Electronic Transactions on Numerical Analysis.
Volume 2, pp. 183-193, December 1994.
Copyright © 1994, Kent State University.
ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

CONVERGENCE OF INFINITE PRODUCTS OF MATRICES AND INNER–OUTER ITERATION SCHEMES*

RAFAEL BRU†, L. ELSNER‡, AND M. NEUMANN§

Dedicated to Wilhelm Niethammer on the occasion of his sixtieth birthday.

Abstract. We develop conditions under which a product $\prod_{i=0}^{\infty} T_i$ of matrices chosen from a possibly infinite set of matrices $\mathcal{S} = \{T_j | j \in J\}$ converges. We obtain the following conditions which are sufficient for the convergence of the product: There exists a vector norm such that all matrices in $\mathcal{S}$ are nonexpansive with respect to this norm and there exists a subsequence $\{i_k\}_{k=0}^{\infty}$ of the sequence of the nonnegative integers such that the corresponding sequence of operators $\left\{T_{i_k}\right\}_{k=0}^{\infty}$ converges to an operator which is paracontracting with respect to this norm. We deduce the continuity of the limit of the product of matrices as a function of the sequences $\{i_k\}_{k=0}^{\infty}$. But more importantly, we apply our results to the question of the convergence of inner–outer iteration schemes for solving **singular** consistent linear systems of equations, where the outer splitting is regular and the inner splitting is weak regular.

Key words. iterative methods, infinite products, contractions.

AMS subject classification. 65F10.

1. Introduction.

Given a system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n,n}$ and x and b are n-vectors, the standard iterative method for solving the system is induced by the splitting of A into

$$A \; = \; P - Q, \tag{1.2}$$

where P is a nonsingular matrix. Then, beginning with an arbitrary vector x_0, the recurrence relation

$$Px_{k+1} \; = \; Qx_k + b \tag{1.3}$$

is used to compute a sequence of iterations whose limit is hoped to be a solution to (1.1).

If A is a nonsingular matrix, often the reason for preferring an iterative method generated by the recurrence relation (1.3) over a direct method of solution is due to the convenience of solving (1.3) for the approximation x_k over direct solution of (1.1). In several instances authors have shown that, when A is nonsingular, to obtain

* Received April 21, 1994. Accepted for publication November 8, 1994. Communicated by V. Mehrmann. Corrected January 20, 1996. The original manuscript is stored in vol.2.1994/pp183-193.dir/pp183-193org.ps

† Departament de Matemàtica Aplicada, Universitat Politècnica de València, 46071 València, Spain. This research was supported by grants CICYT #TIC91–1157–C03–01 DGICYT and #PB91–0535.

‡ Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501–Bielefeld, Federal Republic of Germany. This research was supported by Sonderforschungsbereich 343. "*Diskrete Strukturen in der Mathematik*", Universität Bielefeld.

§ Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269–3009, USA. This research was supported by NSF Grants Nos. DMS–9007030 and DMS–9306357 and by Sonderforschungsbereich 343 "*Diskrete Strukturen in der Mathematik*", Universität Bielefeld.

ETNA
Kent State University
etna@mcs.kent.edu

a good approximation to the solution of (1.1), one **need not** even solve the system (1.3) exactly for each x_{k+1}. Rather, they suggest that for each $k \geq 1$, we solve the system (1.3) itself by iterations. For this purpose they split the matrix P into

$$P \;=\; F - G, \tag{1.4}$$

where the matrix F is invertible. Then, beginning with $y_0 := z_k$, p_k *inner iterations*

$$y_j \;=\; F^{-1}Gy_{j-1} + F^{-1}d, \quad d \;=\; Qz_k + b, \quad j = 1, \ldots, p_k, \tag{1.5}$$

are computed after which one resets $z_{k+1} = y_{p_k}$. The entire inner–outer iteration process can then be expressed as follows[1]:

$$\begin{aligned} z_{k+1} &= (F^{-1}G)^{p_k} z_k + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} b \\ &= \{(F^{-1}G)^{p_k} + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} Q\} z_k + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} b \\ &= T_{p_k} z_k + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} b, \end{aligned} \tag{1.6}$$

where

$$T_{p_k} \;:=\; (F^{-1}G)^{p_k} + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} Q, \quad k = 1, 2, \ldots \tag{1.7}$$

