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# A NEW GERŠGORIN-TYPE EIGENVALUE INCLUSION SET*

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**Abstract.** We give a generalization of a less well-known result of Dashnic and Zusmanovich [2] from 1970, and show how this generalization compares with related results in this area.

**Key words.** Geršgorin theorem, Brauer Cassini ovals, nonsingularity results.

**AMS subject classifications.** 15A18, 65F15.

**1. Introduction.** Our interest here is in nonsingularity results for matrices and their equivalent eigenvalue inclusion sets in the complex plane. As examples of this, we have the famous result of Geršgorin [3]:

THEOREM 1. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ and for any eigenvalue $\lambda$ of $A$, there is a positive integer $k$ in $N := \{1, 2, \cdots, n\}$ such that*

$$|\lambda - a_{k,k}| \le r_k(A) := \sum_{j \in N \setminus \{k\}} |a_{k,j}|. \tag{1.1}$$

*Consequently, if $\sigma(A)$ denotes the collection of all eigenvalues of $A$, then*

$$\sigma(A) \subseteq \Gamma(A) := \bigcup_{i=1}^{n} \Gamma_i(A), \text{ where } \Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \le r_i(A)\}. \tag{1.2}$$

Here, $\Gamma_i(A)$ is the $i$-th **Geršgorin disk**, and $\Gamma(A)$ is the **Geršgorin set** for the matrix $A$. The equivalent nonsingularity result for this is

THEOREM 2. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ which is strictly diagonally dominant, i.e.,*

$$|a_{i,i}| > r_i(A) \quad (\text{all } i \in N), \tag{1.3}$$

*it follows that $A$ is nonsingular.*

Similarly, there is the following nonsingularity result of Ostrowski [5]:

THEOREM 3. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \ge 2$, with*

$$|a_{i,i}| \cdot |a_{j,j}| > r_i(A) \cdot r_j(A) \quad (\text{all } i \ne j \text{ in } N), \tag{1.4}$$

*it follows that $A$ is nonsingular.*

Its equivalent eigenvalue inclusion set is the following result of Brauer [1]:

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THEOREM 4. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, and for any eigenvalue $\lambda$ of $A$, there is a pair of distinct integers $i$ and $j$ in $N$ such that*

$$\lambda \in K_{i,j}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A)\}. \tag{1.5}$$

*Consequently,*

$$\sigma(A) \subseteq \mathcal{K}(A) := \bigcup_{\substack{i,j\in N\\ i\neq j}} K_{i,j}(A). \tag{1.6}$$

The quantity $K_{i,j}(A)$ of (1.5) is called the $(i,j)$-th **Brauer Cassini oval**, and $\mathcal{K}(A)$ of (1.6) is called the **Brauer set** for the matrix $A$. (For further results about these sets, see Varga [6].)

**2. New results.** To describe our first result here, let $S$ denote a nonempty subset of $N = \{1, 2, \cdots, n\}, n \geq 2$, and let $\overline{S} := N\backslash S$ denote its complement in $N$. Then, given any matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, split each row sum, $r_i(A)$ from (1.1), into two parts, depending on $S$ and $\overline{S}$, i.e.,

$$\left\{ \begin{array}{l} r_i(A) := \displaystyle\sum_{j\in N\backslash\{i\}} |a_{i,j}| = r_i^S(A) + r_i^{\overline{S}}(A), \text{ where} \\ r_i^S(A) := \displaystyle\sum_{j\in S\backslash\{i\}} |a_{i,j}|, \text{ and } r_i^{\overline{S}}(A) := \displaystyle\sum_{j\in \overline{S}\backslash\{i\}} |a_{i,j}| \quad (\text{all } i \in N). \end{array} \right. \tag{2.1}$$

DEFINITION 1. *Given any matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, and given any nonempty subset $S$ of $N$, then $A$ is an $S$-**strictly diagonally dominant matrix** if*

$$\left\{ \begin{array}{l} i) \;\; |a_{i,i}| > r_i^S(A) \;\; (\text{all } i \in S), \quad \text{and} \\ ii) \;\; (|a_{i,i}| - r_i^S(A)) \cdot (|a_{j,j}| - r_j^{\overline{S}}(A)) > r_i^{\overline{S}}(A) \cdot r_j^S(A) \; (\text{all } i \in S, \text{ all } j \in \overline{S}). \end{array} \right. \tag{2.2}$$

