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ORTHOGONAL POLYNOMIALS AND RAMANUJAN'S q-CONTINUED FRACTIONS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We give new and simple proofs to some famous q-continued fraction identities of Ramanujan by using the theory of orthogonal polynomials.

Key words. orthogonal polynomials, continued fraction

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1. Introduction. The main purpose of this short article is to announce some alternative approaches to proving several famous q-continued fraction identities of Ramanujan. An important ingredient in our approaches is the use of the orthogonal polynomials. Indeed, several results come out so easily from the perspective of orthogonal polynomials that we cannot help but wondering if Ramanujan had some equivalence of orthogonal polynomials (in his own way).

Among the identities we can verify, here are some examples: (The entry numbers refer to those as given in Berndt's books [4, 5].)

Identity 1. (Entry 15 on page 30 of [4]) *If* $|q| < 1$, *then*

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{b^k q^{k^2}}{(aq)_k (q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{b^k q^{k(k+1)}}{(aq)_k (q)_k}} = 1 + \frac{bq}{1-aq} + \frac{bq^2}{1-aq^2} + \frac{bq^3}{1-aq^3} + \cdots.$$

This identity was proved by several authors, c.f. [4], using functional relations and the general theory of continued fractions.

Identity 2. (Entry 16 on page 31 of [4]) *For each positive integer* n, *let*

$$\mu = \mu_n(a,q) = \sum_{k=0}^{[(n+1)/2]} \frac{a^k q^{k^2} (q)_{n-k+1}}{(q)_k (q)_{n-2k+1}}$$

and

$$\nu = \nu_n(a,q) = \sum_{k=0}^{[n/2]} \frac{a^k q^{k(k+1)} (q)_{n-k}}{(q)_k (q)_{n-2k}}.$$

Then

$$\frac{\mu}{\nu} = 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^n}{1}.$$

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By introducing an auxiliary variable x, we can obtain the above identity from a familiar identity of orthogonal polynomials. Note that, by letting $n \to \infty$, we can obtain a special case of Identity 1. In fact, our approach will allow us to find a "complete" version of Identity 2 that includes Identity 1 as its limiting case.

Identity 3. (Entry 12 on page 24 of [4]) *Suppose that a, b, and q are complex numbers with $|ab| < 1$ and $|q| < 1$ or that $a = bq^{2m+1}$ for some integer m. Then*

$$\frac{(a^2q^3;q^4)_\infty (b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty (b^2q;q^4)_\infty} = \frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(q^2+1)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(q^4+1)} + \cdots.$$

In our proof of this identity, we obtained a generalization of the Roger-Ramanujan identity that involves q-continued fractions and q-products. We will give the precise statement of the result and its proof in a forthcoming paper.

Several more identities of Ramanujan involving q-continued fractions and q-products can be derived by our method. Here are some of them:

Identity 4. (Entry 11 on page 21 of [4]) *Suppose that a, b, and q are complex numbers with $|q| < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Then*

$$\begin{aligned} &\frac{(-a)_\infty (b)_\infty - (a)_\infty (-b)_\infty}{(-a)_\infty (b)_\infty + (a)_\infty (-b)_\infty} \\ &= \frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \cdots. \end{aligned}$$

Identity 5. (Entry 22 on page 50 of [5]) *For $|q| < 1$,*

$$\frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty} = \frac{1}{1} + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots.$$

The first published proof of Identity 5 was given by Selberg, c.f. [5, p. 50]. It is related to the Ramanujan-Gőllnitz-Gordon continued fraction.

Identity 6. (Entry 19 on page 46 of [5]) *For $|q| < 1$,*

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \frac{q^7}{1+q^4} - \cdots.$$

This is a difficult identity to verify. The first proof as given in [5] was given by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere in 1992 and used deep theorem of Andrews in [1]. There is a recent paper by Andrews et. al. [2] on this continued fraction. Among other things, they provided another proof of this identity using functional relations, q difference equations, and several identities including one of Ramanujan.

Identity 7. (Entry 18 on page 45 of [5]) *For $|q| < 1$,*

$$\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = \frac{1}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \cdots.$$

Some identities of Ramanujan require careful interpretation. Here is one of them to which a nice clarification has been provided by Zagier (see [5, p. 40]).

