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A POLYNOMIAL COLLOCATION METHOD FOR CAUCHY SINGULAR INTEGRAL EQUATIONS OVER THE INTERVAL*

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Abstract. In this paper we consider a polynomial collocation method for the numerical solution of a singular integral equation over the interval. More precisely, the operator of our integral equation is supposed to be of the form $aI + \mu^{-1}bS\mu I + K$ with S the Cauchy integral operator, with piecewise continuous coefficients a and b, with a regular integral operator K, and with a Jacobi weight μ. To the equation $[aI + \mu^{-1}bS\mu I + K]u = f$ we apply a collocation method, where the collocation points are the Chebyshev nodes of the second kind and where the trial space is the space of polynomials multiplied by another Jacobi weight. For the stability and convergence of this collocation in weighted L^2 spaces, we derive necessary and sufficient conditions.

Key words. Cauchy singular integral equation; polynomial collocation method; stability.

AMS subject classifications. 45L10; 65R20; 65N38.

1. Introduction.

Discretization schemes including collocation based on polynomial approximation are the most popular numerical methods for the numerical solution of the Cauchy singular integral equations (cf. e.g. [1, 2, 4, 6, 7, 8, 9, 13, 18] and [28, Chapter 9]). These methods are based on well-known invariance properties for polynomial spaces with respect to the integral operators if the latter are multiplied by a correctly chosen weight function. Thus polynomial methods are spectral methods and exhibit optimal convergence properties.

On the other hand, the mentioned methods are restricted to integral operators the coefficients of which satisfy some smoothness properties. Moreover, the construction of the weight functions, of the orthogonal polynomials, and of the collocation nodes is not so simple if the coefficients of the integral operators are not constant. Therefore, it is natural to use Chebyshev nodes even if the intrinsic weight function of the operator is different from the Chebyshev weight. Moreover, if additional fixed singularities occur, the invariance property holds only for the Cauchy singular part and not for the whole operator. Consequently, the usual approximation arguments do not apply, and, there is no motivation to choose complicated weights. Furthermore, iterative methods with integral equations, the coefficient functions of which change in every step of iteration, suggest to choose fixed collocation nodes independently of the coefficient functions (cf. [15]). In comparison to spline methods or trigonometric approaches, numerical experiments for various equations (cf. [24, 25]) promise better approximation results for the polynomial collocation.

Polynomial methods have been considered for Mellin convolution operators e.g. in [21, 23, 25]. These methods at least together with slight modifications are expected to converge for all invertible operators. Similarly, for properly chosen weight functions and the corresponding nodes, the invertibility of the Cauchy singular integral operator is the only condition needed to ensure the stability of the polynomial collocation. However, if the collocation nodes are chosen independently of the intrinsic weights, then there arise additional stability conditions expressed in form of the invertibility of related operators (cf. the special case treated in [17, 19]). Local principles and Banach algebra techniques are the main tools to prove such results.

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In this paper we analyze the polynomial collocation method for an integral equation of the type

$$a(x)u(x) + \frac{b(x)}{\mu(x)} \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{\mu(y)u(y)}{y-x}\,\mathrm{d}y + \int_{-1}^{1} k(x,y)u(y)\,\mathrm{d}y = f(x)\,, \tag{1.1}$$

$-1 < x < 1$, where $a, b : [-1,1] \longrightarrow \mathbb{C}$ stand for given piecewise continuous[1] coefficient functions, where the weight function μ is defined by $\mu(x) := (1-x)^{\gamma}(1+x)^{\delta}$ with real numbers $-1 < \gamma, \delta < 1$, where the kernel $k : (-1,1) \times (-1,1) \mapsto \mathbb{C}$ is supposed to be continuous, where the right-hand side function f is given in the weighted L^2 space $\mathbf{L}^2_\sigma$, and where u stands for the unknown solution. The Hilbert space $\mathbf{L}^2_\sigma$ is defined as the space of all functions $u : (-1,1) \longrightarrow \mathbb{C}$ which are square integrable with respect to the weight $\sigma(x) := v^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta},\ -1 < \alpha, \beta < 1$. The inner product of this space is defined by

$$\langle u, v \rangle_\sigma := \int_{-1}^{1} u(x)\overline{v(x)}\sigma(x)\,\mathrm{d}x$$

and the norm by $\|u\|_\sigma := \sqrt{\langle u, u \rangle_\sigma}$. Note that the condition $-1 < \alpha, \beta < 1$ for the exponents of the classical Jacobi weight guarantees that the singular integral operator $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is continuous, i.e. $S \in \mathcal{L}(\mathbf{L}^2_\sigma)$ (see [11, Theorem I.4.1]). In short operator notation (1.1) takes the form

$$Au := (aI + b\mu^{-1}S\mu I + K)u = f\,. \tag{1.2}$$

Here $aI : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ denotes the multiplication operator defined by $(au)(x) := a(x)u(x)$, the operator $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is the Cauchy singular integral operator given by

$$(Su)(x) := \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{u(y)}{y-x}\,\mathrm{d}y\,,$$

and $K : \mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma$ stands for the integral operator with kernel $k(x,y)$.

For the numerical solution of the singular integral equation (1.2), we consider the polynomial collocation method

$$a(x^{\varphi}_{jn})u_n(x^{\varphi}_{jn}) + \frac{b(x^{\varphi}_{jn})}{\mu(x^{\varphi}_{jn})} \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{\mu(y)u_n(y)}{y - x^{\varphi}_{jn}}\,\mathrm{d}y + \int_{-1}^{1} k(x^{\varphi}_{jn}, y)u_n(y)\,\mathrm{d}y = f(x^{\varphi}_{jn}), \quad j = 1, \ldots, n\,, \tag{1.3}$$

where the collocation points $x^{\varphi}_{jn} := \cos\dfrac{j\pi}{n+1}$, $j = 1, \ldots, n$, are the Chebyshev nodes of the second kind corresponding to the weight function $\varphi(x) := \sqrt{1-x^2}$ and where the trial function u_n is sought in the space of all functions $u_n = \vartheta p_n$ with p_n a polynomial of degree less than n and with the Jacobi weight $\vartheta := v^{\frac{1}{4}-\frac{\alpha}{2}, \frac{1}{4}-\frac{\beta}{2}}$. To formulate our main result on the convergence of the method (1.3), we have to write it in the operator form

$$A_n u_n = M_n f, \quad u_n \in \operatorname{im} L_n. \tag{1.4}$$

Here L_n denotes the orthogonal projection of $\mathbf{L}^2_\sigma$ onto the n dimensional trial space $\operatorname{im} L_n$ of polynomials multiplied by ϑ. By M_n we denote the interpolation projection defined by

[1] For definiteness, we assume that the function values coincide with the limits from the left and that the functions are continuous at the point -1. The set of piecewise continuous functions on $[-1,1]$ is denoted by $\mathbf{PC}$.

$M_n f \in \operatorname{im} L_n$ and $(M_n f)(x^{\varphi}_{jn}) = f(x^{\varphi}_{jn}),\ j = 1,\dots,n$. Finally, the discretized integral operator $A_n : \operatorname{im} L_n \to \operatorname{im} L_n$ is given by $A_n := M_n A|_{\operatorname{im} L_n}$. In accordance with e.g. [28, Chapter 1], we call the collocation method stable if the operators A_n are invertible at least for sufficiently large n and if the norms of the inverse operators A_n^{-1} are bounded uniformly with respect to n. Of course, the norm is the operator norm in the space $\operatorname{im} L_n$ if the last is equipped with the restriction of the $\mathbf{L}^2_\sigma$ norm. We call the collocation method (1.4) convergent if, for any right-hand side $f \in \mathbf{L}^2_\sigma$ and for any approximating sequence f_n with $\|f - f_n\|_\sigma \to 0$, the approximate solutions u_n obtained by solving $A_n u_n = f_n$ converge to the exact solution u of (1.2) in the norm of $\mathbf{L}^2_\sigma$. Note that the stability implies bounded condition numbers for the matrix representation of A_n in a convenient basis, and, together with the consistency relation $A_n L_n \to A$, it implies the convergence.

To formulate our main result, we need some notation and a few assumptions. For the exponents in the weight functions μ and σ, we suppose

$$-1 < \alpha - 2\gamma < 1, \quad -1 < \beta - 2\delta < 1, \tag{1.5}$$

$$\alpha_0 := \gamma + \frac{1}{4} - \frac{\alpha}{2} \neq 0, \quad \beta_0 := \delta + \frac{1}{4} - \frac{\beta}{2} \neq 0\,. \tag{1.6}$$

Note that condition (1.5) ensures the boundedness of the integral operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ whereas (1.6) is needed to derive strong limits for the discrete operators in Lemma 3.10. Furthermore, we introduce the numbers

$$\kappa_\pm := \frac{1}{2\pi} \arg \frac{a(\pm 1) \mp b(\pm 1)}{a(\pm 1) \pm b(\pm 1)} \in \left(-\frac{1}{2} - \epsilon_\pm, \frac{1}{2} - \epsilon_\pm \right), \tag{1.7}$$

where

$$\epsilon_+ := \frac{\alpha}{2} - \gamma\,, \quad \epsilon_- := \frac{\beta}{2} - \delta\,.$$

For the definition of $\kappa_\pm$, instead of $(-1/2 - \epsilon_\pm, 1/2 - \epsilon_\pm)$ any interval of length one can be used. Our choice, however, is natural since the invertibility of the operator $A : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ implies $\kappa_- \neq \pm 1/2 - \epsilon_-$ and $\kappa_+ \neq \pm 1/2 - \epsilon_+$.

In the subsequent analysis, we will show that there exist limit operators of the matrices corresponding to the linear systems (1.3). These operators $W_\omega\{A_n\},\ \omega = 3,4$ will be introduced in the Lemmata 3.8 and 3.10, and the invertibility of $W_\omega\{A_n\},\ \omega = 3,4$ will turn out to be necessary for the stability of the collocation method. The condition for $W_\omega\{A_n\},\ \omega = 3,4$ to be Fredholm and to have a vanishing index can be expressed by the condition

$$\left| \kappa_\pm + \frac{1}{4} \right| < \frac{1}{2}. \tag{1.8}$$

Using the just introduced notation, the main result is

THEOREM 1.1. *Suppose that the conditions (1.5) and (1.6) are satisfied, that the coefficient functions a and b are piecewise continuous over $[0,1]$, and that the kernel function $k(x,y)$ divided by $(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}$ is continuous on $[-1,1]\times[-1,1]$ (or satisfies the weaker assumption met in Corollary 2.6). The polynomial collocation method (1.3) for the approximate solution of (1.1) is stable and convergent if and only if*

i) the operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ is invertible,

ii) the condition (1.8) holds,

iii) the null spaces $\ker W_3\{A_n\}$ *and* $\ker W_4\{A_n\}$ *are trivial.*

Unfortunately, the verification of the condition iii) seems to be hopeless. Therefore, it is good to know that condition iii) is not "essential" in the following sense: The case that condition ii) is fulfilled but condition iii) not is very rare and exceptional. Indeed, we get

REMARK 1.2. *Fix* $b(\pm 1)$, μ, *and* σ. *Consider the set* $\Sigma_\pm$ *of all complex numbers* $z = a(\pm 1)$ *such that condition ii) holds. The set of points in* $\Sigma_\pm$ *such that condition iii) is violated is countable and the accumulation points belong to* $\mathbb{C}\setminus\Sigma_\pm$. *This fact is a simple consequence of the general theory of analytic families of Fredholm operators.*

REMARK 1.3. *If the exceptional case should occur, then the numerical method should be modified slightly. One way to do this is the so called* i_* *modification introduced in [12] and used also e.g. in [20, 28]. For a stability proof of such a modified method, we refer to [28], Sections 11.30 and 12.46.*

REMARK 1.4. *In the particular case of singular integral operators* $A = aI + b\mu^{-1}S\mu I$ *with* $\mu(x) := \sqrt{\sigma(x)\varphi(x)}$ *the condition iii) is satisfied whenever condition ii) holds. Indeed, in this case the operators* $\mathbf{A}^\mu_\pm$ *are zero (cf. the subsequent Lemma 3.10) and [28], Theorem 11.19 applies. Different proofs of this fact can be found in [17, 19, 31] (cf. also [14, Cor. 3.3]).*