We note, from (2.2 *i*), that as $|a_{i,i}| - r_i^S(A) > 0$ for all $i \in S$, then on dividing by this term in (2.2 *ii*) gives

$$\left(|a_{j,j}| - r_j^{\overline{S}}(A)\right) > \frac{r_i^{\overline{S}}(A) \cdot r_j^S(A)}{(|a_{i,i}| - r_i^S(A))} \geq 0 \quad (\text{all } j \in \overline{S}),$$

so that we also have

$$|a_{j,j}| - r_j^{\overline{S}}(A) > 0 \quad (\text{all } j \in \overline{S}). \tag{2.3}$$

If $S = N$, so that $\overline{S} = \emptyset$, then the conditions of (2.2 *i*) reduce to $|a_{i,i}| > r_i(A)$ (all $i \in N$), and this is just the familiar statement that $A$ is **strictly diagonally dominant**.

Our first result here is

THEOREM 5. *Let $S$ be a nonempty subset of $N$, and let $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, be $S$-strictly diagonally dominant. Then, $A$ is nonsingular.*

*Proof.* If $S = N$, then, as we have seen, $A$ is strictly diagonally dominant, and thus nonsingular from Theorem 2. Next, we assume that $S$ is a nonempty subset of $N$ with $\overline{S} \neq \emptyset$.

The idea of the proof is to construct a positive diagonal matrix $W$ such that $AW$ is strictly diagonally dominant. Now, define $W$ as $W = \text{diag}[w_1, w_2, \cdots, w_n]$, where

$$w_k := \begin{cases} \gamma, & \text{for all } k \in S, \text{ where } \gamma > 0, \text{ and} \\ 1, & \text{for all } k \in \overline{S}. \end{cases}$$

It then follows that $AW := [\alpha_{i,j}] \in \mathbb{C}^{n\times n}$ has its entries given by

$$\alpha_{i,j} := \begin{cases} \gamma a_{i,j}, & \text{if } j \in S, \text{ all } i \in N, \text{ and} \\ a_{i,j}, & \text{if } j \in \overline{S}, \text{ all } i \in N. \end{cases}$$

Then, the row sums of $AW$ are, from (2.1), just

$$r_\ell(AW) = r_\ell^S(AW) + r_\ell^{\overline{S}}(AW) = \gamma r_\ell^S(A) + r_\ell^{\overline{S}}(A) \ \ (\text{all } \ell \in N),$$

and $AW$ is then strictly diagonally dominant if

$$\begin{cases} \gamma|a_{i,i}| > \gamma r_i^S(A) + r_i^{\overline{S}}(A) \ \ (\text{all } i \in S), \text{ and} \\ |a_{j,j}| > \gamma r_j^S(A) + r_j^{\overline{S}}(A) \ \ (\text{all } j \in \overline{S}). \end{cases}$$

The above inequalities can be also expressed as

$$\begin{cases} i)\ \gamma(|a_{i,i}| - r_i^S(A)) > r_i^{\overline{S}}(A) \ \ (\text{all } \ i \in S), \text{ and} \\ ii)\ \ |a_{j,j}| - r_j^{\overline{S}}(A) > \gamma r_j^S(A) \ \ (\text{all } j \in \overline{S}), \end{cases} \tag{2.4}$$

which, upon division, can be further reduced to

$$\frac{r_i^{\overline{S}}(A)}{|a_{i,i}| - r_i^S(A)} < \gamma \ (\text{all } i \in S), \text{ and } \gamma < \frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} \ (\text{all } j \in \overline{S}), \tag{2.5}$$

where the final fraction in (2.5) is defined to be $+\infty$ if $r_j^S(A) = 0$ for some $j \in \overline{S}$. The inequalities of (2.4) will all be satisfied if there is a $\gamma > 0$ for which

$$0 \le B_1 := \max_{i\in S} \frac{r_i^{\overline{S}}(A)}{|a_{i,i}| - r_i^S(A)} < \gamma < \min_{j\in \overline{S}} \frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} =: B_2. \tag{2.6}$$

But since (2.2 *ii*) exactly gives that $B_2 > B_1$, then, for any $\gamma > 0$ with $B_1 < \gamma < B_2$, $AW$ is strictly diagonally dominant and hence nonsingular. Then, as $W$ is nonsingular, so is $A$. □

As is now familiar, the nonsingularity in Theorem 2 then gives, by negation, the following equivalent eigenvalue inclusion set in the complex plane.