For nonsingular systems three papers which have considered the convergence of the **inner–outer** iteration scheme which we would like to mention here are Nichols [12], Lanzkron, Rose, and Szyld [9], and Frommer and Szyld [8, Theorem 4.4]. Nichols seems to be the first to have shown that if the spectral radius of both $P^{-1}Q$ and $F^{-1}G$ are smaller than 1 so that the powers of both iteration matrices converge to zero, then for sufficiently large positive integer p we have that if $p_k \geq p$, for all $k \geq 1$, the sequence $\{z_k\}$ produced by the inner–outer iterations converges to the solution to (1.1) from all initial vectors z_0. Lanzkron, Rose, and Syzld [9] show, however, that if A and P are monotone matrices (that is, both have a nonnegative inverse) and both iteration matrices $P^{-1}Q$ and $F^{-1}G$ are nonnegative matrices, with the former induced by a **regular splitting** of A and the latter induced by a **weak regular splitting** of P, then the sequence $\{z_k\}$ converges to the solution of (1.1) whenever $p_k = p$ for all $k \geq 1$ with no restrictions on p. This means that very crude approximations z_k for x_k at each stage of the solution of (1.3) will suffice for the convergence of the inner–outer iteration process. Frommer and Syzld [8] show that under the aforementioned conditions on the splittings, varying the number of inner iterations will still result in the convergence of the inner–outer process.

In this paper we wish to extend some of the results of Lanzkron, Rose, and Szyld [9] and of Frommer and Szyld [8] on inner–outer iterations for solving **nonsingular** systems **to** the solution of **singular** systems. In the case of the latter, although as of

[1] we shall normally reserve the subscripted letter z to denote approximations generated by the inner–outer iteration scheme and use different subscripted letters in conjunction with other iteration schemes

the former, the process of inner–outer iterations can be represented by means of an iteration matrix at every stage, the spectral radius of such a matrix can no longer be less than 1. Furthermore, even if the spectral radius of the iteration matrix at each stage is 1, this does not ensure the convergence of the inner–outer iteration process even if a fixed number of iterations are used between every two outer iterations. The problem is further compounded if the number of inner iterations is allowed to vary between every two outer iterations. Such a situation resembles the so called "chaotic iterations" studied by the authors in previous papers, see for example [4] and [7]. Here we shall also both sharpen and extend some of our previous results on the general problem of convergence of chaotic iterations. We shall further examine some connections between our work here and problems of convergence of infinite products of matrices such as considered recently by Daubechies and Lagarias in [5].

As a motivation for the fundamental assumptions that we shall make in our main conclusions (see Corollary 3.2) we say this: If one is going to employ the inner–outer iteration scheme, then it is very reasonable that often between any two outer iterations only a relatively small number of inner iterations will be computed and only in rare cases many more inner iterations will be allowed. This effectively means that there is a number $m \geq 1$, such that infinitely often at most m inner iterations will be carried out between any two outer ones. This implies that there exists an index $1 \leq m_0 \leq m$ such that for an infinite subsequence i_k of the positive integers, $p_{i_k} = m_0$, viz., infinitely often, $T_{p_{i_k}} = T_{m_0}$. What we shall prove is that under certain convergence properties of T_{m_0}, such as T_{m_0} is paracontracting with respect to a vector norm in respect of which all the T_i's are nonexpansive, the inner–outer iteration (1.6) for any initial vector z_0. This implies that the inner–outer iteration scheme is convergent when the system (1.1) is consistent.

Actually we shall prove a more general result (Theorem 3.1) than that in Corollary 3.2. It is as follows: Suppose we have a (possibly infinite) set of matrices $\mathcal{S} = \{T_j | j \in J\}$, and there exists a vector norm $\|\cdot\|$ on $\mathbb{C}^n$ such that each matrix in $\mathcal{S}$ is nonexpansive with respect to $\|\cdot\|$. From $\mathcal{S}$ select an infinite sequence of matrices $\{T_i\}_{i=0}^{\infty}$. Then if $\{T_i\}_{i=0}^{\infty}$ contains a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ which converges to a matrix H which is paracontracting with respect to $\|\cdot\|$ and such that the nullspace $N(I-H)$ is contained in the intersection of the nullspaces $N(I-T_j)$, $j \in J$, then

$$\exists \lim_{i \to \infty} T_i T_{i-1} \cdots T_0.$$

A by–product of this result will be a conclusion concerning the convergence of an infinite product of nonnegative stochastic matrices.

Finally, let $\mathcal{D}$ be the set of all sequences $(d) = \{d_i\}_{i=0}^{\infty}$ of integers such that each sequence (d) contains an integer $k = k^{(d)}$ such that $d_i = k$ for infinitely many i's. Then, according to Theorem 3.1, if corresponding to the sequence (d), the matrix T_k is paracontracting, then

$$\exists \lim_{i \to \infty} T_{d_i} \cdots T_{i_0} \; =: \; T^{(d)}.$$

We shall show that the function:

$$f : (d) \; \rightarrow \; T^{(d)}$$

is continuous.