REMARK 1.5. *It is not hard to see that the investigation of the collocation method can be restricted to the case where* $\sigma(x)$ *is the Chebyshev weight of the first kind. Indeed, if* $L^0_n : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$, $u \mapsto \sum_{k=0}^{n-1} \langle u, \widetilde{u}^0_k\rangle_{\varphi^{-1}} \widetilde{u}^0_k$, $\widetilde{u}^0_k = \varphi U_k$, *and* $M^0_n = \varphi L^\varphi_n \varphi^{-1} I$ *(for the definition of* U_k *and* L^φ_n, *cf. Section 2), then the stability of* $M_n A L_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ *is equivalent to the stability of* $M^0_n \rho A \rho^{-1} L^0_n : \operatorname{im} L^0_n \longrightarrow \operatorname{im} L^0_n$, *where* $\rho(x)$ *is equal to* $(1-x)^{\frac{1}{4}+\frac{\alpha}{2}}(1+x)^{\frac{1}{4}+\frac{\beta}{2}}$ *(cf. [14], Cor. 3.3). Nevertheless, we retain the notation introduced above. This does not cause additional technical difficulties and is important for further generalizations. Moreover, a wider class of kernels for the operator* K *can be treated.*

REMARK 1.6. *Another goal of the present paper is to prepare a subsequent paper devoted to polynomial collocation for Cauchy singular integral equations with perturbation kernels having fixed singularities ([16]). These further results enable the application of transformation techniques to improve the convergence rate. In comparison to the corresponding spline methods, we expect smaller constants in the error estimates for the polynomial collocation and, consequently, faster convergence.*

The remainder of the present paper is devoted to the proof of Theorem 1.1. To show stability and convergence of (1.4), we shall apply a general technique due to Roch and Silbermann (cf. e.g. [30] and [28, Sections 10.31-10.41]). We shall introduce a special Banach algebra $\mathcal{F}$ of sequences of discretized operators such that the stability of a sequence is equivalent to the invertibility of four limit operators and to the invertibility of a corresponding coset in a suitable quotient algebra $\mathcal{F}/\mathcal{J}$ (cf. Section 2). In particular, the collocation sequence $\{A_n\}$ will be shown to be an element of the algebra $\mathcal{F}$ (cf. Section 3). To show the invertibility of the corresponding element in the quotient algebra, we shall introduce a subalgebra $\mathcal{A}/\mathcal{J}$ of this quotient algebra (cf. Section 4) and a subalgebra in the center of $\mathcal{A}/\mathcal{J}$ (cf. Section 5), and, using the local principle of Allan and Douglas (cf. Theorem 5.2), we shall reduce the invertibility to localized problems. These local invertibility problems will be solved in the Sections 6 and 7. Concerning the invertibility of the limit operators, we shall show in Section 8 that the invertibility of the four limit operators is just the stability criterion in Theorem 1.1.

Finally, we note that the setting for the proof enables the treatment of equations (1.1) including kernel functions k with fixed singularities of Mellin convolution type. We will analyze these classes of equations in a subsequent paper. Having solved the stability and convergence problems for singular integral equations with and without fixed singularities,

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the next essential task is to design algorithms for the assembling of the matrix of the corresponding collocation equations and for the efficient solution of the arising linear systems of equations. These issues will be stressed in future work.

2. Stability Reformulated as the Invertibility in a Banach Algebra. In this section we introduce the Banach algebra of approximate operators together with some auxiliary notation. We formulate the theorem of Roch and Silbermann on the stability of operator sequences in this algebra. This theorem is based on several assumptions which will be verified for our special application while introducing the setting.

For the definition of the algebra, we need some new spaces and operator sequences defined with the help of special basis functions. By T_n and U_n, $n = 0, 1, 2, \ldots$, we denote the normalized Chebyshev polynomials

$$T_0 := \sqrt{\frac{1}{\pi}}\,, \quad T_n(\cos s) := \sqrt{\frac{2}{\pi}}\cos ns\,, \quad n = 1, 2, \ldots\,,$$

and

$$U_n(\cos s) := \sqrt{\frac{2}{\pi}}\frac{\sin(n+1)s}{\sin s}\,, \quad n = 0, 1, 2, \ldots\,,$$

of first and second kind, respectively. In particular, the U_n are orthogonal polynomials with respect to the Chebyshev weight of second kind $\varphi(x)$, and the points x^{φ}_{jn} are the zeros of U_n. We set

$$\widetilde{u}_n(x) := \vartheta(x)U_n(x)\,, \quad n = 0, 1, 2, \ldots\,,$$

with $\vartheta := \sqrt{\sigma^{-1}\varphi} = v^{\frac{1}{4}-\frac{\alpha}{2},\frac{1}{4}-\frac{\beta}{2}}$. Then the solution of (1.4) can be represented by

$$u_n(x) = \sum_{k=0}^{n-1} \xi_{kn}\widetilde{u}_k(x),$$

and, with respect to the orthonormal system $\{\widetilde{u}_n\}_{n=0}^{\infty}$ in $\mathbf{L}^2_\sigma$, the orthogonal projection L_n takes the form

$$L_n u = \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k\rangle_\sigma\, \widetilde{u}_k.$$

The projection M_n is the weighted interpolation operator $M_n := \vartheta L^{\varphi}_n \vartheta^{-1} I$, where L^{φ}_n denotes the polynomial interpolation operator with respect to the nodes x^{φ}_{jn}, $j = 1, \ldots, n$. By ℓ^2 we denote the Hilbert space of all square summable sequences $\xi := \{\xi_k\}_{k=0}^{\infty}$ of complex numbers equipped with the inner product $\langle \xi, \eta\rangle_{\ell^2} := \sum_{k=0}^{\infty} \xi_k\overline{\eta_k}$. Finally, we introduce the Christoffel numbers with respect to the weight $\varphi(x)$ by

$$\lambda^{\varphi}_{kn} := \frac{\pi[\varphi(x^{\varphi}_{kn})]^2}{n+1}\,, \quad k = 1, \ldots, n\,,$$

and the discrete weights

$$\omega_{kn} := \sqrt{\frac{\pi}{n+1}\,\varphi(x^{\varphi}_{kn})\sigma(x^{\varphi}_{kn})} = \sqrt{\frac{\pi}{n+1}}\, v^{\frac{1}{4}+\frac{\alpha}{2},\frac{1}{4}+\frac{\beta}{2}}(x^{\varphi}_{kn})\,.$$

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Now we are in the position to define the four limit operators. We introduce the index set $T := \{1,2,3,4\}$, and, for $\omega \in T$, we define projections $L_n^{(\omega)}$ on the Hilbert spaces $\mathbf{X}_\omega$ and operators $E_n^{(\omega)} : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n^{(\omega)}$. The limit operators (belonging to $\mathcal{L}(\mathbf{X}_\omega)$) of the sequence $\{A_n\}$ are the strong limits $W_\omega\{A_n\} := \lim_{n\to\infty} E_n^{(\omega)} A_n (E_n^{(\omega)})^{-1} L_n^{(\omega)}$. In particular, we define the spaces $\mathbf{X}_\omega$, the projections $L_n^{(\omega)}$, and the operators $E_n^{(\omega)}$ by $\mathbf{X}_1 := \mathbf{X}_2 := \mathbf{L}^2_\sigma$, $\mathbf{X}_3 := \mathbf{X}_4 := \ell^2$, $L_n^{(1)} := L_n^{(2)} := L_n$, $L_n^{(3)} := L_n^{(4)} := P_n$, $E_n^{(1)} := L_n$, $E_n^{(2)} := W_n$, $E_n^{(3)} := V_n$, $E_n^{(4)} := \widetilde{V}_n$, where

$$\begin{aligned} W_n u &:= \sum_{k=0}^{n-1} \langle u, \widetilde{u}_{n-1-k}\rangle_\sigma \, \widetilde{u}_k \,, \\ P_n\{\xi_0,\xi_1,\xi_2,\ldots\} &:= \{\xi_0,\ldots,\xi_{n-1},0,0,\ldots\}\,, \\ V_n u &:= \{\omega_{1n}u(x^\varphi_{1n}),\ldots,\omega_{nn}u(x^\varphi_{nn}),0,0,\ldots\}\,, \\ \widetilde{V}_n u &:= \{\omega_{nn}u(x^\varphi_{nn}),\ldots,\omega_{1n}u(x^\varphi_{1n}),0,0,\ldots\}\,. \end{aligned} \tag{2.1}$$

The operators involved in the last definitions have the following important properties. Immediately from the definitions, we conclude that $(E_n^{(1)})^{-1} = L_n$, $(E_n^{(2)})^{-1} = W_n$ and

$$(E_n^{(3)})^{-1}\xi = \sum_{k=1}^n \frac{\xi_{k-1}}{\omega_{kn}}\,\widetilde{\ell}^\varphi_{kn}\,, \quad (E_n^{(4)})^{-1}\xi = \sum_{k=1}^n \frac{\xi_{n-k}}{\omega_{kn}}\,\widetilde{\ell}^\varphi_{kn}\,,$$

where

$$\widetilde{\ell}^\varphi_{kn}(x) := \frac{\vartheta(x)}{\vartheta(x^\varphi_{kn})}\,\ell^\varphi_{kn}(x) = \frac{\vartheta(x)U_n(x)}{\vartheta(x^\varphi_{kn})(x-x^\varphi_{kn})U_n'(x^\varphi_{kn})}\,.$$

The matrix of the operator W_n with respect to the interpolation basis $\{\widetilde{\ell}^\varphi_{kn}\}_{k=1}^n$ takes the form

$$E_n^{(3)} W_n (E_n^{(3)})^{-1} = \Big((-1)^{j+1}\delta_{k,j}\Big)_{k,j=1}^n. \tag{2.2}$$

Furthermore, the operators $E_n^{(\omega)}$ are isometries, i.e.

$$(E_n^{(\omega)})^* = (E_n^{(\omega)})^{-1}\,, \quad \omega \in T\,. \tag{2.3}$$

For $\omega = 1,2$, this is obvious. In case $\omega = 3$ we have, for $u = \vartheta v \in \operatorname{im} L_n$ and $\xi \in \operatorname{im} P_n$,

$$\begin{aligned} \langle V_n u, \xi\rangle_{\ell^2} &= \sum_{k=1}^n \omega_{kn} u(x^\varphi_{kn})\overline{\xi_{k-1}} = \sum_{k=1}^n \sqrt{\frac{\pi}{n+1}}\,\varphi(x^\varphi_{kn}) v(x^\varphi_{kn})\overline{\xi_{k-1}} \\ &= \sum_{k=1}^n \lambda^\varphi_{kn} \frac{v(x^\varphi_{kn})}{\sqrt{\frac{\pi}{n+1}}\varphi(x^\varphi_{kn})}\,\overline{\xi_{k-1}} = \left\langle v, \sum_{k=1}^n \frac{\xi_{k-1}}{\sqrt{\frac{\pi}{n+1}}\varphi(x^\varphi_{kn})}\,\ell^\varphi_{kn}\right\rangle_\varphi \\ &= \left\langle u, \sum_{k=1}^n \frac{\xi_{k-1}\vartheta(x^\varphi_{kn})}{\sqrt{\frac{\pi}{n+1}}\varphi(x^\varphi_{kn})}\,\widetilde{\ell}^\varphi_{kn}\right\rangle_\sigma = \left\langle u, V_n^{-1}\xi\right\rangle_\sigma\,. \end{aligned}$$

Analogously, we get (2.3) for the case $\omega = 4$. Finally, we observe the property

LEMMA 2.1. *The sequences* $\left\{E_n^{(\omega_1)}(E_n^{(\omega_2)})^{-1}L_n^{(\omega_2)}\right\}$ *converge weakly to zero for all indices* $\omega_1,\omega_2 \in T$ *with* $\omega_1 \neq \omega_2$.

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Proof. We prove the weak convergence to zero for all operator sequences outside the main diagonal of the following table.