Identity 8. (Entry 13 on page 36 of [5])

$$
\begin{aligned}
&1 - \frac{qx}{1}\,+\,\frac{q^2}{1}\,-\,\frac{q^3x}{1}\,+\,\frac{q^4}{1}\,-\,\frac{q^5x}{1}\,+\cdots \\
&= \frac{q}{x}\,+\,\frac{q^4}{x}\,+\,\frac{q^8}{x}\,+\,\frac{q^{12}}{x}\,+\cdots \quad \text{nearly}
\end{aligned} \tag{1.1}
$$

In identifying the continued fraction on the left side of (1.1), the following continued fraction identity (a continued fraction found in Ramanujan's "lost notebook") is used in [5, p. 37]: Let

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1}\,+\,\frac{bq+\lambda q^2}{1}\,+\,\frac{aq^2+\lambda q^3}{1}\,+\,\frac{bq^2+\lambda q^4}{1}\,+\cdots$$

and

$$G(a,b,\lambda,q) = \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n q^{(n^2+n)/2} a^n}{(q;q)_n(-bq;q)_n}, \quad |q|<1.$$

Then

$$F(a,b,\lambda,q) = \frac{G(a,b,\lambda,q)}{G(aq,b,\lambda q,q)}.$$

Ramanujan gave three more related continued fractions. A generalization (with one more parameter) has been considered by Hirschhorn [9]:

$$\frac{1}{1-b+aq}\,+\,\frac{b+\lambda q}{1-b+aq^2}\,+\cdots\frac{b+\lambda q^n}{1-b+aq^{n+1}}\,+\cdots =: F_H(a,b,\lambda,q).$$

Bhargava and Adiga [6] gave another generalization:

$$F_{BA}(a,b,\lambda,q) = \frac{1}{1}\,+\,\frac{aq+\lambda q}{1-aq+bq}\,+\cdots\frac{aq+\lambda q^n}{1-aq+bq^{n+1}}\,+\cdots.$$

Finally, Ramanujan himself also considered:

$$F_R(a,b,\lambda,q) = \frac{1}{1+aq}\,+\,\frac{\lambda q - abq^2}{1+q(aq+b)}\,+\cdots\frac{\lambda q^n - abq^{2n}}{1+q^n(aq+b)}\,+\cdots.$$

Furthermore,

$$F_R(a,b,\lambda,q) = \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)},$$

when $|q|<1$. These results have all been treated by Bhargava and Adiga in a unified way via several canonical functional relations. We believe that the orthogonal polynomial approach derives these relations in more natural ways (so it may help us answer the question: where do these functions/relations come from?).

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2. Orthogonal Polynomials Associated with Roger-Ramanujan Continued Fraction. The original idea of our approach comes from Al-Salam and Ismail in [3]. Consider the three-term recurrence relation:

$$U_{n+1}(x;a,b) = x(1+aq^n)U_n(x;a,b) - bq^{n-1}U_{n-1}(x;a,b), \;\; n \geq 1, \tag{2.1}$$

with $0 < q < 1$, $0 < b$, $-1 < a$ and

$$U_0(x;a,b) = 1, \qquad U_1(x;a,b) = x(1+a).$$

The polynomials of the second kind $U_n^*(x;a,b)$ satisfy (2.1) with the initial conditions

$$U_0^*(x;a,b) = 0, \qquad U_1^*(x;a,b) = 1+a.$$

It is easy to verify that

$$U_n^*(x;a,b) = (1+a)U_{n-1}(x;aq,bq), \;\; n \geq 1. \tag{2.2}$$

The continued fraction associated with (2.1) is

$$\frac{1+a}{x(1+a)} - \frac{b}{x(1+aq)} - \frac{bq}{x(1+aq^2)} - \frac{bq^2}{x(1+aq^3)} - \cdots \tag{2.3}$$

whose n-th convergent is given by $\dfrac{U_n^*(x;a,b)}{U_n(x;a,b)}$. Thus, by (2.2),

$$\frac{1}{x(1+a)} - \frac{b}{x(1+aq)} - \frac{bq}{x(1+aq^2)} - \cdots - \frac{bq^{n-1}}{x(1+aq^n)} = \frac{U_n(x;aq,bq)}{U_{n+1}(x;a,b)} \tag{2.4}$$

THEOREM 2.1 ([3], Theorem 3.1). *The polynomials $\{U_n(x;a,b)\}$ have the generating function*

$$\sum_{n=0}^{\infty} U_n(x;a,b)t^n = \sum_{m=0}^{\infty} \frac{\left(\frac{bt}{xa};q\right)_m}{(xt;q)_{m+1}}(xat)^m q^{m(m-1)/2}. \tag{2.5}$$

Using the analogy with the q-Lommel polynomials, Al-Salam and Ismail [3] found that

$$U_n(x;a,b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-a;q)_{n-k}(q;q)_{n-k}x^{n-2k}(-b)^k}{(-a;q)_k(q;q)_k(q;q)_{n-2k}} q^{k(k-1)}. \tag{2.6}$$

Taking $n \to \infty$, we obtain

$$U_n(x;a,b) \approx x^n(-a;q)_\infty F\left(\frac{b}{x^2};a\right),$$

where

$$F(x;a) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(q;q)_k(-a;q)_k} q^{k(k-1)}.$$

