b$, so $|z|^{-n} b^n \to 0$. (9.15) implies that as $n \to \infty$,

$$z^{-n} u_n^+ \to 1 \qquad z^{-n} u_n^- \to \frac{1}{z-b}.$$

By (8.1), we conclude that

$$f_1(z) + f_2(z)(z-b)^{-1} = \overline{D(1/\bar{z})}^{-1}. \tag{9.17}$$

Again, by analyticity, this holds in all of the region (9.1) except $z = b$. It follows that $D(z)^{-1}$ has a pole at b^{-1} and otherwise is analytic in $\{z \mid |z| < b^{-1}\Delta_2^{-1}\}$. (9.17) also determines the residue to be given by (1.12).

Equation (9.16) becomes

$$\varphi_n(z) = [\overline{D(1/\bar{z})}^{-1} + \bar{C}(z-b)^{-1} D(z)^{-1}] u_n^+(z) - \bar{C} D(z)^{-1} u_n^-(z).$$

Then (9.2) follows from this result, boundedness of f_1, f_2, and (9.15). ☐

10. Asymptotics of the Nevai-Totik Zeros. In this section, we prove Theorems 4.6 and 4.8. They will be simple consequences of Theorems 8.1 and 9.1.

Proof. [Proof of Theorem 4.6] If $\overline{D(1/\bar{z})}^{-1}$ has a zero of order exactly k, at z_0 there is some $C_1 > 0$ and $\delta > 0$ so that

$$|z - z_0| < \delta \Rightarrow |D(1/\bar{z})| \geq C_1 |z - z_0|^k. \tag{10.1}$$

For n large, Hurwitz's theorem implies $|z_n - z_0| < \delta$. By (8.2) and (10.1), we have

$$C_1 |z_n - z_0|^k \leq C_d z_n^{-n} \left(b + \frac{d}{2}\right)^n$$

for small d and n large. Picking d so $z_0^{-1}(b + \frac{d}{2}) < 1$, we see

$$|z_n - z_0|^k \leq C_2 e^{-2k\varepsilon n},$$

for some $\varepsilon > 0$. This implies (4.9) for n large. ☐

Proof. [Proof of Theorem 4.8] Pick Δ_2 to be given by Proposition 9.3. Define $Q(z)$ near z_0 by

$$(z_n - z_0)^k Q(z) = \overline{D(1/\bar{z})}^{-1},$$

so $Q(z_0) \neq 0$ and Q is analytic near z_0. $\varphi_n(z_n) = 0$ and (9.2) implies

$$(z_n - z_0)^k Q(z_n) = C(z_n - b)^{-1} D(z_n)^{-1} \frac{b^n}{z_n^n} + O\left(\frac{(b\Delta_1)^n}{z_n^n}\right). \tag{10.2}$$

By Theorem 4.6, $z_n - z_0 = O(e^{-\varepsilon n})$ so (10.2) becomes

$$(z_n - z_0)^k = C_1 \frac{b^n}{z_0^n} + O\left(\frac{(b\Delta_1)^n}{z_0^n}\right) + O\left(\frac{b^n}{z_0^n} e^{-\varepsilon n}\right) \tag{10.3}$$

since $\bar{C}(z_n - b)^{-1} D(z_n)^{-1} Q(z_n)^{-1}$ can be replaced by its value at z_0 plus an $O(e^{-\varepsilon n})$ error. (10.3) implies (4.10). ☐

11. Zeros Near Regular Points. We call a point z with $|z| = b$ singular if either $z = b$ or $D(1/\bar{z})^{-1} = 0$. Regular points are all not singular points on $\{z \mid |z| = b\}$. There are at most a finite number of singular points. In this section, we will analyze zeros of $\varphi_n(z)$ near regular points. In the next section, we will analyze the neighborhood of singular points.

We will use Rouché's theorem to reduce zeros of $\varphi_n(z)$ to zeros of $(z/b)^n - g(z_0 b)$ (g defined in (11.1)) for $z - z_0 b$ small. Suppose bz_0 with $z_0 \in \partial\mathbb{D}$ is a regular point. Define

$$g(z) = \frac{\bar{C}\,\overline{D(1/\bar{z})}}{D(z)(b-z)}, \tag{11.1}$$

which is regular and nonvanishing at z_0, so

$$g(bz_0) = ae^{i\psi}$$

with $a > 0$ and $\psi \in [0, 2\pi)$. Pick $\delta < b\Delta_2^{-1}$ so that

$$|z - bz_0| < \delta \Rightarrow |g(z) - g(bz_0)| < \frac{a}{4}. \tag{11.2}$$

Define

$$h_1^{(n)}(z) = \left(\frac{z}{b}\right)^n - g(z_0 b),$$
$$h_2^{(n)}(z) = \left(\frac{z}{b}\right)^n - g(z),$$
$$h_3^{(n)}(z) = \frac{\varphi_n(z)\,\overline{D(1/\bar{z})}}{b^n}.$$