ETNA
Kent State University
etna@mcs.kent.edu

2. Preliminaries. Let $B \in \mathbb{C}^{n,n}$. By $N(B)$ and $R(B)$ we shall denote, respectively, the *nullspace of* B and the *range* of B. Recall that the Jordan blocks of B corresponding to 0 are 1×1 if and only if $N(B)\cap R(B) = \{0\}$ and $N(B)+R(B) = \mathbb{C}^n$, a situation which we shall write as $N(B) \oplus R(B) = \mathbb{C}^n$. Recall further that according to Oldenburger [16] the powers of a matrix $B \in \mathbb{C}^{n,n}$ converge if and only if $N(I-B) \oplus R(I-B) = \mathbb{C}^n$ and

$$\gamma(B) \;:=\; \max\{|\lambda| \mid \lambda \in \sigma(B), \lambda \neq 1\} \;<\; 1,$$

where $\sigma(\cdot)$ denotes the spectrum of a matrix.

For a vector $x \in \mathbb{R}^n$ we shall write that $x \gg 0$ $(x > 0)$ $(x \geq 0)$ if all the entries of x are positive numbers (nonnegative numbers, but $x \neq 0$) (nonnegative numbers). We shall use similar notations for real matrices.

Let $||\cdot||$ denote a vector norm in $\mathbb{C}^n$. An $n \times n$ matrix B is **nonexpansive with respect to** $||\cdot||$ if for all $x \in \mathbb{C}^n$,

$$||Bx|| \;\leq\; ||x||.$$

B is called **paracontracting with respect to** $||\cdot||$ if for all $x \in \mathbb{C}^n$,

$$Bx \neq x \Leftrightarrow ||Bx|| \;<\; ||x||.$$

We denote by $\mathcal{N}(||\cdot||)$ the set of all matrices in $\mathbb{C}^{n,n}$ which are paracontracting with respect to $||\cdot||$. Two examples of paracontracting matrices are as follows. For the Euclidean norm it is known that any Hermitian matrix whose eigenvalues lie in $(-1, 1]$ is paracontracting. Suppose now that B is an $n \times n$ positive matrix whose spectral radius is 1 and with a Perron vector $x \gg 0$. We claim that such a matrix is paracontracting with respect to $\|\cdot\|_x$, the monotonic vector norm induced by x. For let $y \in \mathbb{R}^n$ be any vector satisfying $y \neq By$ or, equivalently, not being a multiple of x. We know that

$$\|y\|_x \;=\; \min\{\; \delta > 0 \mid \; -\delta x \leq y \leq \delta x\}.$$

By the positivity of B and because $Bx = x$, it follows that for any δ such that $-\delta x \leq y \leq \delta x$, $-\delta x \ll By \ll \delta x$, so that $\|By\|_x \;< \|y\|_x$.

The concept of paracontraction was introduced by Nelson and Neumann [11] who showed that the product of any number of matrices in $\mathcal{N}(||\cdot||)$ is again an element of $\mathcal{N}(||\cdot||)$. Moreover, they used a result of Mott and Schneider [10] to show that the powers of any matrix $B \in \mathcal{N}(||\cdot||)$ converge. Thus, in particular such matrix has the property that $N(I-B) \oplus R(I-B) = \mathbb{C}^{n,n}$.

Finally, recall that a splitting of A into $A = P - Q$ is called **regular** if P is nonsingular, $P^{-1} \geq 0$, and $Q \geq 0$. Regular splittings where introduced by Varga, [20], who showed that for a regular splitting, $\rho(P^{-1}Q) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$. A splitting $A = P - Q$ is called **weak regular** if P is nonsingular, $P^{-1} \geq 0$, and $P^{-1}Q \geq 0$. This concept was introduced by Ortega and Rheinboldt [15] who showed that, even allowing for this weakening of the assumption on regular splitting, $\rho(P^{-1}Q) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$. Some of Varga's results for regular splittings of nonsingular matrices A were generalized to regular splittings of singular matrices. Neumann and Plemmons [13] showed that if $A = P-Q$ is a regular splitting of A, then $\rho(P^{-1}Q) \leq 1$ and $R(I-P^{-1}Q) \oplus N(I-P^{-1}Q) = \mathbb{R}^n$ if and only if A is **range monotone**, that is, $[Ax \geq 0$ and $x \in R(A)] \Rightarrow x \geq 0$. Moreover they showed that if there exists a vector $x \gg 0$ such that $P^{-1}Qx \leq x$, then $\rho(P^{-1}Q) \leq 1$ and $R(I-P^{-1}Q) \oplus N(I-P^{-1}Q) = \mathbb{R}^n$, and such a positive vector always exists if A is a singular and irreducible M–matrix.