$\omega_1 \backslash \omega_2$	1	2	3	4
1	L_n	W_nL_n	$V_n^{-1}P_n$	$\widetilde{V}_n^{-1}P_n$
2	W_nL_n	L_n	$W_nV_n^{-1}P_n$	$W_n\widetilde{V}_n^{-1}P_n$
3	V_nL_n	$V_nW_nL_n$	P_n	$V_n\widetilde{V}_n^{-1}P_n$
4	$\widetilde{V}_nL_n$	$\widetilde{V}_nW_nL_n$	$\widetilde{V}_nV_n^{-1}P_n$	P_n

Table of $E_n^{(\omega_1)}(E_n^{(\omega_2)})^{-1}L_n^{(\omega_2)}$

First we remark that all sequences in this table are uniformly bounded. Thus, the weak convergence of W_nL_n follows from

$$\langle f, W_nL_n\widetilde{u}_m\rangle_\sigma = \langle f, \widetilde{u}_{n-1-m}\rangle_\sigma \longrightarrow 0, \quad n \longrightarrow \infty\,,$$

which holds for all $f \in \mathbf{L}^2_\sigma$ and $m \in \mathbb{N}$. Setting $e_m := \{\delta_{km}\}_{k=0}^\infty$, we get, for $n > \max\{m,j\}$,

$$\begin{aligned}\langle e_{j-1}, V_nL_n\widetilde{u}_m\rangle_{\ell^2} &= \omega_{jn}\widetilde{u}_m(x_{jn}^\varphi) = \sqrt{\frac{\pi}{n+1}}\varphi(x_{jn}^\varphi)U_m(x_{jn}^\varphi)\\ &= \sqrt{\frac{2}{n+1}}\sin\frac{(m+1)j\pi}{n+1}\,,\end{aligned}$$

and the weak convergence of V_nL_n follows. Analogously we proceed with $\widetilde{V}_nL_n$, $V_nW_nL_n$, and $\widetilde{V}_nW_n$. The weak convergence of $V_n^{-1}P_n$ follows from

$$\begin{aligned}\left\langle \widetilde{u}_m, V_n^{-1}P_ne_{j-1}\right\rangle_\sigma &= \frac{1}{\omega_{jn}}\left\langle \widetilde{u}_m, \widetilde{\ell}_{jn}^\varphi\right\rangle_\sigma = \frac{1}{\omega_{jn}\vartheta(x_{jn}^\varphi)}\left\langle U_m, \ell_{jn}^\varphi\right\rangle_\varphi\\ &= \frac{\lambda_{jn}^\varphi}{\omega_{jn}\vartheta(x_{jn}^\varphi)}U_m(x_{jn}^\varphi) = \sqrt{\frac{2}{n+1}}\sin\frac{(m+1)j\pi}{n+1}\end{aligned}$$

which is valid for $n > \max\{m,j\}$. Analogously we get

$$\left\langle \widetilde{u}_m, W_nV_n^{-1}P_ne_{j-1}\right\rangle_\sigma = \left\langle \widetilde{u}_{n-1-m}, V_n^{-1}P_ne_{j-1}\right\rangle_\sigma = \sqrt{\frac{2}{n+1}}\sin\frac{(n-m)j\pi}{n+1}\,.$$

The relation

$$\left\langle e_{m-1}, \widetilde{V}_nV_n^{-1}P_ne_{j-1}\right\rangle_{\ell^2} = \left\langle e_{m-1}, e_{n+1-j}\right\rangle_{\ell^2}$$

shows the weak convergence to zero of the sequence $\left\{\widetilde{V}_n V_n^{-1} P_n\right\}$. For the sequences $\left\{\widetilde{V}_n^{-1} P_n\right\}$, $\left\{W_n \widetilde{V}_n^{-1} P_n\right\}$, and $\left\{V_n \widetilde{V}_n^{-1} P_n\right\}$, we can proceed in an analogous way. $\square$

Next we define the algebra of operator sequences - the basic algebra for our further considerations. By $\mathcal{F}$ we denote the set of all sequences $\{A_n\} = \{A_n\}_{n=1}^{\infty}$ of linear operators $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$, for which there exist operators $W_\omega\{A_n\} \in \mathcal{L}(\mathbf{X}_\omega)$ such that, for all $\omega \in T$,

$$E_n^{(\omega)} A_n (E_n^{(\omega)})^{-1} L_n^{(\omega)} \longrightarrow W_\omega\{A_n\}, \quad \left(E_n^{(\omega)} A_n (E_n^{(\omega)})^{-1} L_n^{(\omega)}\right)^* \longrightarrow W_\omega\{A_n\}^*$$

holds in the sense of strong convergence for $n \to \infty$. If we define

$$\lambda_1\{A_n\} + \lambda_2\{B_n\} := \{\lambda_1 A_n + \lambda_2 B_n\}, \; \{A_n\}\{B_n\} := \{A_n B_n\}, \; \{A_n\}^* := \{A_n^*\},$$

and

$$\|\{A_n\}\|_{\mathcal{F}} := \sup\left\{\|A_n L_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} : n = 1, 2, \ldots\right\},$$

then it is not hard to see that $\mathcal{F}$ becomes a C^*-algebra with unit element $\{L_n\}$. From Lemma 2.1 and (2.3) we conclude (cf. [28, Lemma 10.34])

COROLLARY 2.2. *For all $\omega \in T$ and all compact operators $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$, the sequences $\left\{A_n^{(\omega)}\right\} = \left\{(E_n^{(\omega)})^{-1} L_n^{(\omega)} T_\omega E_n^{(\omega)}\right\}$ belong to $\mathcal{F}$, and, for $\omega_1 \neq \omega_2$, we get the strong limits*

$$E_n^{(\omega_1)} A_n^{(\omega_2)} (E_n^{(\omega_1)})^{-1} L_n^{(\omega_1)} \longrightarrow 0, \quad \left(E_n^{(\omega_1)} A_n^{(\omega_2)} (E_n^{(\omega_1)})^{-1} L_n^{(\omega_1)}\right)^* \longrightarrow 0.$$

Using Corollary 2.2, we define the subset $\mathcal{J} \subset \mathcal{F}$ of all sequences of the form

$$\sum_{\omega=1}^{4} \left\{(E_n^{(\omega)})^{-1} L_n^{(\omega)} T_\omega E_n^{(\omega)}\right\} + \{C_n\},$$

where $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$ and where $\{C_n\}$ is in the ideal $\mathcal{N} \subset \mathcal{F}$ of all sequences $\{C_n\}$ tending to zero in norm, i.e. of all sequences with $\|C_n L_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \longrightarrow 0$. Now, the following theorem is crucial for our stability and convergence analysis.

THEOREM 2.3 ([28], Theorem 10.33). *The set $\mathcal{J}$ forms a two-sided closed ideal of $\mathcal{F}$. A sequence $\{A_n\} \in \mathcal{F}$ is stable if and only if the operators $W_\omega\{A_n\} : \mathbf{X}_\omega \longrightarrow \mathbf{X}_\omega$, $\omega \in T$, are invertible and if the coset $\{A_n\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$.*

In addition to the operator sequences corresponding to the collocation method applied to compact operators, the sequences of quadrature discretizations of integral operators with continuous kernels are contained in $\mathcal{J}$, too. Indeed, we can formulate the following lemma.

LEMMA 2.4. *Suppose the function $k(x,y)/\rho(y)$, where $\rho = \sqrt{\sigma\varphi} = \vartheta^{-1}\varphi$, is continuous on $[-1,1] \times [-1,1]$ and that K is the integral operator with kernel $k(x,y)$. Then $\{M_n K L_n\} \in \mathcal{J}$. Moreover, if the approximations $K_n \in \mathcal{L}(\operatorname{im} L_n)$ are defined by*

$$K_n = (E_n^{(3)})^{-1} \left(\frac{\pi}{n+1}\, \rho(x^\varphi_{(i+1)n}) k(x^\varphi_{(i+1)n}, x^\varphi_{(j+1)n}) \vartheta(x^\varphi_{(j+1)n})\right)_{i,j=0}^{n-1} E_n^{(3)},$$

then the norms of the operators $K_n - L_n K|_{\operatorname{im} L_n}$ tend to zero, and $\{K_n\}$ is in $\mathcal{J}$.

Proof. The operators K_n can be written as $M_n \widetilde{K}_n$, where

$$\left(\widetilde{K}_n u_n\right)(x) = \int_{-1}^{1} \varphi(y) L_n^\varphi \left[k(x,.)\varphi^{-1} u_n\right](y)\, \mathrm{d}y.$$

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Obviously, due to the Arzelà-Ascoli theorem the operator $K : \mathbf{L}^2_\sigma \longrightarrow \mathbf{C}[-1,1]$ is compact. Hence, $\lim_{n\to\infty} \|M_n K L_n - L_n K L_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} = 0$ (see Lemma 2.5 below) and it is sufficient to show that $\lim_{n\to\infty} \left\|\widetilde{K}_n L_n - K L_n\right\|_{\mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{C}[-1,1])} = 0$. To this end, we consider an arbitrary $u \in \mathbf{L}^2_\sigma$ and get $L_n u = \vartheta p_n$, where p_n is a certain polynomial of degree less than n. By $k_n(x,y)$ we refer to the best uniform approximation to $k(x,y)/\rho(y)$ in the space of polynomials with degree less than n in both variables. Due to the exactness of the Gauß rule we have

$$\left(\widetilde{K}_n L_n u\right)(x) = \int_{-1}^{1} \varphi(y) L_n^\varphi[k(x,.)\rho^{-1}](y) p_n(y)\,\mathrm{d}y,$$

and so

$$\begin{aligned}
\left|\left(\widetilde{K}_n L_n u - K L_n u\right)(x)\right| &= \left|\int_{-1}^{1} \varphi(y)\Big(L_n^\varphi[k(x,.)\rho^{-1}](y) - k(x,y)/\rho(y)\Big) p_n(y)\,\mathrm{d}y\right| \\
&\le \left|\sum_{j=1}^{n} \lambda_{jn}^\varphi \left[k(x,x_{jn}^\varphi)/\rho(x_{jn}^\varphi) - k_n(x,x_{jn}^\varphi)\right] p_n(x_{jn}^\varphi)\right| \\
&\qquad + \left|\int_{-1}^{1} \varphi(y)\left[k_n(x,y) - k(x,y)/\rho(y)\right] p_n(y)\,\mathrm{d}y\right| \\
&\le C_n \left(\sqrt{\sum_{j=1}^{n} \lambda_{jn}^\varphi \left|p_n(x_{jn}^\varphi)\right|^2} + \|p_n\|_\varphi\right) = 2C_n \|p_n\|_\varphi \\
&= 2C_n \|L_n u\|_\sigma
\end{aligned}$$

with $C_n := \|k(x,y)/\rho(y) - k_n(x,y)\|_\infty \|1\|_\varphi$ and $\lim_{n\to\infty} C_n = 0$. $\square$

LEMMA 2.5 ([19], Lemma 3.1). *If the function $f : (-1,1) \longrightarrow \mathbb{C}$ is locally Riemann integrable and if, for some $\chi > 0$,*

$$|f(x)| \le C\, v^{\chi - \frac{1+\alpha}{2}, \chi - \frac{1+\beta}{2}}(x), \quad -1 < x < 1,$$

then $\lim_{n\to\infty} \|M_n f - f\|_\sigma = 0$ and

$$\|M_n f\|_\sigma \le C \sup\{|f(x) v^{-\chi + \frac{1+\alpha}{2}, -\chi + \frac{1+\beta}{2}}(x)| : -1 < x < 1\}.$$

COROLLARY 2.6. *Due to Lemma 2.5 the condition on $k(x,y)$ in Lemma 2.4 can be relaxed. In fact, it is sufficient to assume that $v^{\frac{1+\alpha}{2}-\chi, \frac{1+\beta}{2}-\chi}(x) k(x,y)/\rho(y)$ is continuous on $[-1,1] \times [-1,1]$ for some $\chi > 0$, and the assertion of Lemma 2.4 remains true.*

3. The Operator Sequence of the Collocation Method as an Element of the Banach Algebra $\mathcal{F}$. We have to show that the sequence of discretized operators $A_n := M_n A|_{\operatorname{im} L_n}$ is an element of $\mathcal{F}$. At first we summarize some well-known results (cf. the Lemmata 3.1–3.6 and Remark 3.7) which will be needed in the following. We start with recalling the well-known relations between the Chebyshev polynomials of first and second kind

(3.1) $$S\varphi U_n = \mathrm{i} T_{n+1}, \quad n = 0,1,2,\dots,$$

and

(3.2) $$T_{n+1} = \frac{1}{2}(U_{n+1} - U_{n-1}), \quad n = 0,1,2,\dots, \quad U_{-1} \equiv 0.$$

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LEMMA 3.1 ([26], Theorem 9.25). *Suppose μ and ν are classical Jacobi weights with $\mu\nu \in \mathbf{L}^1(-1,1)$, and fix $j \in \mathbb{N}$. Then, for each polynomial $q(x)$ with $\deg q \le jn$,*

$$\sum_{k=1}^{n} \lambda_{kn}^{\mu} \, |q(x_{kn}^{\mu})| \, \nu(x_{kn}^{\mu}) \le C \int_{-1}^{1} |q(x)| \mu(x) \nu(x) \, \mathrm{d}x \,,$$

where the constant C does not depend on n and q.

Now we consider an η with $0 < \eta \le 1$. By $\mathbf{C}^{0,\eta} := \mathbf{C}^{0,\eta}[-1,1]$ we denote the Banach space of all Hölder continuous functions $f : [-1,1] \longrightarrow \mathbb{C}$ with respect to the exponent η. The norm in this space is defined by

$$\|f\|_{\mathbf{C}^{0,\eta}} := \|f\|_\infty + \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|^\eta} : x,y \in [-1,1],\, x \ne y \right\},$$

where $\|f\|_\infty := \sup\{|f(x)| : -1 \le x \le 1\}$.