Theorem 9.1 implies that

$$|z - bz_0| < \delta \Rightarrow |h_2^{(n)}(z) - h_3^{(n)}(z)| \le D_2 \Delta_1^n. \tag{11.3}$$

THEOREM 11.1. *Let z_0 be a regular point and $\delta < b\Delta_2^{-1}$ so that (11.2) holds. Let $j_1 \le j_2$ be integers with $|j_k| < (n-1)\delta/2$. Let*

$$I_n = \left\{ z \,\middle|\, \left|\frac{|z|}{b} - 1\right| < \frac{\delta}{2b},\ \arg\left(\frac{z}{z_0}\right) \in \left(\frac{\psi}{n} + \frac{2\pi j_1}{n} - \frac{\pi}{n}, \frac{\psi}{n} + \frac{2\pi j_2}{n} + \frac{\pi}{n}\right)\right\}$$

Then for n large, I_n has exactly $(j_2 - j_1) + 1$ zeros $\{z_\ell^{(n)}\}_{\ell=1}^{j_2-j_1+1}$ of φ_n. Moreover,

(a)

$$|z_\ell^{(n)}| = b\left(1 + \frac{1}{n}\log|g(z_\ell^{(n)})| + O\left(\frac{1}{n^2}\right)\right), \tag{11.4}$$

$$= b\left(1 + \frac{1}{n}\log a + O\left(\frac{\delta}{n}\right) + O\left(\frac{1}{n^2}\right)\right). \tag{11.5}$$

(b)

$$\arg z_\ell^{(n)} = \frac{\arg g(z_k^{(n)})}{n} + \frac{2\pi\ell}{n} + O\left(\frac{1}{n^2}\right), \tag{11.6}$$

$$= \frac{\psi}{n} + \frac{2\pi\ell}{n} + O\left(\frac{\delta}{n}\right) + O\left(\frac{1}{n^2}\right). \tag{11.7}$$

ETNA
Kent State University
etna@mcs.kent.edu

(c) *If* $|j_k| \leq J$, *(11.5) and (11.7) hold without the* $O(\delta/n)$ *term.*

Proof. Note first that in I_n, since $||z|-b| \leq \frac{\delta}{2}$ and $|\arg(z/z_0)| \leq \frac{\delta}{2}$, we have $|z - z_0 b| < \delta$. Consider the boundary of the region I_n which has two arcs at $|z| = b(1 \pm \delta)$ and two straight edges are $\arg(\frac{z}{z_0}) = \frac{\psi}{n} + \frac{2\pi j_1}{n} - \frac{\pi}{n}$ and $\arg(\frac{z}{z_0}) = \frac{\psi}{n} + \frac{2\pi j_2}{n} + \frac{\pi}{n}$.

We claim on ∂I_n, we have for n large that

$$|h_j(z) - h_1(z)| \leq \tfrac{1}{2}\,|h_1(z)| \qquad j = 2, 3. \tag{11.8}$$

Consider the 4 pieces of ∂I_n:

$|z| = b(1+\delta)$. It holds that $|h_1^{(n)}(z)| \geq (1+\delta)^n - a > a/2$ for n large so, by (11.2) in this region,

$$|h_1^{(n)} - h_2^{(n)}| \leq \tfrac{a}{4} < \tfrac{1}{2}\,|h_1^{(n)}|,$$

and clearly, by (11.3) for n large,

$$|h_1^{(n)} - h_3^{(n)}| < \tfrac{1}{2}\,|h_1^{(n)}|.$$

$|z| = b(1-\delta)$. It holds that $|h_1^{(n)}(z)| \geq a - (1-\delta)^n > a/2$ for n large so, as above, (11.8) holds there.

$|\arg z - \frac{\psi}{n} - \frac{2\pi j_\ell}{n}| = \frac{\pi}{n}$. It holds that $z^n = -e^{i\psi}|z|^n$ so

$$|h_1^{(n)}(z)| = \left|\frac{z}{b}\right|^n + a > a,$$

and the argument follows the ones above to get (11.8). Thus, (11.8) holds.

By Rouché's theorem, $h_1^{(n)}, h_2^{(n)}, h_3^{(n)}$ have the same number of zeros in each I_n. By applying this to each region with $j_1 = j_2$ and noting that $h_1^{(n)}$ has exactly one zero in such a region, we see that each $h_k^{(n)}$ has $j_1 + j_2 + 1$ zeros in each I_n, and there is one each in each pie slice of angle $2\pi/n$ about angles $\psi/n + 2\pi\ell/n$.