3. Main Results. Most of the results in this paper are consequences of the following theorem:

THEOREM 3.1. *Let $\mathcal{S} = \{T_j | j \in J\}$ be a set of matrices in $\mathbb{C}^{n,n}$, let $\{T_i\}_{i=0}^{\infty}$ be a sequence of matrices chosen from $\mathcal{S}$, and consider the iteration scheme*

$$x_{i+1} \;=\; T_i x_i, \qquad i = 0, 1, 2, \ldots \tag{3.1}$$

Suppose that all $T_j \in \mathcal{S}$ are nonexpansive with respect to the same vector norm $\|\cdot\|$ and there exists a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of the sequence $\{T_i\}_{i=0}^{\infty}$ such that

$$\lim_{k\to\infty} T_{i_k} \;=\; H, \tag{3.2}$$

where H is a matrix with the following properties:

(i) *H is paracontracting with respect to $\|\cdot\|$,*

and

(ii) $N(I-H) \subseteq \bigcap_{j\in J} N(I-T_j)$.

Then for any $x_0 \in \mathbb{C}^n$ the sequence (3.1) *is convergent and*

$$\lim_{i\to\infty} x_i \in \mathcal{N}(I-H) \;\subseteq\; \bigcap_{j\in J} \mathcal{N}(I-T_j).$$

Proof. Let $x_0 \in \mathbb{C}^n$ be an arbitrary, but fixed vector, and consider the subsequence of vectors $\{x_{i_k}\}_{k=0}^{\infty}$ of the sequence $\{x_i\}_{i=0}^{\infty}$ generated by the iteration scheme (3.1) from x_0. As it is bounded, it contains a convergent subsequence which, without loss of generality can be taken to be $\{x_{i_k}\}_{k=0}^{\infty}$ itself. Assume therefore that

$$\lim_{k\to\infty} x_{i_k} \;=\; \xi.$$

Because of the nonexpansiveness of the T_j's, the sequence $\{\|x_i\|\}_{i=0}^{\infty}$ is monotonically nonincreasing. Hence we have that

$$\lim_{i\to\infty} \|x_i\| \;=\; \lim_{k\to\infty} \|x_{i_k}\| \;=\; \|\xi\|.$$

We now claim that ξ is a fixed point of H. From the equality

$$\lim_{k\to\infty} \{H\xi - T_{i_k} x_{i_k}\} \;=\; \lim_{k\to\infty} \{(H - T_{i_k})\,\xi + T_{i_k}(\xi - x_{i_k})\} \;=\; 0$$

we have that

$$\|H\xi\| \;=\; \lim_{k\to\infty} \|T_{i_k} x_{i_k}\| \;=\; \|\xi\|$$

and so, as H is paracontracting, $H\xi = \xi$. By (ii) it follows that ξ is also a fixed point of each T_i. To complete the proof we shall now show that $x_i \to \xi$. For any $\epsilon > 0$ choose a positive integer $k(\epsilon)$ such that

$$\|x_{i_{k(\epsilon)}} - \xi\| \;<\; \epsilon.$$

Then for any $i > i_{k(\epsilon)}$, we obtain using the nonexpansiveness of the T_j's that

$$\|x_i - \xi\| \;=\; \|T_{i-1}(x_{i-1} - \xi)\| \;\le\; \|x_{i-1} - \xi\| \;\le \ldots \le \|x_{i_{k(\epsilon)}} - \xi\| \;<\; \epsilon.$$

ETNA
Kent State University
etna@mcs.kent.edu

Our proof is now complete. □

Often the $T_i's$ come from some finite or infinite pool of matrices. If one of these operators appears infinitely often, then condition (3.2) is satisfied. This leads to the following corollary:

COROLLARY 3.2. *Consider the iteration scheme*

$$x_{i+1} \;=\; T_i x_i, \qquad i = 0, 1, 2, \ldots$$

If the $T_i's$ are nonexpansive with respect to the same vector norm $\|\cdot\|$ and if there is a matrix T such that $T_i \;=\; T$ for infinitely many i's, where T is paracontracting with respect to $\|\cdot\|$, and if

$$N(I-T) \;\subseteq\; \bigcap_{i=0}^{\infty} N(I-T_i),$$

then $\lim_{i\to\infty} x_i$ exists and is in $\bigcap_{i=0}^{\infty} N(I-T_i)$ for any $x_0 \in \mathbb{C}^n$.