LEMMA 3.2 ([31], Lemma 4.13). *If $w \in \mathbf{C}^{0,\eta}$ with $\eta > \frac{1}{2}[1+\max\{\alpha,\beta,0\}]$, then the commutator $wS - SwI$ belongs to $\mathcal{K}(\mathbf{L}^2_\sigma, \mathbf{C}^{0,\lambda})$ for some $\lambda > 0$.*

LEMMA 3.3 ([28], Proposition 9.7, Theorem 9.9). *Assume that $a, b \in \mathbf{C}^{0,\eta}$ are real valued functions, where $\eta \in (0,1)$ and $[a(x)]^2 + [b(x)]^2 > 0$ for $x \in [-1,1]$. Furthermore, assume that the integers $\lambda_\pm$ satisfy the relations*

$$\alpha_0 := \lambda_+ + g(1) \in (-1,1) \quad \text{and} \quad \beta_0 := \lambda_- - g(-1) \in (-1,1)\,,$$

where $g : [-1,1] \longrightarrow \mathbb{R}$ is a continuous function such that

$$a(x) - \mathrm{i}b(x) = \sqrt{[a(x)]^2 + [b(x)]^2}\, e^{\mathrm{i}\pi g(x)}\,.$$

Then there exists a positive function $w \in \mathbf{C}^{0,\eta}$ such that, for each polynomial p of degree n, the function $av^{\alpha_0,\beta_0}wp + \mathrm{i}Sbv^{\alpha_0,\beta_0}wp$ is a polynomial of degree $n-\kappa$, where $\kappa = -\lambda_+ - \lambda_-$ and where, by definition, a polynomial of negative degree is identically zero.

Suppose $\gamma, \delta \ge 0$. By $\mathbf{C}_{\gamma,\delta}$ we denote the Banach space of all continuous functions $f : (-1,1) \longrightarrow \mathbb{C}$, for which $v^{\gamma,\delta} f$ is continuous over $[-1,1]$. Moreover, by $\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}$ we refer to the Banach space of all functions f such that $v^{\alpha,\beta} f$ belongs to $\mathbf{L}^p(-1,1)$. The norms in $\mathbf{C}_{\gamma,\delta}$ and $\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}$ are defined by

$$\|f\|_{\gamma,\delta,\infty} := \left\|v^{\gamma,\delta} f\right\|_\infty, \quad \|f\|_{\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}} := \left\|v^{\alpha,\beta} f\right\|_{\mathbf{L}^p(-1,1)}.$$

We introduce the operator $T_{\gamma,\delta}$ by

$$(T_{\gamma,\delta} u)(x) := \int_{-1}^{1} \left[1 - \frac{v^{\gamma,\delta}(y)}{v^{\gamma,\delta}(x)}\right] \frac{u(y)}{y-x} \, \mathrm{d}y\,, \quad -1 < x < 1.$$

LEMMA 3.4 ([14], Corollary 4.4). *If*

$$p > 2\,, \quad \gamma, \delta \in \left(-\frac{1}{4}, -\frac{1}{p}\right) \cup \left(\frac{1}{p}, 1 - \frac{1}{2p}\right), \quad 0 < \chi < \min\left\{\frac{1}{4} - \frac{1}{2p}, \frac{1}{4} + \gamma, \frac{1}{4} + \delta\right\},$$

then the operator $T_{\gamma,\delta} : \widetilde{\mathbf{L}}^p_{v^{\gamma-\frac{1}{2p},\delta-\frac{1}{2p}}} \longrightarrow \mathbf{C}_{\gamma+\frac{1}{4}-\chi,\delta+\frac{1}{4}-\chi}$ is compact.

LEMMA 3.5 ([14], (2.9)). *The sequence $\{W_n\}$ converges weakly to 0 in the space $\widetilde{\mathbf{L}}^p_\psi$ with $\psi = v^{\frac{1}{4}+\frac{\alpha}{2}-\frac{1}{2p},\frac{1}{4}+\frac{\beta}{2}-\frac{1}{2p}}$.*

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LEMMA 3.6 ([22], relation after Theorem 3.1). *Suppose* $\omega \in \mathbf{L}^2$ *is a Jacobi weight and* $f : (-1,1) \to \mathbb{C}$ *is a function satisfying* $\omega\varphi^{-1}, \omega^{-1}\varphi \in \mathbf{L}^2$ *and* $f\omega, f'\varphi\omega \in \mathbf{L}^2$. *Then the polynomial interpolation projection* L_n^φ *based on the Chebyshev nodes of the second kind satisfies the error estimate*

$$\|\omega(L_n^\varphi f - f)\|_{\mathbf{L}^2} \leq C\, n^{-1}\, \|f'\varphi\omega\|_{\mathbf{L}^2} \ .$$

Finally, we will use the following special case of Lebesgue's dominated convergence theorem.

REMARK 3.7. *If* $\xi, \eta \in \ell^2$, $\xi^n = \{\xi_k^n\}$, $|\xi_k^n| \leq |\eta_k|$ *for all* $k = 0,1,2,\ldots$ *and for all* $n \geq n_0$, *and if* $\lim_{n\to\infty} \xi_k^n = \xi_k$ *for all* $k = 0,1,2,\ldots$, *then* $\lim_{n\to\infty} \|\xi^n - \xi\|_{\ell^2} = 0$.

Now, for the singular integral operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ (cf. (1.2)), we show that the sequence $\{M_n A L_n\}$ belongs to the algebra $\mathcal{F}$, and we compute $W_\omega\{A_n\}$. We prove this fact separately for multiplication operators, for the singular integral operator $\mu^{-1}S\mu$ with a special weight μ, and for $\mu^{-1}S\mu$ with a general μ.

LEMMA 3.8. *Let* $a \in \mathbf{PC}$, $A = aI$, *and* $A_n = M_n a L_n$. *Then* $\{A_n\} \in \mathcal{F}$, *where*

$$W_n A_n W_n = M_n a L_n \quad \text{and} \quad A_n^* = M_n \overline{a} L_n \,, \tag{3.3}$$

which implies $(W_n A_n W_n)^* = M_n \overline{a} L_n$ *and* $W_1\{A_n\} = W_2\{A_n\} = A$. *Moreover,*

$$W_3\{A_n\} = a(1)I \quad \text{and} \quad W_4\{A_n\} = a(-1)I \,. \tag{3.4}$$

Proof. Since the operators $E_n^{(3)} : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n^{(3)}$ are unitary, the system $\{\frac{1}{\omega_{kn}}\widetilde{\ell}_{kn}^\varphi : k = 1,\ldots n\}$ forms an orthonormal basis in $\operatorname{im} L_n$. However, with respect to this Lagrange interpolation basis the matrix of the discretized multiplication operator and its adjoint take the form

$$E_n^{(3)} M_n a L_n (E_n^{(3)})^{-1} = \left(a(x^\varphi_{(k+1)n})\delta_{j,k} \right)_{j,k=0}^{n-1} , \tag{3.5}$$

$$E_n^{(3)} (M_n a L_n)^* (E_n^{(3)})^{-1} = \left(\overline{a}(x^\varphi_{(k+1)n})\delta_{j,k} \right)_{j,k=0}^{n-1} . \tag{3.6}$$

From this representation as diagonal operators and the diagonal representation (2.2) of W_n, we get $W_n M_n a L_n W_n = M_n a L_n$ and the uniform boundedness of the sequence $\{M_n a L_n\}$

$$\|M_n a L_n\|_{\mathcal{L}(L^2_\sigma)} \leq C\, \|a\|_\infty \,. \tag{3.7}$$

This uniform boundedness together with the convergence properties of M_n (cf. Lemma 2.5) implies the convergences $M_n a L_n \longrightarrow aI$, $(M_n a L_n)^* = M_n \overline{a} L_n \longrightarrow \overline{a} I$ as well as $W_n M_n a L_n W_n L_n = M_n a L_n \longrightarrow aI$, $(W_n M_n a L_n W_n)^* L_n = M_n \overline{a} L_n \longrightarrow \overline{a} I$. The limits in (3.4) follow easily from (3.5). Similarly, the adjoints to the operators in (3.4) are the limits of the sequences of adjoint operators due to (3.6). □

LEMMA 3.9. *Suppose* $A = \rho^{-1}S\rho I$, *where* $\rho = \vartheta^{-1}\varphi = \sqrt{\sigma\varphi}$ *and* $A_n = M_n A L_n$. *Then* $\{A_n\} \in \mathcal{F}$ *and*

$$W_1\{A_n\} = A\,, \quad W_2\{A_n\} = -A\,, \quad W_3\{A_n\} = A_+\,, \quad W_4\{A_n\} = A_-$$

with

$$A_\pm := \left(\pm \frac{2(k+1)}{\pi \mathrm{i}[(j+1)^2 - (k+1)^2]}\,\widetilde{\delta}_{jk} \right)_{j,k=0}^\infty , \quad \widetilde{\delta}_{jk} := \begin{cases} 2 & \text{if } k \equiv j+1 \bmod 2\,, \\ 0 & \text{if } k \equiv j \bmod 2\,. \end{cases}$$

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Proof. At first we prove the uniform boundedness of the sequence $\{A_n\}$. From (3.1) it follows that $S\rho u_n$ is a polynomial with a degree of at most n for $u_n \in \operatorname{im} L_n$. Hence we can apply Lemma 3.1 together with the boundedness of the operator $S : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ and obtain, for $u_n \in \operatorname{im} L_n$,

$$\left\|M_n\rho^{-1}S\rho u_n\right\|^2_\sigma = \left\|L^\varphi_n\varphi^{-1}S\rho u_n\right\|^2_\varphi = \sum_{k=1}^n \lambda^\varphi_{kn}\varphi^{-2}(x^\varphi_{kn})\,|(S\rho u_n)(x^\varphi_{kn})|^2$$
$$\le C\int_{-1}^1 |(S\rho u_n)(x)|^2\,\varphi^{-1}(x)\,\mathrm{d}x \le C\,\|\rho u_n\|^2_{\varphi^{-1}} = C\,\|u_n\|^2_\sigma\,.$$

Again with the help of (3.1) as well as with the help of Lemma 2.5 we see that, for fixed m and for $n > m$ tending to ∞,

$$M_n\rho^{-1}S\rho\widetilde{u}_m = \mathrm{i}M_n\rho^{-1}T_{m+1} \longrightarrow \mathrm{i}\rho^{-1}T_{m+1} = \rho^{-1}S\rho\widetilde{u}_m$$

in $\mathbf{L}^2_\sigma$. If we additionally take into account (3.2) and (3.3), then we also get

$$W_nM_n\rho^{-1}S\rho W_n\widetilde{u}_m = \frac{\mathrm{i}}{2}W_nM_n\rho^{-1}\vartheta^{-1}W_nW_n(\widetilde{u}_{n-m} - \widetilde{u}_{n-m-2}) = -\mathrm{i}M_n\rho^{-1}T_{m+1}$$
$$= -M_n\rho^{-1}S\rho\widetilde{u}_m \longrightarrow -\rho^{-1}S\rho\widetilde{u}_m$$

in $\mathbf{L}^2_\sigma$. The well-known Poincaré-Bertrand commutation formula implies that, for $u \in \mathbf{L}^2_\sigma$ and $v \in \mathbf{L}^2_{\sigma^{-1}}$,

$$\langle Su, v\rangle = \langle u, Sv\rangle\,, \tag{3.8}$$

where $\langle .,.\rangle$ denotes the $\mathbf{L}^2(-1,1)$ inner product without weight. Consequently, the adjoint operator of $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is equal to $\sigma^{-1}S\sigma I : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$. Again, taking into account that $S\rho L_nu$ is a polynomial with a degree of at most n (cf. (3.1)), we conclude, for $u, v \in \mathbf{L}^2_\sigma$,

$$\left\langle M_n\rho^{-1}S\rho L_nu, v\right\rangle_\sigma = \left\langle L^\varphi_n\varphi^{-1}S\rho L_nu, \vartheta^{-1}L_nv\right\rangle_\varphi$$
$$= \sum_{k=1}^n \lambda^\varphi_{kn}(\varphi\vartheta)^{-1}(x^\varphi_{kn})(S\rho L_nu)(x^\varphi_{kn})(L_nv)(x^\varphi_{kn})$$
$$= \left\langle S\rho L_nu, L^\varphi_n(\varphi\vartheta)^{-1}L_nv\right\rangle_\varphi = \left\langle S\rho L_nu, \vartheta M_n\varphi^{-1}L_nv\right\rangle_\sigma$$
$$= \left\langle u, L_n\vartheta S\rho M_n\varphi^{-1}L_nv\right\rangle_\sigma\,.$$

Hence, since $L_n\vartheta S\rho L_n = M_n\vartheta S\rho L_n$,

$$(M_n\rho^{-1}S\rho L_n)^* = M_n\vartheta S\rho M_n\varphi^{-1}L_n = M_n\varphi L_nM_n\rho^{-1}S\rho L_nM_n\varphi^{-1}L_n\,. \tag{3.9}$$

In view of Lemma 2.5 we obtain the $\mathbf{L}^2_\sigma$ convergence $M_n\varphi^{-1}\widetilde{u}_m \longrightarrow \varphi^{-1}\widetilde{u}_m$ for each fixed $m = 0,1,2,\ldots$, and the strong convergence of $(M_n\rho^{-1}S\rho L_n)^*$ follows from the strong convergence of $M_n\varphi L_n$ and $M_n\rho^{-1}S\rho L_n$. From (3.9) we also get the $\mathbf{L}^2_\sigma$ convergence

$$(W_nM_n\rho^{-1}S\rho W_n)^* = -M_n\varphi L_nM_n\rho^{-1}S\rho L_nM_n\varphi^{-1}L_n \longrightarrow \vartheta S\vartheta^{-1}I = -(\rho^{-1}S\rho I)^*\,.$$

The limit relations for $W_\omega\{A_n\},\ \omega = 1, 2$ are proved.