THEOREM 6. *Let $S$ be any nonempty subset of $N := \{1, 2, \cdots, n\}$, $n \ge 2$, with $\overline{S} := N\backslash S$. Then, for any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, define the Geršgorin-type disks*

$$\Gamma_i^S(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \le r_i^S(A)\} \ (\text{any } i \in S), \tag{2.7}$$

*and the sets*

$$V_{i,j}^S(A) := \{z \in \mathbb{C} : (|z - a_{i,i}| - r_i^S(A)) \cdot (|z - a_{j,j}| - r_j^{\overline{S}}(A)) \le r_i^{\overline{S}}(A) \cdot r_j^S(A)\}, \tag{2.8}$$

*(any* $i \in S$*, any* $j \in \overline{S}$*). Then,*

$$\sigma(A) \subseteq C^S(A) := \left( \bigcup_{i \in S} \Gamma_i^S(A) \right) \cup \left( \bigcup_{i \in S, j \in \overline{S}} V_{i,j}^S(A) \right). \tag{2.9}$$

We remark that Dashnic and Zusmanovich [2] obtained the result of Theorem 5 in the special case that the set $S$ is a singleton, i.e., $S_i := \{i\}$ for some $i \in N$. In this case, we define the associated set, from Theorem 6, as the set $\mathcal{D}_i(A)$, so that, from (2.7) and (2.8),

$$\mathcal{D}_i(A) = \Gamma_i^{S_i}(A) \cup \left( \bigcup_{j \in N \backslash \{i\}} V_{i,j}^{S_i}(A) \right). \tag{2.10}$$

Now, $r_i^{S_i}(A) = 0$ from (2.1) so that $\Gamma_i^{S_i}(A) = \{a_{i,i}\}$ from (2.7). Moreover, we also have, from (2.8) in this case that, for all $j \neq i$ in $N$,

$$V_{i,j}^{S_i}(A) = \{z \in \mathbb{C} : |z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|\ ) \leq r_i(A) \cdot |a_{j,i}|\}. \tag{2.11}$$

But as $z = a_{i,i}$ is necessarily contained in $V_{i,j}^{S_i}(A)$ for all $j \neq i$, we can simply write from (2.11) that

$$\mathcal{D}_i(A) = \bigcup_{j \in N \backslash \{i\}} V_{i,j}^{S_i}(A) \quad (\text{any } i \in N). \tag{2.12}$$

This shows that $D_i(A)$ is determined from $(n-1)$ sets $V_{i,j}^{S_i}(A)$, plus the added information from (2.1) on the partial row sums of $A$. The associated Geršgorin set $\Gamma(A)$, from (1.2), is determined from $n$ disks and the associated Brauer set $\mathcal{K}(A)$, from (1.6) is determined from $\binom{n}{2}$ Cassini ovals. These sets are compared in the next section.

## 3. Comparisons with other eigenvalue inclusion sets.

We first establish the new result of

THEOREM 7. *For any* $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$*, and for any* $i \in N$*, consider* $\mathcal{D}_i(A)$ *of* (2.12)*. Then* (cf. (1.2)),

$$\mathcal{D}_i(A) \subseteq \Gamma(A), \tag{3.1}$$

*and for* $n = 2$*, and for all* $A = [a_{i,j}] \in \mathbb{C}^{2 \times 2}$*, we have (cf. (1.5) and (1.6))*

$$\mathcal{D}_1(A) = \mathcal{D}_2(A) = \mathcal{K}(A) = K_{1,2}(A). \tag{3.2}$$

*But, for any* $n \geq 3$ *and for any* $i \in N$*, there is a matrix* $\tilde{F}$ *in* $\mathbb{C}^{n \times n}$ *for which*

$$\mathcal{D}_i(\tilde{F}) \nsubseteq \mathcal{K}(\tilde{F}) \text{ and } \mathcal{K}(\tilde{F}) \nsubseteq \mathcal{D}_i(\tilde{F}). \tag{3.3}$$

*Proof*. To establish (3.1), fix some $i \in N$ and consider any $z \in \mathcal{D}_i(A)$. Then from (2.12), there is a $j \neq i$ such that $z \in V_{i,j}^{S_i}(A)$, i.e., from (2.11),

$$|z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|) \leq r_i(A) \cdot |a_{j,i}|. \tag{3.4}$$

If $z \notin \Gamma(A)$, then $|z - a_{k,k}| > r_k(A)$ for *all* $k \in N$, so that $|z - a_{i,i}| > r_i(A) \geq 0$, and $|z - a_{j,j}| > r_j(A) \geq 0$. Thus, the left part of (3.4) satisfies

$$|z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|) > r_i(A) \cdot |a_{j,i}|,$$

which contradicts the inequality in (3.4). Thus, $z \in \Gamma(A)$ for each $z \in \mathcal{D}_i(A)$, which establishes (3.1).