At the zeros of $h_3^{(n)}$, we have $|h_2^{(n)}(z)| \leq D_2 \Delta_1^n$ so

$$\begin{aligned}\left|\frac{z_\ell^{(n)}}{b}\right| &= e^{\log|g(z_\ell^{(n)})|/n} + O(\Delta_2^n)\\ &= 1 + \frac{\log|g(z_\ell^{(n)})|}{n} + O\left(\frac{1}{n^2}\right),\end{aligned}$$

proving (11.4). The proof of (11.6) is similar. Since $|g(z) - g(z_0)| \leq C\delta$, we get (11.5) and (11.7). If $|j_k| \leq J$, $|g(z_\ell^{(n)}) - g(z_0)| \leq O(1/n)$ so the $O(\delta/n)$ term is not needed, proving (c). ∎

THEOREM 11.2. *In each region* I_n *of Theorem 11.1, the zeros* $z_\ell^{(n)}$ *obey*

$$|z_\ell^{(n)}| - |z_{\ell+1}^{(n)}| = O\left(\frac{1}{n^2}\right), \tag{11.9}$$

$$\left|\arg z_{\ell+1}^{(n)} - \arg z_\ell^{(n)} - \frac{2\pi}{n}\right| = O\left(\frac{1}{n^2}\right). \tag{11.10}$$

Proof. By (11.5) and (11.6), we have

$$|z_{\ell+1}^{(n)} - z_\ell^{(n)}| \leq \frac{C}{n}$$

(in fact, $C = 2\pi b + O(\delta)$). Thus

$$|g(z_{\ell+1}^{(n)}) - g(z_\ell^{(n)})| \leq \frac{C_1}{n},$$

so (11.4) and (11.6) imply (11.9) and (11.10). ∎

12. Zeros Near Singular Points. We first consider the singular point $z = b$ which is always present, and then turn to other singular points which are quite different:

THEOREM 12.1. *Let $\{\alpha_j\}_{j=0}^\infty$ be a sequence of Verblunsky coefficients obeying BLS asymptotics (1.11). Fix any positive integer, j, with $j < (n-1)\delta/2$ where $\delta < b\Delta_2^{-1}$ is picked so that*

$$|z - b| < \delta \Rightarrow |g(z) - 1| < \tfrac{1}{4}.$$

Then

$$I_n = \left\{ z \;\middle|\; \left| \frac{|z|}{b} - 1 \right| < \frac{\delta}{2b},\ |\arg(z)| < \frac{2\pi j}{n} + \frac{\pi}{n} \right\}$$

has exactly $2j$ zeros $\{z_\ell^{(n)}\}_{\ell=1}^{j} \cup \{z_\ell^{(n)}\}_{\ell=-j}^{-1}$ with

$$||z_\ell^{(n)}| - b| = O\left(\frac{\delta}{n}\right) + O\left(\frac{1}{n^2}\right), \tag{12.1}$$

$$\arg z_\ell^{(n)} = \frac{2\pi\ell}{n} + O\left(\frac{\delta}{n}\right) + O\left(\frac{1}{n^2}\right). \tag{12.2}$$

Moreover, for each fixed $\ell = \pm 1, \pm 2, \ldots,$

$$|z_\ell^{(n)} - be^{2\pi i\ell/n}| = O\left(\frac{1}{n^2}\right). \tag{12.3}$$

REMARK 12.2. *We emphasize again the zero at $z = b$ is "missing," that is, $z_\ell^{(n)}$, $|\ell| \geq 2$, has its nearest zeros in each direction a distance $b2\pi/n + O(1/n^2)$, while $z_\pm^{(n)}$ has a zero on one side at this distance, but on the other at distance $b4\pi/n + O(1/n^2)$.*

Proof. This is just the same as Theorem 11.1! $g(z)$ is regular at $z = b$. Indeed, $g(b) = 1$ since $\overline{D(1/\bar{z})}^{-1}$ and $\bar{C}D(z)/(z-b)$ have to have precisely cancelling poles. $D(1/\bar{z})$ vanishes at $z = b$, so $\varphi_n(z)\overline{D(1/\bar{z})}/b^n$ which, by the argument of Theorem 11.1, has $2j+1$ zeros in I_n, has one at $z = b$ from $D(1/\bar{z})$ and $2j$ from $\varphi_n(z)$, (12.1)–(12.3) follow as did (11.5), (11.7), and (11.9)–(11.10). ∎

By a compactness argument and Theorems 11.1 and 12.1, we have the following strong form of Theorem 4.4 when there are no singular points other than $z = b$:

THEOREM 12.3. *Let $\{\alpha_j\}_{j=0}^\infty$ be a sequence of Verblunsky coefficients obeying the BLS condition (1.11). Suppose $z = b$ is the only singular point, that is, $D(z)^{-1}$ is nonvanishing on $|z| = b^{-1}$. Let $\{w_j\}_{j=1}^J$ be the Nevai-Totik zeros and m_j their multiplicities. Let δ be such that for all $j \neq k$, $|w_j - w_k| > 2\delta$ and $|w_j| > b + 2\delta$, and let $M = \sum_{j=1}^J m_j$ the total multiplicity of the NT zeros. Let $C > \sup_{|z|=b} |\log(|g(z)|)|$. Then*

ETNA
Kent State University
etna@mcs.kent.edu

(1) *For some large N and all $n > N$, the zeros of $\varphi_n(z)$ are m_j zeros in $\{z \mid |z - w_j| < \delta\}$ for $j = 1, \dots, J$ and $n - M$ in the annulus*

$$\big|\, |z| - |b| \,\big| < \frac{C}{n}.$$

(2) *The $n-M$ zeros in the annulus can be labelled by increasing arguments $0 < \arg(z_1^{(n)}) < \cdots < \arg(z_{n-J}^{(n)}) < 2\pi$, and with $\arg(z_{n-J+1}^{(n)}) = \arg(z_1^{(n)}) + 2\pi$ and $|z_{n-J+1}^{(n)}| = |z_1^{(n)}|$, we have*

$$\arg(z_{k+1}^{(n)}) - \arg(z_k^{(n)}) = \frac{2\pi}{n} + O\left(\frac{1}{n^2}\right)$$

and

$$\frac{|z_{k+1}^{(n)}|}{|z_k^{(n)}|} = 1 + O\left(\frac{1}{n^2}\right)$$

uniformly in $k = 1, 2, \dots, n - J + 1$.

REMARK 12.4. *1. By [23], if $\alpha_n = -b^n$, all zeros have $|z| = b$, so it can happen that there are no $O(1/n)$ terms in $|z_k^{(n)}|$. However, since there are n zeros in $(\frac{2\pi}{n}, \frac{n-1}{n}2\pi)$ (i.e., the zero near $z = b$ is missing), there are always either NT zeros or $O(1/n^2)$ corrections in some $\arg(z_n^{(\ell)})$.*

2. While it can happen that there is no $O(1/n)$ term, its absence implies strong restrictions on α_n. For

$$|g(z)| = 1 \quad on \quad |z| = b \tag{12.4}$$

implies $\overline{g(b/\bar{z})}\, g(z) = 1$ near $|z| = b$, and that equation allows analytic continuations of g, and so of D or D^{-1}. In fact, (12.4) implies that $D(z)^{-1}$ is analytic in $\{z \mid |z| < b^{-3}\}$ except for a pole at $z = b^{-1}$ and that implies, by Theorem 7.2.1 of [36], that $\alpha_n = -Cb^n + O(b^{2n})(1-\varepsilon)^n$ for all ε. Thus, if the BLS condition holds but $\liminf |\alpha_n - Cb^n|^{1/n} > b^2$, then there must be $O(1/n)$ corrections to $|z_k^{(n)}| = b$. Note that for the Roger's Szegő polynomials, the poles of $D(z)^{-1}$ are precisely at $z \in \{b^{-2k-1}\}_{k=0}^{\infty}$ (see [36, Eqn. (1.6.59)]), consistent with the $|z| < b^{-3}$ statement above.

We now turn to an analysis of the other singular points. As a warmup, we study zeros of

$$f_n(z) = z^n - K(1 - z)^k, \tag{12.5}$$

where

$$K = ae^{i\psi}$$

is nonzero, $a > 0$, k a fixed positive integer, and we take $n \to \infty$.

We begin by localizing $|z|$ and $|z - 1|$.

PROPOSITION 12.5.

(i) *There are $M > 0$ and N_0 so that if $|z| \geq 1 + M/n$ and $n \geq N_0$, then $f_n(z) \neq 0$.*

(ii) *There is N_1 so if $n \geq N_1$ and*

$$|z| \leq 1 - 2k\,\frac{\log n}{n}, \tag{12.6}$$

then $f_n(z) \neq 0$.

(iii) *There is* N_2 *so if* $n \geq N_2$ *and*

$$|z-1| \leq \frac{k}{2}\frac{\log n}{n}, \tag{12.7}$$

then $f_n(z) \neq 0$.

REMARK 12.6. *If one proceeds formally and lets* $y = 1-z$*, then* $f_n(z) = 0$ *is equivalent to* $(1-y)^n = Ky^k$ *and finds*

$$\begin{aligned} y &= \frac{k}{n}(-\log y) + O(y^2) + \frac{\log a}{n} + \frac{i\psi}{n} \\ &= \frac{k}{n}\log n - \frac{k}{n}\log k - \frac{k}{n}\log(\log n) + O\left(\left(\frac{\log n}{n}\right)^2\right). \end{aligned}$$

It was this formal calculation that caused us to pick $2k\frac{\log n}{n}$ *in* (12.6) *and* $\frac{k}{2}\frac{\log n}{n}$ *in* (12.7).