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To get $W_3\{A_n\}$, we compute the matrix of A_n with respect to the basis $\{\widetilde{\ell}^{\varphi}_{k,n}\}$. With the help of (3.1) and $U_n'(x^{\varphi}_{kn}) = \sqrt{2/\pi}\,(-1)^{k+1}(n+1)/[\varphi(x^{\varphi}_{kn})]^2$ we compute, for $x \in (-1,1)\setminus\{x^{\varphi}_{kn}\}$,

$$\begin{aligned}\left(A\widetilde{\ell}^{\varphi}_{kn}\right)(x) &= \frac{1}{\rho(x)\vartheta(x^{\varphi}_{kn})}\frac{1}{\pi\mathrm{i}}\int_{-1}^{1}\frac{U_n(y)\varphi(y)\,\mathrm{d}y}{(y-x)(y-x^{\varphi}_{kn})U_n'(x^{\varphi}_{kn})}\\ &= \frac{1}{\rho(x)\vartheta(x^{\varphi}_{kn})U_n'(x^{\varphi}_{kn})}\,\frac{1}{x^{\varphi}_{kn}-x}\,\frac{1}{\pi\mathrm{i}}\int_{-1}^{1}\left(\frac{1}{y-x^{\varphi}_{kn}}-\frac{1}{y-x}\right)U_n(y)\varphi(y)\,\mathrm{d}y\\ &= \sqrt{\frac{\pi}{2}}\,\frac{(-1)^{k+1}}{n+1}\,\frac{\rho(x^{\varphi}_{kn})\varphi(x^{\varphi}_{kn})}{\rho(x)}\,\frac{\mathrm{i}}{x^{\varphi}_{kn}-x}\left[T_{n+1}(x^{\varphi}_{kn})-T_{n+1}(x)\right].\end{aligned}$$

In particular, we obtain

$$\left(A\widetilde{\ell}^{\varphi}_{kn}\right)(x^{\varphi}_{jn}) = \frac{\rho(x^{\varphi}_{kn})\varphi(x^{\varphi}_{kn})}{\mathrm{i}(n+1)\rho(x^{\varphi}_{jn})}\,\frac{\widetilde{\delta}_{jk}}{x^{\varphi}_{kn}-x^{\varphi}_{jn}}.$$

Hence, for $n > m$,

$$\begin{aligned}E^{(3)}_n A_n (E^{(3)}_n)^{-1} L^{(3)}_n e_{m-1} &= \left\{\frac{\omega_{jn}}{\omega_{mn}}\left(A\widetilde{\ell}^{\varphi}_{mn}\right)(x^{\varphi}_{jn})\right\}_{j=1}^{n}\\ &= \left\{\frac{\varphi(x^{\varphi}_{mn})}{\mathrm{i}(n+1)}\,\frac{\widetilde{\delta}_{jm}}{x^{\varphi}_{mn}-x^{\varphi}_{jn}}\right\}_{j=1}^{n},\end{aligned} \tag{3.10}$$

where we have taken into account that $\omega_{jn} = \sqrt{\frac{\pi}{n+1}}\rho(x^{\varphi}_{jn})$ and $T'_{n+1}(x) = (n+1)U_n(x)$ (cf. the computation of the diagonal entries). Now we observe that, for fixed k and j with $k \ne j$ and for $n \to \infty$,

$$\frac{\varphi(x^{\varphi}_{kn})}{n+1}\,\frac{1}{x^{\varphi}_{kn}-x^{\varphi}_{jn}} = \frac{\sin\frac{k\pi}{n+1}}{2(n+1)\,\sin\frac{k+j}{2(n+1)}\pi\,\sin\frac{j-k}{2(n+1)}\pi} \longrightarrow \frac{2k}{\pi(j^2-k^2)} \tag{3.11}$$

and, for fixed k and $j = 1,\ldots,n$, $j \ne k$, and $n > 2k$,

$$\frac{\varphi(x^{\varphi}_{kn})}{n+1}\,\frac{1}{|x^{\varphi}_{kn}-x^{\varphi}_{jn}|} \le \frac{\frac{k\pi}{n+1}}{2(n+1)\,\frac{2\sqrt{2}}{3\pi}\,\frac{k+j}{2(n+1)}\pi\,\frac{2}{\pi}\,\frac{|j-k|}{2(n+1)}\pi} = \frac{3\pi k}{2\sqrt{2}|j^2-k^2|}, \tag{3.12}$$

and the same for fixed j and $k = 1,\ldots,n$, $k \ne j$, and $n > 2j$. Using (3.11) and (3.12) together with Remark 3.7, we see that

$$\begin{aligned}&\left\|E^{(3)}_n A_n (E^{(3)}_n)^{-1} L^{(3)}_n e_{m-1} - P_n A_+ e_{m-1}\right\|_{\ell^2}\\ &\quad= \sum_{j=1}^{n}\left|\frac{\varphi(x^{\varphi}_{mn})}{n+1}\,\frac{1}{x^{\varphi}_{mn}-x^{\varphi}_{jn}} - \frac{2m}{\pi(j^2-m^2)}\right|^2 \widetilde{\delta}_{jm} \longrightarrow 0, \quad n \longrightarrow \infty\end{aligned}$$

and

$$\left\|\left(E^{(3)}_n A_n (E^{(3)}_n)^{-1} L^{(3)}_n\right)^{*} e_{m-1} - P_n A_+^{*} e_{m-1}\right\|_{\ell^2}$$

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$$= \sum_{j=1}^{n} \left| \frac{\varphi(x_{jn}^{\varphi})}{n+1} \frac{1}{x_{mn}^{\varphi} - x_{jn}^{\varphi}} - \frac{2j}{\pi(j^2 - m^2)} \right|^2 \widetilde{\delta}_{jm} \longrightarrow 0, \quad n \longrightarrow \infty .$$

The case $\omega = 4$ can be treated analogously. □

Now we deal with the general operator $\mu^{-1} S \mu I$ and the corresponding operator sequence $\{M_n \mu^{-1} S \mu L_n\}$ of the collocation method, where $\mu = v^{\gamma,\delta}$ and where we assume (1.5) – (1.6).

LEMMA 3.10. *Suppose $A = \mu^{-1} S \mu I$ and $A_n = M_n A L_n$, where $\mu = v^{\gamma,\delta}$ satisfies (1.5) and (1.6). Then $\{A_n\} \in \mathcal{F}$ and*

(3.13)
$$W_1\{A_n\} = A\,, \qquad W_2\{A_n\} = -\rho^{-1} S \rho I\,,$$
$$W_3\{A_n\} = A_+ + \mathbf{A}_+^{\mu}\,, \quad W_4\{A_n\} = A_- + \mathbf{A}_-^{\mu}\,.$$

Here $\rho := \vartheta^{-1}\varphi$ and $A_\pm$ are the same as in Lemma 3.9, and

(3.14)
$$\mathbf{A}_\pm^{\mu} := \pm \mathbf{B}_\pm \pm \mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1} \mp \mathbf{A} \mp \mathbf{D}_\pm \mathbf{A} \mathbf{W} \mathbf{V}_\pm \mathbf{D}_\pm^{-1} \pm \mathbf{V}_\pm \mathbf{A} \mathbf{W}$$

with

(3.15)
$$\mathbf{A} := \left(\frac{2(k+1)(1-\delta_{j,k})}{\pi \mathrm{i}[(j+1)^2 - (k+1)^2]} \right)_{j,k=0}^{\infty},$$

(3.16)
$$\mathbf{D}_\pm := \left((k+1)^{2\chi_\pm} \delta_{k,j} \right)_{k,j=0}^{\infty},$$
$$\chi_+ := \frac{1}{4} + \frac{\alpha}{2} - \gamma\,, \quad \chi_- := \frac{1}{4} + \frac{\beta}{2} - \delta\,,$$
$$\mathbf{B}_\pm := \left(b_{(k+1)}^{\pm} \delta_{k,j} \right)_{k,j=0}^{\infty},$$

(3.17)
$$\mathbf{V}_\pm := \left(d_{(k+1)}^{\pm} \delta_{k,j} \right)_{k,j=0}^{\infty}, \quad \mathbf{W} := \left(\frac{(-1)^{(k+1)}}{\sqrt{2\pi}} \delta_{k,j} \right)_{k,j=0}^{\infty}.$$

Moreover, choosing $\zeta_\pm := -\chi_\pm$, the $b_k^\pm$ and $d_k^\pm$ are defined by

(3.18)
$$b_k^{\pm} := \frac{4(-1)^{k+1}k}{\mathrm{i}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1}{[(k\pi)^2 - s^2]^2}\, s \,\sin s \,\mathrm{d}s\,,$$

(3.19)
$$d_k^{\pm} := 2\sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1}{(k\pi)^2 - s^2}\, s \sin s \,\mathrm{d}s$$
$$+4\sqrt{\frac{2}{\pi}} \int_{\frac{\pi}{2}}^{\infty} \cos s \left\{ \frac{s^2\left[\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1\right]}{[(k\pi)^2 - s^2]^2} + \frac{\zeta_\pm \left(\frac{s}{k\pi}\right)^{2\zeta_\pm} + \frac{1}{2}\left[\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1\right]}{(k\pi)^2 - s^2} \right\} \mathrm{d}s\,.$$

Proof. **i)** First we check the strong convergence of A_n towards $W_1\{A_n\}$. We choose integers $\lambda_\pm$ such that $\alpha_0 - \lambda_+$ and $\lambda_- - \beta_0$ are in $(-1, 0)$ (cf. (1.6)). Moreover, by $g(x)$ we denote a polynomial with degree of at most 1 such that $g(1) = \alpha_0 - \lambda_+$ and $g(-1) = \lambda_- - \beta_0$. Then, $\hat{a}(x) := -\cot[\pi g(x)]$ is a continuous function on $[-1, 1]$ and $\hat{a}(x) - \mathrm{i} = \sqrt{[\hat{a}(x)]^2 + 1}\, e^{\mathrm{i}\pi g(x)}$. By Lemma 3.3, there exists a positive function $w \in \bigcap_{\eta \in (0,1)} \mathbf{C}^{0,\eta}$