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It follows that

$$\lim_{n\to\infty} \frac{U_n^*(x;a,b)}{U_n(x;a,b)} = \frac{1}{x} \frac{F(\frac{bq}{x^2};aq)}{F(\frac{b}{x^2};a)} \tag{2.7}$$

as long as $F(\frac{b}{x^2};a) \neq 0$. Since the quotient on the left hand side of (2.7) is the n-th convergent of the continued fraction in (2.3), we obtain

$$\frac{1+a}{x(1+a)} - \frac{b}{x(1+aq)} - \frac{bq}{x(1+aq^2)} - \frac{bq^2}{x(1+aq^3)} - \cdots = \frac{1}{x} \frac{F(\frac{bq}{x^2};aq)}{F(\frac{b}{x^2};a)} \tag{2.8}$$

whenever $F(\frac{b}{x^2};a) \neq 0$.

In [3], F was used to derive many spectral properties of the orthogonality measure. It turns out that, in our approaches, it is more convenient to consider another related function.

Set

$$G(x;c,q) = \sum_{m=0}^{\infty} \frac{(x;q)_m}{(q;q)_m} c^m q^{m(m-1)/2}.$$

Applying Darboux's method in (2.5) gives (as $n \to \infty$)

$$\begin{aligned} U_n(x;a,b) &\approx x^n \sum_{m=0}^{\infty} \frac{\left(\frac{b}{ax^2};q\right)_m}{(q;q)_m} a^m q^{m(m-1)/2} \\ &= x^n G(\frac{b}{ax^2};a,q). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{U_n^*(x;a,b)}{U_n(x;a,b)} = \frac{1+a}{x} \frac{G(\frac{b}{ax^2};aq,q)}{G(\frac{b}{ax^2};a,q)}. \tag{2.9}$$

Using (2.9) in (2.3), we obtain

$$\frac{1}{x(1+a)} - \frac{b}{x(1+aq)} - \frac{bq}{x(1+aq^2)} - \frac{bq^2}{x(1+aq^3)} - \cdots = \frac{1}{x} \frac{G(\frac{b}{ax^2};aq,q)}{G(\frac{b}{ax^2};a,q)}$$

whenever $G(\frac{b}{ax^2};a) \neq 0$.

3. Proofs. To illustrate one aspect of our method, we give proofs of Identities 1 and 2 below. The proofs of other identities require more from the theory of orthogonal polynomials and special functions; they will be given in a forthcoming paper.

3.1. Identity 1. We show that Identity 1 is a special case of (2.8). Replace a by $-a$ and b by $-bq$ in (2.3), we have

$$\frac{1}{x(1-a)}\,+\,\frac{bq}{x(1-aq)}\,+\,\frac{bq^2}{x(1-aq^2)}\,+\cdots=\frac{F(\frac{-bq^2}{x^2};-aq)}{(1-a)xF(\frac{-bq}{x^2};-a)}.$$

From this, it follows that

$$x+\frac{bq}{x(1-aq)}\,+\,\frac{bq^2}{x(1-aq^2)}\,+\,\frac{bq^3}{x(1-aq^3)}\,+\cdots$$

is equal to

$$\frac{(1-a)xF(\frac{-bq}{x^2};-a)+axF(\frac{-bq^2}{x^2};-aq)}{F(\frac{-bq^2}{x^2};-aq)}.$$

By the definition of F, it is easy to verify that

$$F(\frac{-bq^2}{x^2};-aq)=\sum_{k=0}^{\infty}\frac{b^kq^{k^2+k}}{x^{2k}(q;q)_k(aq;q)_k}$$

and

$$(1-a)xF(\frac{-bq}{x^2};-a)+axF(\frac{-bq^2}{x^2};-aq)=\sum_{k=0}^{\infty}\frac{b^kq^{k^2}}{x^{2k-1}(q;q)_k(aq;q)_k}.$$

So, we just verified a generalization of Identity 1:

$$x+\frac{bq}{x(1-aq)}\,+\,\frac{bq^2}{x(1-aq^2)}\,+\,\frac{bq^3}{x(1-aq^3)}\,+\cdots=\frac{\sum_{k=0}^{\infty}\frac{b^kq^{k^2}}{x^{2k-1}(q;q)_k(aq;q)_k}}{\sum_{k=0}^{\infty}\frac{b^kq^{k^2+k}}{x^{2k}(q;q)_k(aq;q)_k}},$$

which implies Identity 1 when we take $x=1$.