Next, to establish (3.2), it can be easily seen from (1.5)-(1.6) and (2.11)-(2.12) that (3.2) is valid for any $A = [a_{i,j}] \in \mathbb{C}^{2\times 2}$.

Finally, to establish (3.3), consider first the specific $3 \times 3$ matrix $E$ of

$$E = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & i & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{3.5}$$

Then, it can be verified that

$$\begin{aligned} \Gamma(E) = \;& \{z \in \mathbb{C} : |z-1| \leq 1\} \cup \{z \in \mathbb{C} : |z-i| \leq 1\} \cup \{z \in \mathbb{C} : |z+1| \leq 1\}, \\ \mathcal{K}(E) = \;& \{z \in \mathbb{C} : |z-1| \cdot |z-i| \leq 1\} \cup \{z \in \mathbb{C} : |z-i| \cdot |z+1| \leq 1\} \\ & \cup \{z \in \mathbb{C} : |z-1| \cdot |z+1| \leq 1\}, \\ \mathcal{D}_1(E) = \;& \{z \in \mathbb{C} : |z-1| \cdot (|z-i| - 1) \leq 0\} \cup \{z \in \mathbb{C} : |z-1| \cdot (|z+1| - 1) \leq 0\}. \end{aligned}$$

It is interesting to note that $\mathcal{D}_1(E)$ reduces to the union of the two disks $\{z \in \mathbb{C} : |z-i| \leq 1\}$ and $\{z \in \mathbb{C} : |z+1| \leq 1\}$, *and* the single point $z = 1$. These above three sets are shown in Fig. 3.1, where we see that the special case $i = 1$ and $n = 3$ of (3.3) is valid.

To establish (3.3) in general, let $n > 3$, and consider the matrix $F$ in $\mathbb{C}^{n\times n}$ which is obtained by adding $n-3$ rows of zeros beneath the matrix $E$ of (3.5) and $n-3$ columns of zeros to the right of $E$, so that $E$ becomes the upper $3 \times 3$ principal submatrix of $F$. From the structure of $F$, it is not difficult to show that (3.3) holds for $F$ in the case $i = 1$, i.e.,

$$\mathcal{D}_1(F) \nsubseteq \mathcal{K}(F) \text{ and } \mathcal{K}(F) \nsubseteq \mathcal{D}_1(F).$$

But, given any $i \in N$, there is a suitable $n \times n$ permutation matrix $P$ such that if $\tilde{F} := P^T F P$, then

$$\mathcal{D}_i(\tilde{F}) \nsubseteq \mathcal{K}(\tilde{F}) \text{ and } \mathcal{K}(\tilde{F}) \nsubseteq \mathcal{D}_i(\tilde{F}),$$

completing the proof of Theorem 7. □

Next, it is evident from (2.9) of Theorem 6 that, for any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$,

$$\sigma(A) \subseteq \mathcal{D}_i(A) \quad (\text{all } i \in N),$$

so that

$$\sigma(A) \subseteq \mathcal{D}(A) := \bigcap_{i \in N} \mathcal{D}_i(A). \tag{3.6}$$

Now, as each $\mathcal{D}_i(A)$, from (2.12), depends on $(n-1)$ oval-like sets $V_{i,j}^{S_i}(A)$, it follows that $\mathcal{D}(A)$ of (3.6) is determined from $n(n-1)$ oval-like sets $V_{i,j}^{S_i}(A)$, which is *twice* the number of Cassini ovals, namely $\binom{n}{2}$, which determine the Brauer set $\mathcal{K}(A)$. This suggests, perhaps, that $\mathcal{D}(A) \subseteq \mathcal{K}(A)$. This inclusion is true, and this new result is established in