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such that $(\hat{a}I + \mathrm{i}S)\mu w u_n$ is a polynomial of degree less than $n-\kappa$ for each $u_n \in \operatorname{im} L_n$, where $\kappa = -\lambda_+ - \lambda_-$. Now we use the decomposition

$$\mu^{-1}S\mu I = \mathrm{i}\hat{a}I - \mathrm{i}(\mu w)^{-1}(\hat{a}I + \mathrm{i}S)\mu w I + (\mu w)^{-1}(wS - SwI)\mu I \tag{3.20}$$

to prove the uniform boundedness of the sequence $\{M_n\mu^{-1}S\mu L_n\}$. The uniform boundedness of $\{M_n\hat{a}L_n\}$ follows from Lemma 3.8. Taking into account Lemma 3.1 and the boundedness of $S : \mathbf{L}^2_{v^{\alpha-2\gamma,\beta-2\delta}} \longrightarrow \mathbf{L}^2_{v^{\alpha-2\gamma,\beta-2\delta}}$ we get, for $u_n \in \operatorname{im} L_n$ and $q_n = (\hat{a}I+\mathrm{i}S)\mu w u_n$,

$$\begin{aligned}\left\|M_n(\mu w)^{-1}q_n\right\|^2_\sigma &= \left\|L_n^\varphi(\vartheta\mu w)^{-1}q_n\right\|^2_\varphi = \sum_{k=1}^n \lambda^\varphi_{kn}[(\vartheta\mu w)(x^\varphi_{kn})]^{-2}|q_n(x^\varphi_{kn})|^2\\ &\le C\int_{-1}^1 \frac{\varphi(x)}{[\vartheta(x)\mu(x)]^2}\,|q_n(x)|^2\,\mathrm{d}x\\ &\le C\int_{-1}^1 \frac{\varphi(x)}{[\vartheta(x)\mu(x)]^2}\,|\mu(x)u_n(x)|^2\,\mathrm{d}x = C\,\|u_n\|^2_\sigma\,,\end{aligned}$$

which proves the uniform boundedness of the second term in (3.20) corresponding to the collocation method. To handle the third term we set $H_w := wS - SwI$. Due to (1.5), we obtain the inequality $\frac{1}{2}[1+\max\{\alpha-2\gamma,\beta-2\delta,0\}] < 1$. Thus, in view of Lemma 3.2, we have $H_w \in \mathcal{K}(\mathbf{L}^2_{\sigma\mu^{-2}})$ which implies $\mu^{-1}H_w\mu I \in \mathcal{K}(\mathbf{L}^2_\sigma)$. Moreover, choosing a $\chi > 0$ such that

$$\chi < \min\left\{\frac{1+\alpha}{2}-\gamma, \frac{1+\alpha}{2}, \frac{1+\beta}{2}-\delta, \frac{1+\beta}{2}\right\}$$

and applying the Lemmata 2.5 and 3.2, we get $\{(M_n - L_n)w^{-1}\mu^{-1}H_w\mu L_n\} \in \mathcal{N}$ and, consequently,

$$\{M_n w^{-1}\mu^{-1}H_w\mu L_n\} \in \mathcal{J}\,. \tag{3.21}$$

Using the decomposition (3.20) together with (1.5) and Lemma 2.5, we infer that, for each fixed $m = 0,1,2,\ldots$,

$$M_n\mu^{-1}S\mu L_n\widetilde{u}_m \longrightarrow \mu^{-1}S\mu\widetilde{u}_m$$

holds in $\mathbf{L}^2_\sigma$. The strong convergence of $\{A_n\}$ to $W_1\{A_n\}$ is shown. Now we prove the strong convergence of the sequence $\{A_n^*\}$, which obviously is uniformly bounded. To show that $A_n^*\widetilde{u}_m$ converges for each $m = 0,1,2,\ldots$ we again use the decomposition (3.20), where the first and the third term are already covered by Lemma 3.8 and by (3.21), respectively. For $u, v \in \mathbf{L}^2_\sigma$ and $q_n = (\hat{a}I + \mathrm{i}S)\mu w L_n u$ we compute

$$\begin{aligned}\left\langle M_n\mu^{-1}(\hat{a}I+\mathrm{i}S)\mu w L_n u, L_n v\right\rangle_\sigma &= \left\langle L_n^\varphi(\vartheta\mu)^{-1}q_n, \vartheta^{-1}L_n v\right\rangle_\varphi\\ &= \sum_{j=1}^n \lambda^\varphi_{jn}\Big((\vartheta\mu)^{-1}q_n\Big)(x^\varphi_{jn})(\vartheta^{-1}L_n v)(x^\varphi_{jn})\\ &= \left\langle q_n, L_n^\varphi(\vartheta\mu)^{-1}\vartheta^{-1}L_n v\right\rangle_\varphi\\ &= \left\langle \vartheta\mu(\hat{a}I+\mathrm{i}\mu^{-1}S\mu)wL_n u, M_n(\vartheta\mu)^{-1}L_n v\right\rangle_\sigma\\ &= \left\langle u, L_n w(\hat{a}I+\mathrm{i}\mu^{-1}S\mu I)^*\vartheta\mu M_n(\vartheta\mu^{-1}L_n v\right\rangle_\sigma\,.\end{aligned}$$

Hence, $(M_n(\mu w)^{-1}(\hat{a}I+\mathrm{i}S)\mu w L_n)^* = L_n w(\hat{a}I+\mathrm{i}\mu^{-1}S\mu I)^*\vartheta\mu M_n(\vartheta\mu)^{-1}L_n M_n w^{-1}L_n$ and it remains to show that $\vartheta\mu M_n(\vartheta\mu)^{-1}\widetilde{u}_m \longrightarrow \widetilde{u}_m$ in $\mathbf{L}^2_\sigma$. For this we write

$$\left\|\vartheta\mu M_n(\vartheta\mu)^{-1}\widetilde{u}_m - \widetilde{u}_m\right\|_\sigma = \left\|\vartheta^2\mu\left[L_n^\varphi(\vartheta^2\mu)^{-1}\widetilde{u}_m - (\vartheta^2\mu)^{-1}\widetilde{u}_m\right]\right\|_\sigma$$
$$= \left\|\mu\varphi\sigma^{-\frac{1}{2}}\left[L_n^\varphi(\vartheta^2\mu)^{-1}\widetilde{u}_m - (\vartheta^2\mu)^{-1}\widetilde{u}_m\right]\right\|_{\mathbf{L}^2}$$

and remark that for $\omega = \mu\varphi\sigma^{-\frac{1}{2}}$ and $f = (\vartheta^2\mu)^{-1}\widetilde{u}_m$ the conditions of Lemma 3.6 are fulfilled.

ii) Since $\{A_n\}$ is uniformly bounded, we need to show the existence of the strong limits $W_\omega\{A_n\}$ with $\omega \in \{2,3,4\}$ for a complete system of functions, only. At first we prove the limit $W_2\{A_n\} = -\rho^{-1}S\rho I$. We write

$$\mu^{-1}S\mu I = \rho^{-1}S\rho I + \mu^{-1}K\mu I \tag{3.22}$$

with $K := S - \rho^{-1}\mu S\rho\mu^{-1}I$. Moreover, for $p \geq 2$, we set

$$\psi := v^{\frac{1}{4}+\frac{\alpha}{2}-\frac{1}{2p},\frac{1}{4}+\frac{\beta}{2}-\frac{1}{2p}}, \quad \widetilde{\psi} := \mu^{-1}\psi = v^{\frac{1}{4}+\frac{\alpha}{2}-\gamma-\frac{1}{2p},\frac{1}{4}+\frac{\beta}{2}-\delta-\frac{1}{2p}}.$$

By assumption (1.5) we have $-\frac{1}{4} < \frac{1}{4}+\frac{\alpha}{2}-\gamma < \frac{3}{4}$, $-\frac{1}{4} < \frac{1}{4}+\frac{\beta}{2}-\delta < \frac{3}{4}$, and together with (1.6) we can apply Lemma 3.4 for sufficiently large p and sufficiently small $\chi > 0$ to conclude the compactness of

$$K_\mu := \mu^{-1}K\mu I : \widetilde{\mathbf{L}}^p_\psi \stackrel{\mu I}{\longrightarrow} \widetilde{\mathbf{L}}^p_{\widetilde{\psi}} \stackrel{K}{\longrightarrow} \mathbf{C}_{\frac{1+\alpha}{2}-\gamma-\chi,\frac{1+\beta}{2}-\delta-\chi} \stackrel{\mu^{-1}I}{\longrightarrow} \mathbf{C}_{\frac{1+\alpha}{2}-\chi,\frac{1+\beta}{2}-\chi}. \tag{3.23}$$

Using the decomposition (3.22) together with Lemma 3.9, it remains to prove that the functions $W_n M_n K_\mu W_n \widetilde{u}_m$ converges to zero in $\mathbf{L}^2_\sigma$ for each fixed $m = 0,1,2,\ldots$. As a consequence of Lemma 2.5 and the compactness of the operator (3.23) we get

$$\lim_{n\to\infty}\|(M_n - I)K_\mu\|_{\widetilde{\mathbf{L}}^p_\psi\to\mathbf{L}^2_\sigma} = 0$$

for some $p > 2$. Together with the uniform boundedness of $W_n : \widetilde{\mathbf{L}}^p_\psi \longrightarrow \widetilde{\mathbf{L}}^p_\psi$ (see Lemma 3.5) this leads to

$$\lim_{n\to\infty}\|W_n(M_n - I)K_\mu W_n\|_{\widetilde{\mathbf{L}}^p_\psi\to\mathbf{L}^2_\sigma} = 0\,.$$

Again Lemma 3.5 and the compactness of the operator (3.23) imply, for some $p > 2$,

$$\lim_{n\to\infty}\|W_n K_\mu W_n u\|_\sigma = 0, \quad u \in \widetilde{\mathbf{L}}^p_\psi\,.$$

It remains to remark that $\widetilde{u}_m \in \widetilde{\mathbf{L}}^p_\psi$ for all $p \geq 1$.

To handle the strong convergence of $\left\{(W_n M_n\mu^{-1}S\mu W_n)^*\right\}$, we first consider sequences of the form $\left\{M_n b_0 b\mu^{-1}S\mu L_n\right\}$, where $b_0 \in \mathbf{PC}$ and b is a differentiable function with $b' \in \mathbf{C}^{0,1}[-1,1]$ and $b(\pm 1) = b'(\pm 1) = 0$. We use the decomposition

$$b\mu^{-1}S\mu I = b\rho^{-1}S\rho I + \mu^{-1}(bS - SbI)\mu I + \mu^{-1}(Sb\mu\rho^{-1}I - b\mu\rho^{-1}S)\rho I$$
$$=: b\rho^{-1}S\rho I + K_1 + K_2\,.$$

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In the same way as for (3.21) one can show that $\{M_nK_jL_n\} \in \mathcal{J}$, $j = 1, 2$. With the help of the Lemmata 3.8 and 3.9 the inclusion $\left\{M_nb_0b\mu^{-1}S\mu L_n\right\} \in \mathcal{F}$ follows. Using this fact and the estimate (cf. (3.7))

$$\left\|M_n(b-\widetilde{b})\mu^{-1}S\mu L_n\right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} = \left\|M_n(b-\widetilde{b})L_nM_n\mu^{-1}S\mu L_n\right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \le C\,\|b-\widetilde{b}\|_\infty$$

we get $\left\{M_nb\mu^{-1}S\mu L_n\right\} \in \mathcal{F}$ for all $b \in \mathbf{PC}$ with $b(\pm 1) = 0$.

Now, choose δ with $0 < \delta < \min\{[-1-\alpha]/2, [-1-\beta]/2\}$. Then the function $f = v^{-\delta,-\delta}\widetilde{u}_m$ fulfills the conditions of Lemma 2.5 such that $M_nv^{-\delta,-\delta}\widetilde{u}_m \longrightarrow v^{-\delta,-\delta}\widetilde{u}_m$. By $W_nM_nv^{-\delta,-\delta}W_n = M_nv^{-\delta,-\delta}L_n$ (cf. (3.3)), we get

$$W_nM_n\mu^{-1}S\mu W_n = M_nv^{-\delta,-\delta}L_nW_nv^{\delta,\delta}\mu^{-1}S\mu W_n\,.$$

Hence, from the previous result and from Lemma 2.5 we conclude

$$\begin{aligned}\lim_{n\to\infty}(W_nM_n\mu^{-1}S\mu W_n)^*\widetilde{u}_m &= \lim_{n\to\infty}(W_nv^{\delta,\delta}\mu^{-1}S\mu W_n)^*M_nv^{-\delta,-\delta}\widetilde{u}_m\\ &= (v^{\delta,\delta}\mu^{-1}S\mu I)^*v^{-\delta,-\delta}\widetilde{u}_m\end{aligned}$$

in $\mathbf{L}^2_\sigma$. The strong convergence of $\left\{(W_nM_n\mu^{-1}S\mu W_n)^*\right\}$ is proved.