If an $n \times n$ matrix T is stochastic, then it easily deduced that it is nonexpansive with respect to the vector norm $\|x\|_\infty \;=\; \max_{i=1,\ldots,n} |x_i|$. It follows from remarks made in Section 2 that, in particular, a stochastic matrix T is paracontracting with respect to $\|\cdot\|_\infty$ if $T \gg O$. Thus another corollary to Theorem 3.1 is the following:

COROLLARY 3.3. *Let $\mathcal{S} = \{T_j | j \in J\}$ be a set of stochastic matrices. If one of the matrices in $\mathcal{S}$, say T, is positive, then any infinite product of the T_j's containing T infinitely often is convergent.*

A third corollary resulting from the above theorem is a slight strengthening, in the sense that it allows an infinite pool of both inner and outer splittings, of a result due to Frommer and Szyld [8], mentioned in the introduction:

COROLLARY 3.4. *Suppose that the $n \times n$ coeffcient matrix A in the system* (1.1) *is monotone. For each $i \geq 1$, let $A = P_i - Q_i$ be a regular splitting of A and $P_i = F_i - G_i$ be a weak regular splitting. Consider the inner–outer iteration process:*

$$z_{i+1} \;= T_{i,p_i} z_i + \sum_{i=0}^{p_i-1} (F_i^{-1} G_i)^i F_i^{-1} Q_i b,$$

where as, in the introduction, $p_i \geq 1$ and

$$T_{i,p_i} \;=\; \left(F_i^{-1} G_i\right)^{p_i} + \sum_{j=0}^{p_i-1} \left(F_i^{-1} G_i\right)^j F_i^{-1} Q_i. \tag{3.3}$$

If there are splittings $A = P - Q$ and $P = F - G$ such that for infinitely many i's $P_i = P$ and $F_i = F$ simultaneously, then for any $z_0 \in \mathbb{R}^n$,

$$\lim_{i\to\infty} z_i \;=\; A^{-1} b.$$

Proof. Using the usual approach of error analysis, it suffices to show that for each vector $w \in \mathbb{R}^n$,

$$\lim_{i\to\infty} T_{i,p_i} \cdots T_{1,p_1} w \;=\; 0.$$

As A is monotone, for the n–vector e of all 1's, we have that $x := A^{-1}e \gg 0$. Now for each $i \geq 1$,

$$I - T_{i,p_i} \;=\; (I - R_i^{p_i})\, P_i^{-1} A \;=\; \sum_{j=0}^{p_i-1} (R_i)^j\, F_i^{-1} A, \tag{3.4}$$

where $R_i = F_i^{-1} G_i \geq 0$ and $F_i^{-1} \geq 0$. As F_i^{-1} must have a positive element in each row and we see that

$$x \;\gg\; T_{i,1}x \;\geq\; T_{i,2}x \;\geq \ldots \geq T_{i,p_i}x, \tag{3.5}$$

and because of the nonnegativity of T_{i,p_i}, which follows from (3.5), we have that for all $i > 0$

$$\|T_{i,p_i}\|_x \;< 1, \tag{3.6}$$

implying that these operators are paracontracting with respect to $\|\cdot\|_x$. For each $i \geq 0$ define

$$q_i \;=\; \begin{cases} 1, & \text{if } P_i = P \text{ and } F_i = F, \\ p_i & \text{otherwise.} \end{cases}$$

Then, as $N(I - T_{i,q_i}) = \{0\}$ for each $i \geq 0$ and as for infinitely many i's, T_{i,p_i} equals a fixed operator, it follows by Theorem 3.1 that

$$\lim_{i\to\infty} T_{i,q_i} \cdots T_{1,q_1} x \;=\; 0,$$

and therefore by (3.5) and the nonnegativity of the $T_{i,j}$'s we obtain that

$$\lim_{i\to\infty} T_{i,p_i} \cdots T_{1,p_1} x \;=\; 0.$$

This completes the proof. □

Another consequence of Theorem 3.1 is this:

THEOREM 3.5. *Suppose that $S = \{T_j | j \in J\}$ is a set of matrices in $\mathbb{C}^{n,n}$ and let $\mathcal{D}$ be the set of all sequences $(d) = \{d_i\}_{i=1}^{\infty}$ of integers such that each (d) contains an integer $k^{(d)}$ such that $d_i = k^{(d)}$ for infinitely many i's. Consider the function*

$$f:\; (d) \;\to\; T^{(d)} \;:=\; \lim_{i\to\infty} T_{d_i} \cdots T_{d_1}. \tag{3.7}$$

Suppose that $N(I - T_j) = M$ for all $j \in J$ and that there exists a vector norm $\|\cdot\|$ such that all T_j's in $\mathcal{S}$ are nonexpansive with respect to $\|\cdot\|$ and such that for each sequence $(d) \in \mathcal{D}$, $T_{k^{(d)}} \in \mathcal{N}(\|\cdot\|)$. Then for any $(d_1) = \{d_i^{(1)}\}_{i=1}^{\infty}$ and $(d_2) = \{d_i^{(2)}\}_{i=1}^{\infty}$ in $\mathcal{D}$ and for any $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that

$$\left\| T^{(d_1)} - T^{(d_2)} \right\| \;<\; \epsilon$$

if

$$\mathrm{dist}((d_1),(d_2)) \;=\; 2^{-r} \;<\; \delta(\epsilon),$$

where r is the smallest integer such that $d_r^{(1)} \neq d_r^{(2)}$.

