Two-level additive Schwarz preconditioners for fourth-order mixed methods.  
M. R. Hanisch.

Abstract.
A two-level additive Schwarz preconditioning scheme for solving Ciarlet-Raviart,  
Hermann-Miyoshi, and Hellan-Hermann-Johnson mixed method equations for the  
biharmonic Dirichlet problem is presented. Using suitably defined mesh-dependent  
forms, a unified approach, with ties to the work of Brenner for nonconforming meth-  
ods, is provided. In particular, optimal preconditioning of a Schur complement for-  
mulation for these equations is proved on polygonal domains without slits, provided  
the overlap between subdomains is sufficiently large.

Key Words.
additive Schwarz preconditioner, mixed finite elements, biharmonic equation, do-  
main decomposition, mesh dependent norms

AMS(MOS) Subject Classifications.
65F10, 65N30, 65N55

Dual variable methods for mixed-hybrid finite element approximation of the poten-  
tial fluid flow problem in porous media.  
M. Arioli, J. Maryška, M. Rozložník, and M. Tůma.

Abstract.
Mixed-hybrid finite element discretization of Darcy’s law and the continuity equa-  
tion that describe the potential fluid flow problem in porous media leads to symmetric  
indefinite saddle-point problems. In this paper we consider solution techniques  
based on the computation of a null-space basis of the whole or of a part of the left  
lower off-diagonal block in the system matrix and on the subsequent iterative solu-  
tion of a projected system. This approach is mainly motivated by the need to solve  
a sequence of such systems with the same mesh but different material properties. A  
fundamental cycle null-space basis of the whole off-diagonal block is constructed  
using the spanning tree of an associated graph. It is shown that such a basis may be  
theoretically rather ill-conditioned. Alternatively, the orthogonal null-space basis of  
the sub-block used to enforce continuity over faces can be easily constructed. In the  
former case, the resulting projected system is symmetric positive definite and so the  
conjugate gradient method can be applied. The projected system in the latter case  
remains indefinite and the preconditioned minimal residual method (or the smoothed  
conjugate gradient method) should be used. The theoretical rate of convergence for  
both algorithms is discussed and their efficiency is compared in numerical experi-  
ments.

Key Words.
saddle-point problem, preconditioned iterative methods, sparse matrices, finite ele-  
ment method
AMS(MOS) Subject Classifications.
65F05, 65F50

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A network programming approach in solving Darcy’s equations by mixed finite-element methods. M. Arioli and G. Manzini.

Abstract.
We use the null space algorithm approach to solve the augmented systems produced by the mixed finite-element approximation of Darcy’s laws. Taking into account the properties of the graph representing the triangulation, we adapt the null space technique proposed in [M. Arioli and L. Baldini, A backward error analysis of a null space algorithm in sparse quadratic programming, SIAM J. Matrix Anal. and Applics., 23 (2001), pp. 425–442], where an iterative-direct hybrid method is described. In particular, we use network programming techniques to identify the renumbering of the triangles and the edges, which enables us to compute the null space without floating-point operations. Moreover, we extensively take advantage of the graph properties to build efficient preconditioners for the iterative algorithm. Finally, we present the results of several numerical tests.

Key Words.
augmented systems, sparse matrices, mixed finite-element, graph theory

AMS(MOS) Subject Classifications.
65F05, 65F10, 64F25, 65F50, 65G05

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Abstract.
In this article, we discuss the numerical solution of the Dirichlet problem for the real elliptic Monge-Ampère equation, in two dimensions, by an augmented Lagrangian based iterative method. To derive the above algorithm, we take advantage of a reformulation of the Monge-Ampère problem as a saddle-point one, for a well-chosen augmented Lagrangian functional over the product of appropriate primal and dual sets. The convergence of the finite element approximation and of the iterative methods described in this article still has to be proved, however, on the basis of numerical experiments reported in this article, it is safe to say that: (i) The augmented Lagrangian methodology discussed here provides a sequence converging to a solution of the Monge-Ampère problem under consideration, if such a solution exists in the space $H^2(\Omega)$. (ii) If, despite the smoothness of its data, the above problem has no solution, the above augmented Lagrangian method solves it in a least-squares sense, to be precisely defined in this article.

Key Words.
elliptic Monge-Ampère equation, augmented Lagrangian algorithms, mixed finite element approximations

AMS(MOS) Subject Classifications.
35J60, 65F10, 65N30
Regularization and stabilization of discrete saddle-point variational problems.  
*P. B. Bochev and R. B. Lehoucq.*

**Abstract.**
Our paper considers parameterized families of saddle-point systems arising in the finite element solution of PDEs. Such saddle point systems are ubiquitous in science and engineering. Our motivation is to explain how these saddle-point systems can be modified to avoid onerous stability conditions and to obtain linear systems that are amenable to iterative methods of solution. In particular, the algebraic and variational perspectives of regularization and stabilization are explained.