iii) Since the limit $W_4\{A_n\}$ can be derived analogously to $W_3\{A_n\}$, we restrict our further consideration to $W_3\{A_n\}$. To get this limit $W_3\{A_n\}$, we consider the structure of the corresponding matrix more closely. Setting $B := \mu^{-1}K\mu I = \mu^{-1}S\mu I - \rho^{-1}S\rho^{-1}I$ and $B_n := M_nBL_n$, we compute, for $x \ne x^\varphi_{kn}$,

$$\left(B\widetilde{\ell}^\varphi_{kn}\right)(x) = \frac{1}{\pi \mathrm{i}}\int_{-1}^{1}\left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\frac{\vartheta(y)U_n(y)\,\mathrm{d}y}{\vartheta(x^\varphi_{kn})(y-x)(y-x^\varphi_{kn})U_n'(x^\varphi_{kn})} \tag{3.24}$$

$$\begin{aligned}
&= \frac{1}{\pi \mathrm{i}}\,\frac{1}{x^\varphi_{kn}-x}\,\frac{1}{\vartheta(x^\varphi_{kn})U_n'(x^\varphi_{kn})}\left\{\int_{-1}^{1}\left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\frac{\vartheta(y)U_n(y)}{y-x^\varphi_{kn}}\,\mathrm{d}y\right.\\
&\qquad\left. - \int_{-1}^{1}\left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\frac{\vartheta(y)U_n(y)}{y-x}\,\mathrm{d}y\right\}\\
&= \frac{1}{x^\varphi_{kn}-x}\,\frac{1}{\vartheta(x^\varphi_{kn})U_n'(x^\varphi_{kn})}\left[\frac{\mu(x^\varphi_{kn})}{\mu(x)} - \frac{\rho(x^\varphi_{kn})}{\rho(x)}\right]\frac{1}{\rho(x^\varphi_{kn})}\left(S\varphi U_n\right)(x^\varphi_{kn})\\
&\quad + \frac{1}{\pi \mathrm{i}}\,\frac{1}{x^\varphi_{kn}-x}\,\frac{1}{\vartheta(x^\varphi_{kn})U_n'(x^\varphi_{kn})} \,*\\
&\quad * \left\{\int_{-1}^{1}\left(\left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\frac{1}{\rho(y)} - \left[\frac{\mu(x^\varphi_{kn})}{\mu(x)} - \frac{\rho(x^\varphi_{kn})}{\rho(x)}\right]\frac{1}{\rho(x^\varphi_{kn})}\right)\frac{\varphi(y)U_n(y)}{y-x^\varphi_{kn}}\,\mathrm{d}y\right.\\
&\qquad\left. - \frac{1}{\mu(x)}\int_{-1}^{1}\left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x)}{\rho(x)}\right]\frac{\varphi(y)U_n(y)}{y-x}\,\mathrm{d}y\right\}\\
&= \left[\frac{\mu(x^\varphi_{kn})}{\mu(x)} - \frac{\rho(x^\varphi_{kn})}{\rho(x)}\right]\frac{\varphi(x^\varphi_{kn})}{\mathrm{i}(n+1)}\,\frac{1}{x^\varphi_{kn}-x}\\
&\quad + \frac{1}{\pi \mathrm{i}}\,\frac{1}{x^\varphi_{kn}-x}\sqrt{\frac{\pi}{2}}\,\frac{\rho(x^\varphi_{kn})}{\mu(x)}\,\frac{\varphi(x^\varphi_{kn})(-1)^{k+1}}{n+1} \,*
\end{aligned}$$

$$* \left\{ \int_{-1}^{1} \left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x_{kn}^\varphi)}{\rho(x_{kn}^\varphi)} \right] \frac{\varphi(y)U_n(y)}{y - x_{kn}^\varphi} \,\mathrm{d}y - \int_{-1}^{1} \left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x)}{\rho(x)} \right] \frac{\varphi(y)U_n(y)}{y-x} \,\mathrm{d}y \right\}$$

$$= \left[\frac{\mu(x_{kn}^\varphi)}{\mu(x)} - \frac{\rho(x_{kn}^\varphi)}{\rho(x)} - \frac{1}{\sqrt{2\pi}} \frac{\mu(x_{kn}^\varphi)}{\mu(x)} (-1)^k d_k^n + \frac{1}{\sqrt{2\pi}} \frac{\rho(x_{kn}^\varphi)}{\rho(x)} (-1)^k d^n(x) \right] *$$

$$* \frac{\varphi(x_{kn}^\varphi)}{\mathrm{i}(n+1)} \frac{1}{x_{kn}^\varphi - x} ,$$

where

$$d^n(x) := \int_{-1}^{1} \left[\frac{\mu(y)\rho(x)}{\rho(y)\mu(x)} - 1 \right] \frac{\varphi(y)U_n(y)}{y-x} \,\mathrm{d}y, \quad d_k^n := d^n(x_{kn}^\varphi) .$$

Consequently, we get

$$E_n^{(3)} B_n (E_n^{(3)})^{-1} = \left(\frac{\omega_{(j+1)n}}{\omega_{(k+1)n}} \left(B\widetilde{\ell}_{(k+1)n}^\varphi \right) (x_{(j+1)n}^\varphi) \right)_{j,k=0}^{n-1}$$

(3.25)

$$= \mathbf{B}_n + \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1} - \mathbf{A}_n - \mathbf{D}_n \mathbf{A}_n \mathbf{W}_n \mathbf{V}_n \mathbf{D}_n^{-1} + \mathbf{V}_n \mathbf{A}_n \mathbf{W}_n$$

with

$$\mathbf{B}_n := \left(\left(B\widetilde{\ell}_{(j+1)n}^\varphi \right) (x_{(j+1)n}^\varphi) \delta_{k,j} \right)_{j,k=0}^{n-1} ,$$

$$\mathbf{A}_n := \left(\frac{\varphi(x_{(k+1)n}^\varphi)}{\mathrm{i}(n+1)} \frac{1 - \delta_{j,k}}{x_{(k+1)n}^\varphi - x_{(j+1)n}^\varphi} \right)_{j,k=0}^{n-1} ,$$

$$\mathbf{W}_n := \left(\frac{(-1)^{j+1}}{\sqrt{2\pi}} \delta_{k,j} \right)_{j,k=0}^{n-1} , \quad \mathbf{V}_n := \left(d_{j+1}^n \delta_{k,j} \right)_{j,k=0}^{n-1} ,$$

$$\mathbf{D}_n := \left(\frac{\rho(x_{(j+1)n}^\varphi)}{\mu(x_{(j+1)n}^\varphi)} \delta_{k,j} \right)_{j,k=0}^{n-1} ,$$

where the diagonal elements in $\mathbf{A}_n$ are equal to zero by definition. We have to show that, for any fixed $m = 1, 2, \ldots$, the sequences

$$\left\{ E_n^{(3)} B_n (E_n^{(3)})^{-1} L_n^{(3)} e_{m-1} \right\} \quad \text{and} \left\{ (E_n^{(3)} B_n (E_n^{(3)})^{-1} L_n^{(3)})^* e_{m-1} \right\}$$

converge in ℓ^2 to $A_+^\mu e_{m-1}$ and $(A_+^\mu)^* e_{m-1}$, respectively.

iv) Now we turn to the limits of $\mathbf{A}_n$ and $\mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}$. The convergence of $\mathbf{A}_n$ to $\mathbf{A}$ follows completely analogously to that of $E_n^{(3)} A_n (E_n^{(3)})^{-1}$ to A_+ in the proof of Lemma 3.9 (cf. (3.10)). Hence, we consider $\mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}$. We introduce $\chi(x) := \rho(x)[\mu(x)]^{-1} = (1-x)^{\chi_+}(1+x)^{\chi_-}$ with

(3.26)
$$\chi_+ := \frac{1}{4} + \frac{\alpha}{2} - \gamma, \quad \chi_- := \frac{1}{4} + \frac{\beta}{2} - \delta$$

and define

$$a_{jk}^{(n)} = \frac{\chi(x_{jn}^\varphi)}{\chi(x_{kn}^\varphi)} \frac{\varphi(x_{kn}^\varphi)}{\mathrm{i}(n+1)} \frac{1 - \delta_{j,k}}{x_{kn}^\varphi - x_{jn}^\varphi}$$

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$$= \left(\frac{\sin \frac{j\pi}{2(n+1)}}{\sin \frac{k\pi}{2(n+1)}} \right)^{2\chi_+} \left(\frac{\cos \frac{j\pi}{2(n+1)}}{\cos \frac{k\pi}{2(n+1)}} \right)^{2\chi_-} \frac{\sin \frac{k\pi}{n+1}}{\mathrm{i}(n+1)} \frac{1-\delta_{j,k}}{2\sin \frac{k+j}{2(n+1)}\pi \, \sin \frac{j-k}{2(n+1)}\pi} .$$

Then, the condition (1.5) is equivalent to

$$-\frac{1}{4} < \chi_\pm < \frac{3}{4} . \tag{3.27}$$

For fixed k and $j = 1, \dots, n$, $j \neq k$, and $n > 2k$, we have the estimate (comp. (3.12))

$$|a_{jk}^{(n)}| \leq C \left(\frac{j}{k} \right)^{2\chi_+} \left(\frac{1-\frac{j}{n+1}}{1-\frac{k}{n+1}} \right)^{2\chi_-} \frac{k}{|j^2-k^2|} ,$$

and the same for fixed j and $k = 1, \dots, n$, $k \neq j$, and $n > 2j$. Thus, for k fixed and $n > 3k$, we get

$$|a_{jk}^{(n)}| \leq \begin{cases} C\, j^{2(\chi_+-1)} & \text{if } j \leq (n+1)/2, \\ C \dfrac{\varphi(x_{kn}^\varphi)}{n} \dfrac{\chi(x_{jn}^\varphi)}{\chi(x_{kn}^\varphi)} & \text{if } j \geq (n+1)/2, \end{cases} \tag{3.28}$$

and, for j fixed and $n > 3j$,

$$|a_{jk}^{(n)}| \leq \begin{cases} C\, k^{-2\chi_+-1} & \text{if } k \leq (n+1)/2, \\ C \dfrac{\varphi(x_{kn}^\varphi)}{n} \dfrac{\chi(x_{jn}^\varphi)}{\chi(x_{kn}^\varphi)} & \text{if } k \geq (n+1)/2. \end{cases} \tag{3.29}$$

Moreover, for fixed j and k, $j \neq k$, we obtain

$$\lim_{n\to\infty} a_{jk}^{(n)} = \left(\frac{j}{k} \right)^{2\chi_+} \frac{2k}{\pi(j^2-k^2)} =: a_{jk}$$

(comp. (3.11)). Together with Remark 3.7 and (3.27) we conclude

$$\lim_{n\to\infty} \left(\sum_{j=1, j\neq k}^{[(n+1)/3]-1} |a_{jk}^{(n)} - a_{jk}|^2 + \sum_{j=[(n+1)/3], j\neq k}^{n} |a_{jk}^{(n)}|^2 + \sum_{j=[(n+1)/3]}^{\infty} |a_{jk}|^2 \right) = 0$$

and

$$\lim_{n\to\infty} \left(\sum_{k=1, k\neq j}^{[(n+1)/3]-1} |a_{jk}^{(n)} - a_{jk}|^2 + \sum_{k=[(n+1)/3], k\neq j}^{n+1} |a_{jk}^{(n)}|^2 + \sum_{k=[(n+1)/3]}^{\infty} |a_{jk}|^2 \right) = 0 ,$$

which imply the ℓ^2 convergences

$$\begin{aligned} \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1} e_{k-1} &\longrightarrow \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} e_{k-1} , \\ (\mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1})^* e_{j-1} &\longrightarrow (\mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1})^* e_{j-1} , \end{aligned} \tag{3.30}$$

where $\mathbf{A}$ and $\mathbf{D}_+$ are defined in (3.15) and (3.16), respectively.

v) Next we compute the limits $d_k^+ := \lim_{n\to\infty} d_k^n$ and $b_k^+ := \lim_{n\to\infty} b_k^n$, where we have set $b_k^n := \left(B\widetilde{\ell}_{kn}^{\varphi}\right)(x_{kn}^{\varphi})$. Note that, analogously, there exist the limits $d_k^- := \lim_{n\to\infty} d_{n+1-k}^n$ and $b_k^- := \lim_{n\to\infty} b_{n+1-k}^n$ which are needed for the limit $W_4\{A_n\}$. In particular, we shall show that, for some $\varepsilon \in (0,1)$,

$$b_k^+ = \lim_{n\to\infty} b_k^n, \quad |b_k^n| \le \frac{C}{\min\{k, n+1-k\}^{\varepsilon}}\,,\ k = 1,\dots,n\,,\ n = 1,2,\dots\,, \tag{3.31}$$

as well as

$$d_k^+ = \lim_{n\to\infty} d_k^n, \quad |d_k^n| \le \frac{C}{\min\{k, n+1-k\}^{\varepsilon}}\,,\ k = 1,\dots,n\,, n = 1,2,\dots\,. \tag{3.32}$$

We define $\zeta(x) = [\rho(x)]^{-1}\mu(x) = [\chi(x)]^{-1} = (1-x)^{\zeta_+}(1+x)^{\zeta_-}$ and $x_n^{\pm} = \pm\cos\frac{\pi}{2(n+1)}$. Using $T'_{n+1}(x) = (n+1)U_n(x)$, $T_{n+1}(x_n^{\pm}) = 0$, and partial integration, we get

$$\begin{aligned} d_k^n &= \frac{1}{\zeta(x_{kn}^{\varphi})}\int_{-1}^{1}\frac{\zeta(y)-\zeta(x_{kn}^{\varphi})}{y-x_{kn}^{\varphi}}\,\varphi(y)U_n(y)\,\mathrm{d}y \\ &= \frac{1}{\zeta(x_{kn}^{\varphi})}\left(\int_{-1}^{x_n^-}+\int_{x_n^+}^{1}\right)\frac{\zeta(y)-\zeta(x_{kn}^{\varphi})}{y-x_{kn}^{\varphi}}\,\varphi(y)U_n(y)\,\mathrm{d}y + \int_{x_n^-}^{x_n^+} F(y,x_{kn}^{\varphi})\,\mathrm{d}y \\ &=: d_{k,-}^{n,1} + d_{k,+}^{n,1} + d_k^{n,2}\,, \end{aligned}$$

where

$$F(y,x) := \frac{T_{n+1}(y)}{(n+1)\zeta(x)}\left[\varphi(y)\,\frac{\zeta(y)-\zeta(x)-\zeta'(y)(y-x)}{(y-x)^2} - \varphi'(y)\,\frac{\zeta(y)-\zeta(x)}{y-x}\right].$$