Proof. (i) For n sufficiently large and $|z| \geq 2$, $|z|^n \geq a(1+|z|)^k$ and for n large, $(1+\frac{M}{n})^n \geq e^{M/2} > a2^k$ if M is suitable, so for such M and n, $|z|^n \geq a(1+|z|)^k$ for $1+\frac{M}{n} \leq |z| \leq 2$.

(ii) If (12.6) holds,

$$|1-z|^k \geq (1-|z|)^k \geq (2k)^k\left(\frac{\log n}{n}\right)^k,$$

while $\log(1+x) \leq x$ implies

$$\begin{aligned} |z|^n &= \exp(n\log(1+(|z|-1))) \\ &\leq \exp(-n(1-|z|)) \\ &\leq \exp(-2k\log n) = n^{-2k}, \end{aligned}$$

so for n large, $a|1-z|^k > |z|^n$.

(iii) If (12.7) holds, then

$$|z| \geq 1 - |z-1| \geq 1 - \frac{k}{2}\frac{\log n}{n},$$

so for n large,

$$\begin{aligned} |z|^n &\geq \exp\left(\frac{3}{4}k\log n\right) \\ &= n^{-\frac{3}{4}k} \\ &> \left(\frac{k}{2}\frac{\log n}{n}\right)^k \\ &\geq |1-z|^k, \end{aligned}$$

so $f_n(z) \neq 0$. ∎

With this, we are able to control ratios of lengths of nearby zeros.

PROPOSITION 12.7.

(i) *If* z, w *are two zeros of* $f_n(z)$ *and*

$$\left|\arg\left(\frac{z}{w}\right)\right| \leq \frac{C_0\log n}{n}, \tag{12.8}$$

then, for n large,

$$\left| \left| \frac{z}{w} \right| - 1 \right| \leq \frac{D_0}{n}. \tag{12.9}$$

(ii) *If z, w are two zeros of $f_n(z)$ and*

$$\left| \arg\left(\frac{z}{w} \right) \right| \leq \frac{C_1}{n}, \tag{12.10}$$

then

$$\left| \left| \frac{z}{w} \right| - 1 \right| \leq \frac{D_1}{n \log n}. \tag{12.11}$$

REMARK 12.8. *In both cases, D_j is a function of C_j, K, and k.*

Proof. Since z and w are both zeros,

$$\left(\frac{z}{w} \right)^n = \frac{(z-1)^k}{(w-1)^k},$$

so, by (iii) of the last proposition, $|w-1| \geq \frac{k}{2} \frac{\log n}{n}$. Thus

$$\begin{aligned} \left| \frac{z}{w} \right| &\leq \left(1 + \frac{|z-w|}{|w-1|} \right)^{k/n} \\ &\leq \left(1 + \frac{2n|z-w|}{\log n} \right)^{k/n}. \end{aligned}$$

If (12.8) holds, $|\frac{z}{w}| \leq (1 + \frac{2C_0}{n})^{k/n} \leq 1 + \frac{\tilde{D}_0}{n}$ for n large. Interchanging z and w yields (12.9). The argument from (12.10) to (12.11) is identical. ∎

As the final step in studying zeros of (12.5), we analyze arguments of nearby zeros.

PROPOSITION 12.9. *Let z_0 be a zero of $f_n(z)$. Fix C_2. Then*

(i) *If w is also a zero and $|w - z_0| \leq C_2/n$, then for some $\ell \in \mathbb{Z}$,*

$$\arg\left(\frac{w}{z} \right) = \frac{2\pi\ell}{n} + O\left(\frac{1}{n \log n} \right) \tag{12.12}$$

with the size of the error controlled by a C_2, K, and k.

(ii) *For each L and n large, there exists exactly one zero obeying (12.12) for each $\ell = 0, \pm 1, \pm 2, \dots, \pm L$.*

(iii) *For any $\psi_0 \in [0, 2\pi)$ and δ, there is N so for $n \geq N$, there is a zero of f_n with $\arg z \in (\psi_0 - \frac{\pi+\delta}{n}, \psi_0 + \frac{\pi+\delta}{n})$.*

REMARK 12.10. *1. These propositions imply that the two nearest zeros to z_0 are $z_0 e^{\pm 2\pi i/n} + O(1/n \log n)$.*

2. If k is odd, the argument of $(z-1)^k$ changes by πk as z moves through 1 along the circle. Thus there are $O(1)$ shifts as zeros swing around the forbidden circle $|z - 1| \sim \frac{k}{n} \log n$. Since there are $\log n$ zeros near that circle, the phases really do slip by $O(1/n \log n)$. This is also why we do not try to specify exact phases, only relative phases.