Proof. By Theorem 3.1 clearly each of the limits $T^{(d_1)}$ and $T^{(d_2)}$ exists. It also follows from that theorem that each column of $T^{(d_1)}$ is a fixed point of each of the operators T_i, $i = 1, 2, \ldots$. Suppose now that $\text{dist}(d_1, d_2) \le 2^{-(r+1)}$ so that $d_i^{(1)} = d_i^{(2)}$, $i = 1, \ldots, r$. Then for all $s > 0$,

$$T_{d_{r+s}^{(2)}} \cdots T_{d_1^{(2)}} - T_{d_r^{(1)}} \cdots T_{d_1^{(1)}} = \left(T_{d_{r+s}^{(2)}} \cdots T_{d_{r+1}^{(2)}} - I\right)\left(T_{d_r^{(1)}} \cdots T_{d_1^{(1)}} - T^{(d_1)}\right)$$

and so

$$\left\|T_{d_{r+s}^{(2)}} \cdots T_{d_1^{(2)}} - T_{d_r^{(1)}} \cdots T_{d_1^{(1)}}\right\| \le 2\left\|T_{d_r^{(1)}} \cdots T_{d_1^{(1)}} - T^{(d_1)}\right\|.$$

On letting $s \to \infty$, we obtain that

$$\left\|T^{(d_2)} - T_{d_r^{(1)}} \cdots T_{d_1^{(1)}}\right\| \le 2\left\|T_{d_r^{(1)}} \cdots T_{d_1^{(1)}} - T^{(d_1)}\right\|.$$

This inequality immediately gives that

$$\left\|T^{(d_2)} - T^{(d_1)}\right\| \le 3\left\|T_{d_r^{(1)}} \cdots T_{d_1^{(1)}} - T^{(d_1)}\right\|. \tag{3.8}$$

We claim that this inequality is all we need in order to establish the continuity of f. Because as $T_{d_i^{(1)}} \cdots T_{d_1^{(1)}}$ tends to $T^{(d_1)}$ as i tends to infinity, for each ϵ there exist an r_0 such that

$$\left\|T_{d_{r_0}^{(1)}} \cdots T_{d_1^{(1)}} - T^{(d_1)}\right\| < \frac{\epsilon}{3}.$$

Thus for any two sequences $(d_1), (d_2) \in \mathcal{D}$ such that $\text{dist}((d_1), (d_2)) \le 2^{-(r_0+1)}$, we obtain readily, via (3.8), that

$$\left\|T^{(d_2)} - T^{(d_1)}\right\| \le \epsilon.$$

□

4. Applications to Singular Systems. In Section 2 we mentioned that if $A = P - Q$ is a regular splitting for $A \in \mathbb{R}^{n,n}$ and A is range monotone, then $\rho(P^{-1}Q) \le 1$ and $N(I - P^{-1}Q) \oplus R(I - P^{-1}Q) = \mathbb{R}^n$. Suppose now that $P = F - G$ is a weak regular splitting for P and consider the inner–outer iteration process

$$\begin{aligned} z_{k+1} &= (F^{-1}G)^{p_k} z_k + \sum_{j=0}^{p_k-1} (F^{-1}G)^j F^{-1} b \\ &= \{(F^{-1}G)^{p_k} + \sum_{j=0}^{p_k-1} (F^{-1}G) F^{-1} Q\} z_k + \sum_{j=0}^{p_k-1} (F^{-1}G)^j F^{-1} b \\ &= T_{p_k} z_k + \sum_{i=0}^{p_k-1} (F^{-1}G)^i F^{-1} b, \end{aligned} \tag{4.1}$$

where

$$T_i = (F^{-1}G)^i + \sum_{j=0}^{i-1} (F^{-1}G)^j F^{-1} Q, \quad i = 1, 2, \ldots \tag{4.2}$$

We observe at once that since $A = P - Q$ is a regular splitting for A and $P = F - G$ is a weak regular splitting for P, any of the inner–outer iteration operators $T_i,\ i \geq 1$, **is a nonnegative matrix**. Already Nichols in [12] essentially showed that the following relation holds:

$$I - T_i \ = \ (I - R^i)(I - P^{-1}Q), \tag{4.3}$$

a relation of which we have made use in Corollary 3.4 where A was assumed to be nonsingular. We now claim the following:

LEMMA 4.1. *Suppose* $A \in \mathbb{R}^{n,n}$ *is range monotone and that* $A = P - Q$ *and* $P = F - G$ *are regular and weak regular splittings for* A *and* P*, respectively. Then* $\rho(T_i) \leq 1$ *and* $N(I - T_i) \oplus R(I - T_i) = \mathbb{R}^n$ *for all* $i \geq 1$.