3.2. Identity 2. Just like the way we derived Identity 1, we can show that Identity 2 is a special case of a more general identity that follows from (2.4). Replacing a by $-a$ and b by $-bq$ in (2.4) we can obtain

$$x(1-a)+\frac{bq}{x(1-aq)}\,+\cdots+\,\frac{bq^n}{x(1-aq^n)}=\frac{U_{n+1}(x;-a,-bq)}{U_n(x;-aq,-bq^2)}.$$

Using the explicit formula (2.6), we have

$$U_{n+1}(x;-a,-bq)=\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\frac{(a;q)_{n+1-k}(q;q)_{n+1-k}x^{n+1-2k}b^k}{(a;q)_k(q;q)_k(q;q)_{n+1-2k}}q^{k^2}$$

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and

$$U_n(x;-aq,-bq^2)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(aq;q)_{n-k}(q;q)_{n-k}x^{n-2k}b^k}{(aq;q)_k(q;q)_k(q;q)_{n-2k}}q^{k(k+1)}.$$

Thus, the following identity holds.

$$x(1-a)+\frac{bq}{x(1-aq)}+\cdots+\frac{bq^n}{x(1-aq^n)}$$
$$=\frac{\displaystyle\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\frac{(a;q)_{n+1-k}(q;q)_{n+1-k}x^{n+1-2k}b^k}{(a;q)_k(q;q)_k(q;q)_{n+1-2k}}q^{k^2}}{\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(aq;q)_{n-k}(q;q)_{n-k}x^{n-2k}b^k}{(aq;q)_k(q;q)_k(q;q)_{n-2k}}q^{k(k+1)}}.$$

Now, we can see that Identity 2 is equivalent to the case when $x=1$ and $a=0$.

4. Further Properties of $G(x;c,q)$. The function $G(x;c;q)$ introduced above plays an important role in deriving some identities involving infinite q-product in our approach. In this section, we discuss some of its properties.

Note that

$$G(x;c,q)-xG(x;cq,q)=(1-x)G(xq;c,q). \tag{4.1}$$

$$\frac{q}{c}G(x;c/q,q)-\frac{q}{c}G(x;c,q)=(1-x)G(xq;c,q). \tag{4.2}$$

Equations (4.1) and (4.2) imply

$$xG(x;cq,q)=\left(1+\frac{q}{c}\right)G(x;c,q)-\frac{q}{c}G(x;c/q,q). \tag{4.3}$$

Iterating (4.3), we obtain

$$x^2G(x;cq^2,q)=\left\{\left(1+\frac{q}{c}\right)\left(1+\frac{1}{c}\right)-\frac{x}{c}\right\}G(x;c,q)-\frac{q}{c}\left(1+\frac{1}{c}\right)G(x;c/q,q)$$

In general,

$$x^nG(x;cq^n,q)=u_n(x;c)G(x;c,q)-v_n(x;c)G(x;c/q,q),$$

with

$$u_1(x;c)=1+\frac{q}{c},\quad v_1(x;c)=\frac{q}{c}$$

$$u_2(x;c)=\left(1+\frac{q}{c}\right)\left(1+\frac{1}{c}\right)-\frac{x}{c},\quad v_2(x;c)=\frac{q}{c}\left(1+\frac{1}{c}\right),$$

and

$$u_{n+1}(x;c)=\left(1+\frac{q}{c}\right)u_n(x;c)-xv_n(x;cq) \tag{4.4}$$

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$$v_{n+1}(x;c) = \frac{q}{c} u_n(x;cq) \tag{4.5}$$

From (4.4) and (4.5), eliminating v_n we get

$$u_{n+1}(x;c) = u_n(x;cq)\left(1+\frac{q}{c}\right) - \frac{x}{c} u_{n-1}(x;cq^2).$$

Replacing q by c/q^{n+1} and letting

$$w_n(x;c) = u_n(x;c/q^n),$$

we have

$$w_{n+1}(x;c) = w_n(x;c)\left(1+\frac{q^{n+2}}{c}\right) - \frac{xq^{n+1}}{c} w_{n-1}(x;c), \tag{4.6}$$

with

$$w_1(x;c) = u_1(x;c/q) = 1 + \frac{q^2}{c}$$

and

$$w_2(x;c) = u_2(x;c/q^2) = \left(1+\frac{q^2}{c}\right)\left(1+\frac{q^3}{c}\right) - \frac{xq^2}{c}.$$

Let

$$U_n(t) = \frac{w_n(x;c)}{x^{n/2}} \qquad \text{with } \ t = 1/\sqrt{x}.$$

Then from (4.6), we obtain

$$U_{n+1}(t) = t\left(1+\frac{q^{n+2}}{c}\right)U_n(t) - \frac{q^{n+1}}{c} U_{n-1}(t).$$

This is a special case of (2.1) when $a = b = q^2/c$.

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