Proof. Since $z_0^n = K(z_0 - 1)^k$, we have

$$\frac{f(z)}{z_0^n} = \left(\frac{z}{z_0} \right)^n - \frac{(z-1)^k}{(z_0-1)^k}.$$

If $|z - z_0| \leq C_2/n$, then since $|z_0 - 1| \geq \frac{1}{2} K \frac{\log n}{n}$ (by Proposition 12.5(iii)),

$$\left| \frac{(z-1)^k}{(z_0 - 1)^k} - 1 \right| \leq \frac{D|z - z_0|}{|z_0 - 1|} \leq \frac{D_2}{\log n}. \tag{12.13}$$

Fix η and any z_0 with $|z_0 - 1| \geq \frac{1}{2} K \frac{\log n}{n}$, we want to look at solutions of

$$\left(\frac{z}{z_0}\right)^n = e^{i\eta} \frac{(z-1)^k}{(z_0 - 1)^k} \tag{12.14}$$

with $|z - z_0| \leq C_2/n$. If $f(z_0) = 0$ and $\eta = 0$, we have solutions of (12.14) are exactly zeros of f.

By (12.13), (12.14) implies

$$n(\arg(z) - \arg(z_0)) = \eta + O\left(\frac{1}{\log n}\right),$$

whose solutions are precisely

$$\arg\left(\frac{z}{z_0}\right) = \frac{2\pi\ell}{n} + \eta + O\left(\frac{1}{\log n}\right), \tag{12.15}$$

which is (12.12) when $\eta = 0$. If we prove that for n large, there is exactly one solution of (12.15) for $\ell = 0, \pm 1, \pm 2, \dots, \pm L$, we have proven (ii). We also have (iii). For given ψ_0, let

$$h(r) = r^n - |K| \, |re^{i\psi_0} - 1|^k.$$

If $n > k$, $h(0) < 0$ and $h(\infty) > 0$, so there is at least one r_0 solving $h(r) = 0$. Let $z_0 = re^{i\psi_0}$ and define η by

$$z_0^n = Ke^{-i\eta}(1 - z_0)^k.$$

Solutions of $f_n(z) = 0$ are then precisely solutions of (12.14), and we have the existence statement in (iii) if we prove existence and uniqueness of solutions of (12.14).

Fix M. Consider the following contour with four parts:

$$C_1 = \left\{ z \;\middle|\; |z| = |z_0|\left(1 + \frac{M}{n}\right);\ \left(\eta - \frac{\pi}{2}\right)\frac{1}{n} \leq \arg\left(\frac{z}{z_0}\right) \leq \left(\eta + \frac{\pi}{2}\right)\frac{1}{n} \right\},$$

$$C_2 = \left\{ z \;\middle|\; \arg\left(\frac{z}{z_0}\right) = \left(\eta + \frac{\pi}{2}\right)\frac{1}{n};\ |z_0|\left(1 - \frac{M}{n}\right) \leq |z| \leq |z_0|\left(1 + \frac{M}{n}\right) \right\},$$

$$C_3 = \left\{ z \;\middle|\; z = |z_0|\left(1 - \frac{M}{n}\right);\ \left(\eta - \frac{\pi}{2}\right)\frac{1}{n} \leq \arg\left(\frac{z}{z_0}\right) \leq \left(\eta + \frac{\pi}{2}\right)\frac{M}{n} \right\},$$

$$C_4 = \left\{ z \;\middle|\; \arg\left(\frac{z}{z_0}\right) = \left(\eta - \frac{\pi}{2}\right)\frac{1}{n};\ |z_0|\left(1 - \frac{M}{n}\right) \leq |z| \leq |z_0|\left(1 + \frac{M}{n}\right) \right\},$$

($\pi/2$ can be replaced by any angle in $(0, \pi)$). For n large, $q(z) \equiv (z/z_0)^n e^{-i\eta}$ follows arbitrarily close to

$$q[C_1] \cong \left\{ w \;\middle|\; |w| = e^M;\ \arg(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\},$$

$$q[C_2] \cong \left\{ w \;\middle|\; e^{-M} \leq |w| \leq e^M;\ \arg(w) = \frac{\pi}{2} \right\},$$

ETNA
Kent State University
etna@mcs.kent.edu

$$q[C_3] \cong \left\{ w \;\middle|\; |w| = e^{-M};\ \arg(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\},$$
$$q[C_4] \cong \left\{ w \;\middle|\; e^{-M} \leq |w| \leq e^{M};\ \arg(w) = -\frac{\pi}{2} \right\},$$

which surrounds $w = 1$ once.