Proof. It follows from the results of Varga [20] and Neumann and Plemmons [13] summarized in Section 2 that $I - R^i$ and $I - P^{-1}Q$ are, respectively, a nonsingular M–matrix and an M–matrix of at most index 1, that is, $\rho(P^{-1}Q) \leq 1$ and $N(I - P^{-1}Q) \oplus R(I - P^{-1}Q) = \mathbb{R}^n$. It now follows by (4.3) and Exercise 5.2 on p.159 of Berman and Plemmons [3] that $I - T_i$ is an M–matrix for all $i \geq 1$. Hence $\rho(T_i) \leq 1$, for all $i \geq 1$. To complete the proof we need to show that $N(I - T_i) \oplus R(I - T_i) = \mathbb{R}^n$. By [14] it suffices to show that the matrix $I - T_i$ possesses a $\{1\}$–inverse Y (see Ben–Israel and Greville [1] for background material on generalized inverses) which is **nonnegative on the range of** $I - T_i$, viz.,

$$x \in R(I - T_i) \text{ and } x \geq 0 \ \Rightarrow \ Yx \geq 0.$$

For that purpose choose $Y = (I - P^{-1}Q)^{\#}(I - R^i)^{-1}$, where $(I - P^{-1}Q)^{\#}$ is the **group generalized inverse** of $I - P^{-1}Q$ which exists by virtue of $R(I - P^{-1}Q)$ and $N(I - P^{-1}Q)$ being complementary subspaces in $\mathbb{R}^n$. Now let $x \geq 0$ be a vector in $R(I - T_i)$, and observe that by (4.3) and the nonnegativity of the matrix $(I - R^i)^{-1}$, the vector $(I - R^i)^{-1}x$ is a nonnegative vector in $R(I - P^{-1}Q)$. But as $I - P^{-1}Q$ is an M–matrix of index at most 1, it follows that $(I - P^{-1}Q)^{\#}$ is monotone on $R(I - P^{-1}Q)$ showing that $Yx \geq 0$ and our proof is done. □

Suppose, as in the above lemma, that $A = P - Q$ and $P = F - G$ are a regular and weak regular splittings for A and P, respectively. Note that in the lemma, the range monotonicity of A was used only to deduce that $I - P^{-1}Q$ is an M–matrix of index at most 1. Another condition which ensures that $I - P^{-1}Q$ is an M–matrix of index at most 1 is, according to [13], that there exists a positive vector x such that $Ax \geq 0$. For then $P^{-1}Qx \leq x$. Furthermore, such a vector exists when A is a singular and irreducible M–matrix. When A is such an M– matrix, then, in fact, there exists a positive vector x such that $Ax = 0$. But then also $0 = P^{-1}Ax = x - P^{-1}Qx$ so that $x = P^{-1}Qx$, and hence

$$T_i x \ = \ P^{-1}Qx + R^i(I - P^{-1}Q)x \ = \ P^{-1}Qx \ = \ x.$$

We can thus conclude that when A is an irreducible M–matrix, not only the conclusions of the above lemma hold, but $T_i x = x$ so that $\|T_i\|_x = 1$. Hence for each $i \geq 1$, T_i is nonexpansive with respect to the norm $\|\cdot\|_x$. We also see that

$$0 \ = \ F^{-1}Ax \ = \ x - F^{-1}Gx - F^{-1}Qx \ \leq \ x - F^{-1}Gx \ = \ x - Rx.$$

Now we know that $Q \geq 0$. Thus if either $F^{-1}Qx \gg 0$ or $F^{-1}Gx \gg 0$, then it follows that $x \gg Rx$ so that inductively, $1 > \|R\|_x \geq \|R\|_x^2 \geq \ldots$. Let $H := P^{-1}Q$. Then from the relation

$$T_i - H \ = \ R^i(I - H)$$

we see that, not only

$$\lim_{i\to\infty} T_i \;=\; H, \tag{4.4}$$

a fact that already follows from $\rho(R) < 1$, but that the rate of convergence behaves as $\|R\|_x$.