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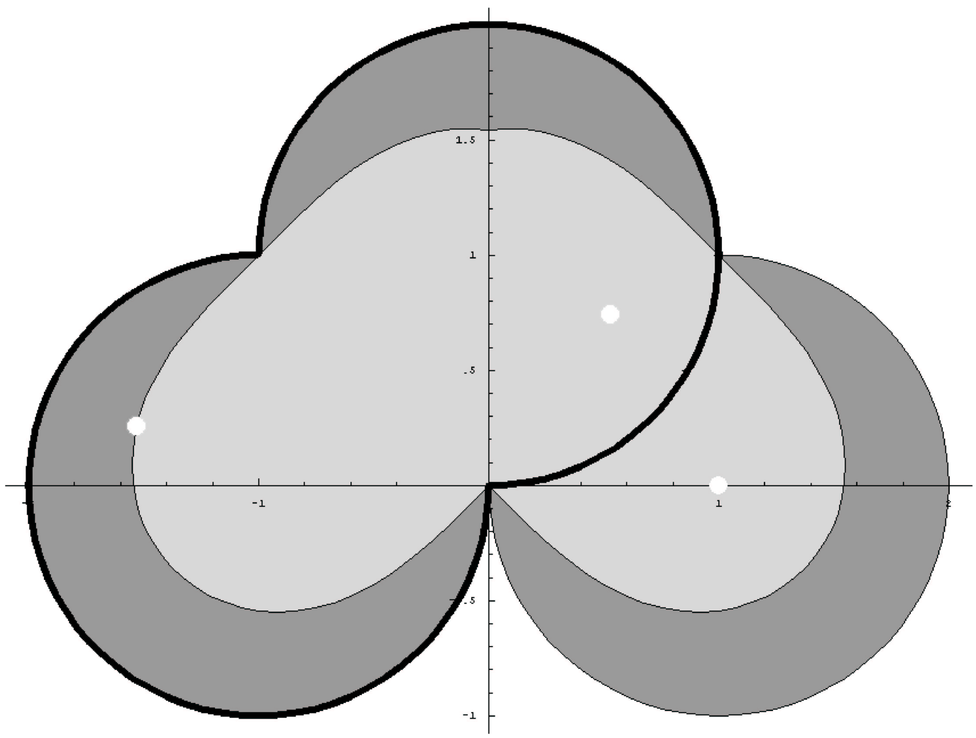


FIG. 3.1. *The sets* $\Gamma(E)$ *(shaded dark gray),* $\mathcal{K}(E)$ *(shaded light gray),* $\mathcal{D}_1(E)$ *(two disks with the bold boundary and the point* $z = 1$*) for the matrix* $E$ *of (3.5). The white dots are the eigenvalues of* $E$*.*

THEOREM 8. *For any* $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, *the associated sets* $\mathcal{D}(A)$, *of* (3.6), *and* $\mathcal{K}(A)$, *of* (1.6), *satisfy*

$$\mathcal{D}(A) \subseteq \mathcal{K}(A). \tag{3.7}$$

*Proof.* First, we observe, from (3.1), that as $\mathcal{D}_i(A) \subseteq \Gamma(A)$ for each $i \in N$, then $\mathcal{D}(A)$, as defined in (3.6), evidently satisfies

$$\mathcal{D}(A) \subseteq \Gamma(A). \tag{3.8}$$

To establish (3.7), consider any $z \in \mathcal{D}(A)$ so that, for each $i \in N$, $z \in \mathcal{D}_i(A)$. Hence, from (2.12), for each $i \in N$, there is $j \in N \backslash \{i\}$ so that $z \in V_{i,j}^{S_i}(A)$, i.e. the inequality of (3.4) is valid. But from (3.8), $\mathcal{D}(A) \subseteq \Gamma(A)$ implies that there is a $k \in N$ with $|z - a_{k,k}| \leq r_k(A)$. For this index $k$, there is a $t \in N \backslash \{k\}$ such that $z \in V_{k,t}^{S_i}(A)$, i.e.,

$$|z - a_{k,k}| \left(|z - a_{t,t}| - r_t(A) + |a_{t,k}|\right) \leq r_k(A) \cdot |a_{t,k}|.$$

This can be rewritten as

$$\begin{aligned} |z - a_{k,k}| \cdot |z - a_{t,t}| &\leq |z - a_{k,k}| \cdot (r_t(A) - |a_{t,k}|) + r_k(A) \cdot |a_{t,k}| \\ &\leq r_k(A)(r_t(A) - |a_{t,k}|) + r_k(A) \cdot |a_{t,k}| = r_k(A) \cdot r_t(A), \end{aligned}$$

that is,

$$|z - a_{k,k}| \cdot |z - a_{t,t}| \leq r_k(A) \cdot r_t(A).$$

Hence, from (1.5) and (1.6), $z \in K_{k,t}(A) \subseteq \mathcal{K}(A)$. As this is true for each $z \in \mathcal{D}(A)$, then $\mathcal{D}(A) \subseteq \mathcal{K}(A)$. $\square$

We remark that the set $\mathcal{D}(A)$ of (3.5) was also considered in Dashnic and Zusmanovich [2], but with no comparisons with $\Gamma(A)$ or $\mathcal{K}(A)$.