**Key Words.**
constrained minimization, saddle point systems, mixed finite elements, regularization, stabilization, penalty, Stokes problem, Darcy flow problem

**AMS(MOS) Subject Classifications.**

Preconditioners for saddle point linear systems with highly singular (1,1) blocks.  
*Chen Greif and Dominik Schötzau.*

**Abstract.**
We introduce a new preconditioning technique for the iterative solution of saddle point linear systems with (1,1) blocks that have a high nullity. The preconditioners are block diagonal and are based on augmentation, using symmetric positive definite weight matrices. If the nullity is equal to the number of constraints, the preconditioned matrices have precisely two distinct eigenvalues, giving rise to immediate convergence of preconditioned MINRES. Numerical examples illustrate our analytical findings.

**Key Words.**
saddle point linear systems, high nullity, augmentation, block diagonal preconditioners, Krylov subspace iterative solvers

**AMS(MOS) Subject Classifications.**
65F10

Combinatorial algorithms for computing column space bases that have sparse inverses.  
*Ali Pinar, Edmond Chow, and Alex Pothen.*

**Abstract.**
This paper presents a new combinatorial approach towards constructing a sparse, implicit basis for the null space of a sparse, under-determined matrix $A$. Our approach is to compute a column space basis of $A$ that has a sparse inverse, which could be used to represent a null space basis in implicit form. We investigate three different algorithms for computing column space bases: two greedy algorithms implemented using graph matchings, and a third, which employs a divide and conquer strategy implemented with hypergraph partitioning followed by a matching. Our results show that for many matrices from linear programming, structural analysis, and circuit simulation, it is possible to compute column space bases having sparse inverses, contrary to conventional wisdom. The hypergraph partitioning method yields sparser basis inverses and has low computational time requirements, relative to the greedy approaches. We also discuss the complexity of selecting a column
space basis when it is known that such a basis exists in block diagonal form with a
given small block size.

**Key Words.**

sparse column space basis, sparse null space basis, block angular matrix, block di-
agonal matrix, matching, hypergraph partitioning, inverse of a basis

**AMS(MOS) Subject Classifications.**

65F50, 68R10, 90C20

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Parallel fully coupled Schwarz preconditioners for saddle point problems. *Feng-Nan
Hwang and Xiao-Chuan Cai.*

**Abstract.**

We study some parallel overlapping Schwarz preconditioners for solving Stokes-like
problems arising from finite element discretization of incompressible flow problems. Most of the existing methods are based on the splitting of the velocity and pressure
variables. With the splitting, fast solution methods are often constructed using var-
ious fast Poisson solvers for one of the variables. More recently, several papers
have investigated the so-called fully coupled approaches in which the variables are
not separated. The fully coupled approach has some advantages over the variable
splitting method when solving Stokes-like equations with many variables, where a
good splitting may be hard to obtain. In this paper we systematically study the par-
allel scalability of several versions of the fully coupled Schwarz method for both
symmetric and nonsymmetric Stokes-like problems. We show numerically that the
performance of a two-level method with a multiplicative iterative coarse solver is
superior to the other variants of Schwarz preconditioners.

**Key Words.**

saddle point problem, two-level Schwarz preconditioning, fully coupled methods,
finite element, parallel processing

**AMS(MOS) Subject Classifications.**

65F10, 65N30, 65N55

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Probing methods for saddle-point problems. *Chris Siefert and Eric de Sturler.***

**Abstract.**

Several Schur complement-based preconditioners have been proposed for solving
(generalized) saddle-point problems. We consider matrices where the Schur com-
plement has rapid decay over some graph known *a priori.* This occurs for many
matrices arising from the discretization of systems of partial differential equations,
and this graph is then related to the mesh. We propose the use of probing methods to
approximate these Schur complements in preconditioners for saddle-point problems.
We demonstrate these techniques for the block-diagonal and constraint precondition-
ers proposed by [Murphy, Golub and Wathen '00], [de Sturler and Liesen '04] and
[Siefert and de Sturler '05]. However, these techniques are applicable to many other
preconditioners as well. We discuss the implementation of probing methods, and
we consider the application of those approximations in preconditioners for Navier-
Stokes problems and metal deformation problems. Finally, we study eigenvalue
clustering for the preconditioned matrices, and we present convergence and timing
results for various problem sizes. These results demonstrate the effectiveness of the
proposed preconditioners with probing-based approximate Schur complements.
Key Words.
saddle-point systems, constraint preconditioners, Krylov methods, Schur complements, probing

AMS(MOS) Subject Classifications.
65F10, 65F50, 05C15