From the analysis above and from Theorem 3.1 we can now state the following result concerning the convergence of the inner–outer iteration process:

THEOREM 4.2. *Let $A \in \mathbb{R}^{n,n}$ and suppose that $A = P - Q$ and $P = F - G$ are a regular splitting and a weak regular splitting for A and P, respectively, and consider the inner–outer iteration process* (4.1) *for solving the consistent linear system $Ax = b$. Suppose there exists a vector $x \gg 0$ such that $Ax \geq 0$ and one of the following conditions is satisfied:*

(i) *For some integer j, T_j is paracontracting and for infinitely many integers k, $p_k = j$.*

or:

(ii) *$P^{-1}Q$ is paracontracting with respect to $\|\cdot\|_x$, the sequence $\{p_k\}_{k=0}^{\infty}$ is unbounded, and either $F^{-1}Qx \gg 0$ or $F^{-1}Gx \gg 0$.*

Then the sequence of iterations $\{z_k\}_{k=1}^{\infty}$ generated by the scheme given by (4.1) *converges to a solution to the system $Ax = b$.*

Proof. Similar to (3.4), we have the identity that

$$I - T_i \;=\; \sum_{j=0}^{i-1} (R_i)^j \, F^{-1} A$$

from which it follows that x is a positive vector for which

$$x \;\geq\; T_1 x \;\geq T_2 x \;\geq \ldots,$$

showing that for each $i \geq 1$, T_i is nonnexpansive with respect to the monotonic vector norm induced by x.

The validity of part (i) is an immediate consequence of Theorem 3.1. The proof of part (ii) also follows readily from Theorem 3.1 because the unboundedness of the sequence $\{p_k\}_{k=0}^{\infty}$ together with the existence of the limit in (4.4) now means that the sequence of matrices $\{T_{p_k}\}_{k=0}^{\infty}$ contains an infinite subsequence of matrices which converges to the paracontracting matrix H. ▯

REFERENCES

[1] A. BEN-ISRAEL AND T.N.E. GREVILLE, *Generalized Matrix Inverses: Theory and Applications*, John Wiley & Sons, New York, 1974.

[2] M. A. BERGER AND YANG WANG, *Bounded Semigroups of Matrices*, Linear Algebra Appl., 166 (1992), pp. 21–27.

[3] A. BERMAN AND R. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[4] R. BRU, L. ELSNER, AND M. NEUMANN, *Models of Parallel Chaotic Iteration Methods*, Linear Algebra Appl., 102 (1988), pp. 175–192.

[5] I. DAUBECHIES AND J.C. LAGARIAS, *Sets of Matrices All Infinite Products of Which Converge*, Linear Algebra Appl., 161 (1992), pp. 227–263.

[6] L. Elsner, *Infinite products and paracontracting matrices*, Preprint 93–058, Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik", Universität Bielefeld, 1993.

[7] L. Elsner, I. Koltracht, and M. Neumann, *On the Convergence of Asynchronous Paracontractions with Applications to Tomographic Reconstruction from Incomplete Data*, Linear Algebra Appl., 130 (1990), pp. 65–82.

[8] A. Frommer and D. B. Szyld, *H–splittings and two-stage iterative methods*, Numer. Math., 63 (1992), pp. 345–356.

[9] P.J. Lanzkron, D.J. Rose, and D.B. Szyld, *Convergence of Nested Classical Iterative Methods for Linear Systems*, Numer. Math., 58 (1991), pp. 658-702.

[10] J. L. Mott and H. Schneider, *Matrix algebras and groups relatively bounded in norm*, Arch. Math., 10 (1959), pp.1–6.

[11] S. Nelson and M. Neumann, *Generalization of the Projection Method with Applications to SOR Method for Hermitian Positive Semidefinite Linear Systems*, Numer. Math., 51 (1987), pp. 123–141.

[12] N.K. Nichols, *On the Convergence of Two–Stage Iterative Processes for Solving Linear Equations*, SIAM J. Numer. Anal., 10 (1973), pp. 460-469.

[13] M. Neumann and R. J. Plemmons, *Convergent Nonnegative Matrices and Iterative methods for consistent linear systems*, Numer. Math., 31 (1978), pp. 265-279.

[14] M. Neumann and R. J. Plemmons, *Generalized inverse–positivity and splittings of M–Matrices*, Linear Algebra Appl., 23 (1979), pp. 21-35.

[15] J. M. Ortega and W. Rheinboldt, *Monotone iterations for nonlinear equations with applications to Gauss–Seidel methods*, SIAM J. Numer. Anal., 4 (1967), pp. 171–190.

[16] R. Oldenburger, *Infinite Powers of Matrices and Characteristic Roots*, Duke Math. J., 6 (1940), pp. 357–361.

[17] U.G. Rothblum, *Algebraic Eigenspaces of Nonnegative Matrices*, Linear Algebra Appl., 12 (1975), pp. 281–292.

[18] G.–C. Rota and W.G. Strang, *A note on the joint spectral radius*, Indagationes, 22 (1960), pp. 379–381.

[19] H. Schneider, *Theorems on M-splittings of a singular M–matrix which depend on graph structure*, Linear Algebra Appl., 58 (1984), pp. 407-424.

[20] R. S. Varga, *Matrix Iterative Analysis*, Prentice–Hall, Englewood Cliffs, N. J., 1961.