By (12.13), if $\tilde{q}(z) = e^{-i\eta}(z/z_0)^n - (z-1)^k/(z_0-1)^k$, then $\tilde{q}[C]$ surrounds $w = 0$ once. It follows that (12.14) has one solution inside C. This proves existence and uniqueness within C. By part (i), there are no other solutions with $|z - z_0| < C/n$. ∎

These ideas immediately imply

THEOREM 12.11. *Let $\{\alpha_j\}_{j=0}^{\infty}$ be a sequence of Verblunsky coefficients obeying BLS asymptotics (1.11). Let bz_0 be a singular point so that $\overline{D(1/\bar{z})}^{-1}$ has a zero of order k at bz_0. Then there is $\delta > 0$ with $\delta < b\Delta_2^{-1}$ and N large so that for $n \geq N_0$, all zeros of $\varphi_n(z)$ in*

$$S \equiv \left\{ z \;\middle|\; \Delta_2^{-1} < \left|\frac{z}{b}\right| < \Delta_2;\ \left|\arg\left(\frac{z}{z_0}\right)\right| < \delta \right\}$$

obey

$$1 - \frac{D}{n} \leq \left|\frac{z}{b}\right| \leq 1 + 2k\,\frac{\log N}{n}, \tag{12.16}$$

for some D. Moreover, for each $z_1 \in S$ and $n > N$, there is a zero, z, in S with

$$\left|\arg\left(\frac{z}{z_1}\right)\right| \leq \frac{2\pi}{n}. \tag{12.17}$$

If z_ℓ is a zero in S, then the two nearest zeros to z_ℓ obey

$$\arg\left(\frac{z_{\ell\pm1}}{z_\ell}\right) = \pm\frac{2\pi}{n} + O\left(\frac{1}{n\log n}\right),$$
$$\left|\frac{z_{\ell+1}}{z_\ell}\right| = 1 + O\left(\frac{1}{n\log n}\right),$$
$$|bz_0 - z_\ell| \geq \frac{k}{2}\,\frac{\log n}{n}. \tag{12.18}$$

REMARK 12.12. *$2k$ in (12.16) can be replaced by any number strictly bigger than k (if we take N_0 large enough). Similarly in (12.17), 2π can be any number strictly bigger than π and $k/2$ in (12.18) can be any number strictly less than k.*

Proof. This follows by combining the analysis of f_n above with the ideas used to prove Theorem 11.1, picking $h_1^{(z)} = (b/z)^n - K(z - z_0 b)^k$, $h_2^{(z)} = (b/z)^n - g(z)^{-1}$, and $h_3^{(z)} = (z-b)D(z)\varphi_n(z)/\bar{C}z^n$. ∎

By compactness of $b[\partial\mathbb{D}]$ and Theorems 11.1, 12.1, and 12.11, we get the following precise form of Theorem 4.4:

THEOREM 12.13. *Let $\{\alpha_j\}_{j=0}^{\infty}$ be a sequence of Verblunsky coefficients obeying BLS asymptotics (1.11). The total multiplicity J of Nevai-Totik zeros is finite. There exists K, $D > 0$, so that for all n large, the $n - J$ zeros not near the Nevai-Totik points can be labelled $z_j^{(n)} = |z_j^{(n)}|e^{i\theta_j^{(n)}}$, $j = 1, \ldots, n - J$, with $0 = \theta_0^{(n)} < \theta_1^{(n)} < \cdots < \theta_{n-J+1}^{(n)} = 2\pi$ so that*
(1)

$$1 - \frac{D}{n} \leq \inf_{j=1,\ldots,n-J} \frac{|z_j^{(n)}|}{b} \leq \sup_{j=1,\ldots,n-J} \frac{|z_j^{(n)}|}{b} \leq 1 + K\,\frac{\log n}{n}.$$

(2)

$$\sup_{j=0,\dots,n-J} n\left|\theta_{j+1}^{(n)}-\theta_j^{(n)}-\frac{2\pi}{n}\right| = O\left(\frac{1}{\log n}\right).$$

(3) *Uniformly in j in $0,\dots,n-J$,*

$$\frac{|z_{j+1}^{(n)}|}{|z_j|} = 1+O\left(\frac{1}{\log n}\right),$$

where $|z_0^{(n)}|$ and $|z_{n-J+1}^{(n)}|$ are symbols for b. Moreover, for L fixed, uniformly in $\ell = 1,\dots,L,n-J,n-J-1,\dots,n-J-L+1$,

$$z_\ell^{(n)} = \begin{cases} be^{2\pi i\ell/n} & \ell = 1,\dots,L \\ be^{-2\pi i(n-J-\ell+1)} & \ell = n-J,n-J-1,\dots,n-J-L+1. \end{cases}$$

13. Comments. One question the reader might have is whether singular points other than $z = b$ ever occur and whether there might be a bound on k. In fact, there are no restrictions for

PROPOSITION 13.1. *If $f(z)$ is nonvanishing and analytic in a neighborhood of $\bar{\mathbb{D}}$ and*

$$\int |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = 1, \tag{13.1}$$

then there is a measure $d\mu$ in the Szegő class with $D(z;d\mu) = f(z)$.