It is interesting also to mention that Huang [4] similarly breaks $N = \{1, 2, \cdots, n\}$ into disjoint subsets $S$ and $\overline{S}$, but assumes a variant of the inequalities of (2.2). Now, if $S = \{i_1, i_2, \cdots, i_k\}$, then $A_{S,S} := [a_{i_j, i_\ell}]$ (all $i_j, i_\ell$ in $S$) is its associated $k \times k$ principal submatrix of $A$, whose associated **comparison matrix** is given by

$$\mathcal{M}(A_{S,S}) := \begin{bmatrix} +|a_{i_1,i_1}| & -|a_{i_1,i_2}| & \cdots & -|a_{i_1,i_k}| \\ -|a_{i_2,i_1}| & +|a_{i_2,i_2}| & \cdots & -|a_{i_2,i_k}| \\ \vdots & & & \vdots \\ -|a_{i_k,i_1}| & -|a_{i_k,i_2}| & \cdots & +|a_{i_k,i_k}| \end{bmatrix}, \tag{3.9}$$

and it is assumed by Huang that $\mathcal{M}(A_{S,S})$ is a **nonsingular** $M$**-Matrix** (or equivalently, that $A_{S,S}$ is a nonsingular $H$-matrix), with the additional assumption (in analogy with (2.6)) that if $\mathbf{r}^{\overline{S}}(A) := [r_{i_1}^{\overline{S}}(A), r_{i_2}^{\overline{S}}(A), \cdots, r_{i_k}^{\overline{S}}(A)]^T$, then

$$||\mathcal{M}^{-1}(A_{S,S}) \cdot \mathbf{r}^{\overline{S}}(A)||_\infty < B_2 := \min_{j \in \overline{S}} \left( \frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} \right), \tag{3.10}$$

where $B_2$ is defined in (2.6). We note that our earlier assumption in (2.2 *i*) makes the associated principal submatrix $A_{S,S}$ a strictly diagonally dominant matrix, so that $\mathcal{M}(A_{S,S})$ in our case is necessarily a nonsingular $M$-matrix.

The result of Huang [4] is more general than the result of our Theorem 5, but it comes with the added expense of having to explicitly determine $\mathcal{M}^{-1}(A_{S,S})$ for use in (3.10).

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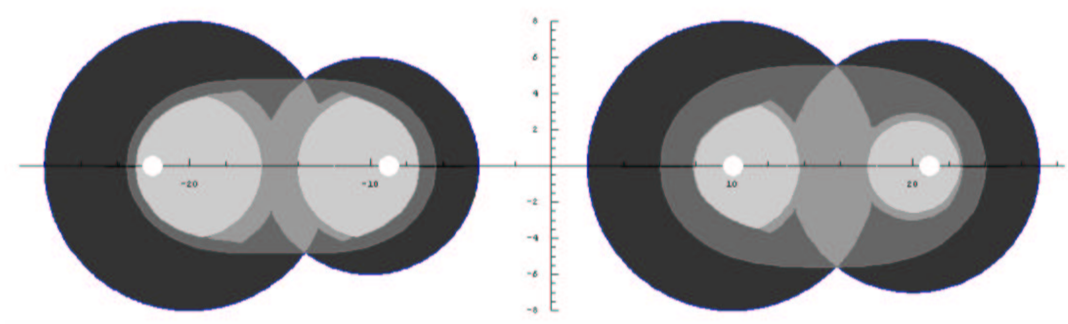


FIG. 4.1. *Considered localization sets referring to the matrix $G$.*

**4. Numerical example.** Finally, we give an example of possible improvement in the eigenvalue localization for a given matrix. For the matrix

$$G = \begin{bmatrix} 10 & 0 & 3 & 5 \\ 0 & -10 & 2 & 4 \\ 2 & 5 & 20 & 0 \\ 4 & 4 & 0 & -20 \end{bmatrix},$$

Fig. 4.1 shows the sets $\Gamma(G)$, $\mathcal{K}(G)$, $\mathcal{D}(G)$and $\mathcal{C}(G) := \bigcap_{S \subset N} C^S(G)$ of (1.2), (1.6), (3.6) and (2.9) respectively, shaded decreasingly. Exact eigenvalues are marked by white dots.

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