Proof. Just take $d\mu = |f(e^{i\theta})|^2 \frac{d\theta}{2\pi}$. □

EXAMPLE 13.2. *Pick disjoint points $z_1,\dots,z_\ell$ with $\arg(z_j) \neq 0$ and $|z_j| = b^{-1}$ and positive integers $k_1,\dots,k_\ell$. Let*

$$f(z) = c\left[\prod_{j=1}^{\ell}(z-z_j)^{-k_j}\right](z-b^{-1}),$$

where c is chosen so that (13.1) holds. Then D^{-1} has a single pole at b^{-1} and so, by Theorem 1.2, $\alpha_j(d\mu)$ obeys the BLS condition. By Proposition 13.1,

$$D(z)^{-1} = c^{-1}\frac{1}{z-b^{-1}}\prod_{j=1}^{\ell}(z-z_j)^{k_j}$$

and so has zeros at z_j of order k_j. Thus we have singular points at $1/\bar{z}_j$ of order k_j.

Let me emphasize that this procedure also lets one move Nevai-Totik zeros without changing the leading asymptotics of the Verblunsky coefficients or the error estimate.

Secondly, we want to prove that in a suitable generic sense, the only singular point is $z = b$. Fix b and Δ in $(0,1)$. Let $\mathcal{V}_{b,\Delta}$ be the space of sequences α_j in $\times\mathbb{D}$ so that for some $C \neq 0$,

$$|C| + \sum |(\alpha_j - Cb^j)\Delta^{-j}b^{-1}| \equiv \|\alpha\| < \infty. \tag{13.2}$$

Obviously, any $\alpha \in \mathcal{V}_{b,\Delta}$ obeys the BLS condition.

THEOREM 13.3. $\{\alpha \in \mathcal{V}_{b,\Delta} \mid$ *The only single point is $z = b\}$ is a dense open set of $\mathcal{V}_{b,\Delta}$.*

Proof. The proof of Theorem 7.2.1 of [36] shows that the finiteness condition of (13.2) is equivalent to $D(z)^{-1} - C(z-b^{-1})^{-1}$ lying in the Wiener class of the disk $\{z \mid |z| \leq (\Delta b)^{-1}\}$ and that $\|\cdot\|$ convergence is equivalent to Wiener convergence of $(z-b^{-1})D(z)^{-1}$.

That the set of no singular points other than $z = b$ is open follows from the fact that the just mentioned Wiener convergence implies uniform convergence, and so $D(z)^{-1}$ is nonvanishing on $\{z \mid |z| = b^{-1}\}$ is an open set.

As noted in Example 13.2, it is easy to move zeros by multiplying $D(z)^{-1}$ by $\frac{z-z_1}{z-z_2}$. This shows that any $D(z)^{-1}$ in the Wiener space is a limit of such D's nonvanishing on $\{z \mid |z| = b^{-1}\}$ and shows that the set of nonsingular α is dense. ☐

It is a worthwhile and straightforward exercise to compute the change of $\arg(g)$ along $\{z \mid |z| = b\}$ and so verify that the number of zeros on the critical circle is exactly n minus the total order of the Nevai-Totik zeros.

Notes added in proofs. 1. As I stated after Theorem 7.1, I was sure the result must be known. This is correct. Precise asymptotics of zeros of Jacobi P_n to order $O(n^{-2})$ is found in P. Vértesi (Studia Sci. Math. Hungar. **25** (1990), 401–405) for zeros away from ± 1 (there are also asymptotic results near ± 1). See also J. Szabados and P. Vértesi (*Interpolation of Functions*, World Scientific, 1990); and Vértesi (Ann. Num. Math. **4** (1997), 561–577; Acta Math. Hungar. **85** (1999), 97–130).

2. Independently and approximately simultaneous with my work on OPUC obeying the BLS condition, Martínez-Finkelshtein, McLaughlin, and Saff (in preparation) produced a study of asymptotics of OPUC obeying the BLS condition using Riemann-Hilbert methods. Their results overlap ours in Sections 8–12.

3. Last-Simon (in preparation) have a different and generally stronger approach to the question of controlling the uniqueness question, that is, only one zero of P_n near the zeros of the Jost-Szegő asymptotic limit. For example, they can prove a clock behavior away from ± 2 if (4.4) is replaced by $\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty$.

4. In connection with Theorem 4.1, Golinskii [16] proves C/n upper and lower bounds (different C's) when the measure is purely absolutely continuous and the weight is bounded and bounded away from zero, and these hypotheses follow from (4.1) and Baxter's theorem.

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