Consider $n \ge 2k-1$. The term $d_{k,-}^{n,1}$ can be estimated by

$$\begin{aligned} |d_{k,-}^{n,1}| &\le C\int_{-1}^{x_n^-}\left[\frac{(1+y)^{\zeta_-}}{(1-x_{kn}^{\varphi})^{\zeta_+}}+1\right]\mathrm{d}y \le C\left[\frac{(1+x_n^-)^{\zeta_-+1}}{(1-x_{kn}^{\varphi})^{\zeta_+}} + x_n^- + 1\right] \\ &\le C\left[\frac{\left(\frac{1}{n^2}\right)^{\zeta_-+1}}{\left(\frac{k}{n}\right)^{2\zeta_+}} + \frac{1}{n^2}\right] = C\left(\frac{1}{k^{2\zeta_+}n^{2(1+\zeta_--\zeta_+)}} + \frac{1}{n^2}\right), \end{aligned}$$

such that $\lim_{n\to\infty} d_{k,-}^{n,1} = 0$ and

$$|d_{k,-}^{n,1}| \le \frac{C}{k^{2\min\{1,1+\zeta_-\}}}\,. \tag{3.33}$$

To consider $d_{k,+}^{n,1}$ we use the substitution $y = \cos\frac{s}{n+1}$ and get

$$d_{k,+}^{n,1} = \sqrt{\frac{2}{\pi}}\int_0^{\frac{\pi}{2}} H(s,x_{kn}^{\varphi})\,\mathrm{d}s \tag{3.34}$$

with

$$H(s,x) = \frac{1}{\zeta(x)(n+1)}\,\frac{\zeta\left(\cos\frac{s}{n+1}\right)-\zeta(x)}{\cos\frac{s}{n+1}-x}\,\sin\frac{s}{n+1}\,\sin s\,.$$

For $0 \le s \le \frac{\pi}{2}$, we can estimate

$$|H(s,x_{kn}^{\varphi})| \le \frac{C}{\left(\frac{k}{n}\right)^{2\zeta_+}} \frac{\left(\frac{s}{n}\right)^{2\zeta_+} + \left(\frac{k}{n}\right)^{2\zeta_+}}{\frac{(k\pi)^2-s^2}{n^2}} \frac{s^2}{n^2} \le C \frac{\left(\frac{s}{k}\right)^{2\zeta_+}+1}{k^2} s^2 .$$

Consequently,

$$|d_{k,+}^{n,1}| \le \frac{C}{k^{2\min\{1,1+\zeta_+\}}} , \tag{3.35}$$

and the functions $H(s,x_{kn}^{\varphi})$ possess an integrable majorant. Thus, we can change the order between the limit and the integration and obtain

$$\lim_{n\to\infty} d_{k,+}^{n,1} = \sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \lim_{n\to\infty} H(s,x_{kn}^{\varphi})\,\mathrm{d}s = 2\sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_+} - 1}{(k\pi)^2 - s^2} s\sin s\,\mathrm{d}s .$$

Furthermore, we write

$$\begin{aligned} d_k^{n,2} &= \left(\int_{x_n^-}^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{\widetilde{x}_{2k,n}^{\varphi}} + \int_{\widetilde{x}_{2k,n}^{\varphi}}^{\frac{1}{2}(1+x_{kn}^{\varphi})} + \int_{\frac{1}{2}(1+x_{kn}^{\varphi})}^{x_n^+} \right) F(y,x_{kn}^{\varphi})\,\mathrm{d}y \\ &=: I_{1,k}^n + I_{2,k}^n + I_{3,k}^n + I_{4,k}^n , \end{aligned}$$

where $\widetilde{x}_{2k,n}^{\varphi} = \max\left\{-\frac{1}{2}, x_{2k,n}^{\varphi}\right\}$. Similarly, from

$$U_n'(x_{kn}^{\varphi}) = \sqrt{2/\pi}(n+1)(-1)^{k+1}[\varphi(x_{kn}^{\varphi})]^{-2}$$

and from (3.24) with $x = x_{kn}^{\varphi}$ we obtain

$$\begin{aligned} b_k^n &= \left(\int_{-1}^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{\widetilde{x}_{2k,n}^{\varphi}} + \int_{\widetilde{x}_{2k,n}^{\varphi}}^{\frac{1}{2}(1+x_{kn}^{\varphi})} + \int_{\frac{1}{2}(1+x_{kn}^{\varphi})}^{1} \right) \widetilde{F}(y,x_{kn}^{\varphi})\,\mathrm{d}y \\ &=: J_{1,k}^n + J_{2,k}^n + J_{3,k}^n + J_{4,k}^n , \end{aligned}$$

where

$$\widetilde{F}(y,x) := \frac{(-1)^{k+1}\varphi(x)}{(n+1)\zeta(x)\pi\mathrm{i}} \frac{\zeta(y)-\zeta(x)}{(y-x)^2} \sin s, \quad y = \cos\frac{s}{n+1} .$$

We observe $x_{kn}^{\varphi} \ge 0$ for $n \ge 2k-1$. For $x_n^- < y < -\frac{1}{2}$, we have $2 > |y - x_{kn}^{\varphi}| > \frac{1}{2}$ and $2 > 1 - y > \frac{3}{2}$. Thus,

$$\begin{aligned} |I_{1,k}^n| \le \frac{C}{n(1-x_{kn}^{\varphi})^{\zeta_+}} \int_{x_n^-}^{-\frac{1}{2}} &\Big\{ (1+y)^{\frac{1}{2}} \left[(1+y)^{\zeta_- -1} + (1-x_{kn}^{\varphi})^{\zeta_+} \right] \\ &+ (1+y)^{-\frac{1}{2}} \left[(1+y)^{\zeta_-} + (1-x_{kn}^{\varphi})^{\zeta_+} \right] \Big\}\,\mathrm{d}y \\ \le \frac{C}{n(1-x_{kn}^{\varphi})^{\zeta_+}} &\left[(1-x_{kn}^{\varphi})^{\zeta_+} + \int_{x_n^-}^{-\frac{1}{2}} (1+y)^{\zeta_- - \frac{1}{2}}\mathrm{d}y \right] \end{aligned}$$

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$$\leq \frac{C}{n} \begin{cases} 1 + \left(\frac{k}{n}\right)^{-2\zeta_+} & \text{if} \quad \zeta_- > -\frac{1}{2}\,, \\ 1 + \left(\frac{k}{n}\right)^{-2\zeta_+} n^{1+2\zeta_-} & \text{if} \quad \zeta_- < -\frac{1}{2}\,, \\ 1 + \left(\frac{k}{n}\right)^{-2\zeta_+} \log n & \text{if} \quad \zeta_- = -\frac{1}{2}\,. \end{cases}$$

Consequently, for some $\varepsilon \in (0,1)$, we arrive at

$$|I_{1,k}^n| \leq C \begin{cases} \dfrac{1}{n^\varepsilon} & \text{if} \qquad \zeta_+ \leq 0\,, \\ \dfrac{1}{k^{2\zeta_+} n^\varepsilon} & \text{if} \quad \zeta_+ > 0\,,\ \zeta_- \geq -\frac{1}{2}\,, \\ \dfrac{1}{n} + \dfrac{1}{k^{2\zeta_+} n^{-2(\zeta_+ + \zeta_-)}} & \text{if} \quad \zeta_+ > 0\,,\ \zeta_- < -\frac{1}{2}\,, \end{cases}$$

and, since in the last case $\zeta_+ + \zeta_- < 0$ (recall (3.27) and $\zeta_\pm = -\chi_\pm$),

$$|I_{1,k}^n| \leq \frac{C}{k^\varepsilon}\,. \tag{3.36}$$

Moreover, taking into account that $\zeta_+ < \frac{1}{4}$ (recall (3.27) and $\zeta_+ = -\chi_+$), we conclude $\lim_{n\to\infty} I_{1,k}^n = 0$ and

$$d_k^{+,2} := \lim_{n\to\infty} d_k^{n,2} = \lim_{n\to\infty} \left(I_{2,k}^n + I_{3,k}^n + I_{4,k}^n\right) = \lim_{n\to\infty} \int_{\frac{\pi}{2}}^\infty G(s, x_{kn}^\varphi)\,\mathrm{d}s\,, \tag{3.37}$$

where

$$G(s,x) := \begin{cases} \dfrac{1}{n+1} F\left(\cos\dfrac{s}{n+1}, x\right) \sin\dfrac{s}{n+1} & \text{if} \quad \dfrac{\pi}{2} < s < \dfrac{2\pi}{3}(n+1) \\ 0 & \text{if} \quad \dfrac{2\pi}{3}(n+1) < s\,. \end{cases}$$

Analogously, we get

$$|J_{1,k}^n| \leq \frac{C}{n(1-x_{kn}^\varphi)^{\zeta_+ - \frac{1}{2}}} \int_{-1}^1 \left[(1+y)^{\zeta_-} + (1-x_{kn}^\varphi)^{\zeta_+}\right]\,\mathrm{d}y \leq \frac{C}{n} \leq \frac{C}{k}\,. \tag{3.38}$$

Thus, $\lim_{n\to\infty} J_{1,k}^n = 0$ and

$$b_k^+ = \lim_{n\to\infty} \left(J_{2,k}^n + J_{3,k}^n + J_{4,k}^n\right) = \lim_{n\to\infty} \int_0^\infty \widetilde{G}(s, x_{kn}^\varphi)\,\mathrm{d}s\,, \tag{3.39}$$

where

$$\widetilde{G}(s,x) := \begin{cases} \dfrac{1}{n+1} \widetilde{F}\left(\cos\dfrac{s}{n+1}, x\right) \sin\dfrac{s}{n+1} & \text{if} \quad 0 < s < \dfrac{2\pi}{3}(n+1)\,, \\ 0 & \text{if} \quad \dfrac{2\pi}{3}(n+1) < s\,. \end{cases}$$

According to the splittings $I_{2,k}^n + I_{3,k}^n + I_{4,k}^n$ and $J_{2,k}^n + J_{3,k}^n + J_{4,k}^n$, we distinguish three cases.

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In the first case, we consider $\frac{1}{2}(1+x_{kn}^{\varphi}) < y = \cos\frac{s}{n+1} < x_n^+$ for $I_{4,k}^n$ and $\frac{1}{2}(1+x_{kn}^{\varphi}) < y = \cos\frac{s}{n+1} < 1$ for $J_{4,k}^n$. This is equivalent to the restriction $\frac{\pi}{2} < s < c_{kn}\frac{k\pi}{\sqrt{2}}$ and $0 < s < c_{kn}\frac{k\pi}{\sqrt{2}}$, respectively, where $\lim_{n\to\infty} c_{kn} = 1$ and $\frac{2\sqrt{2}}{\pi} \le c_{kn} \le \frac{\pi}{2\sqrt{2}}$. Then $y - x_{kn}^{\varphi} > \frac{1}{2}(1-x_{kn}^{\varphi})$ and

$$\begin{aligned}|G(s,x_{kn}^{\varphi})| &\le \frac{C}{n^2}\,\frac{1}{(1-x_{kn}^{\varphi})^{\zeta_+}}\left[(1-y)^{\frac{1}{2}}\left(\frac{(1-y)^{\zeta_+}+(1-x_{kn}^{\varphi})^{\zeta_+}}{(1-x_{kn}^{\varphi})^2}+\frac{(1-y)^{\zeta_+-1}}{1-x_{kn}^{\varphi}}\right)\right.\\ &\qquad \left.+(1-y)^{-\frac{1}{2}}\frac{(1-y)^{\zeta_+}+(1-x_{kn}^{\varphi})^{\zeta_+}}{1-x_{kn}^{\varphi}}\right]\frac{s}{n}\\ &\le \frac{C}{n^3}\left[\frac{\left(\frac{s}{n}\right)^{1+2\zeta_+}}{\left(\frac{k}{n}\right)^{4+2\zeta_+}}+\frac{\frac{s}{n}}{\left(\frac{k}{n}\right)^4}+\frac{\left(\frac{s}{n}\right)^{2\zeta_+-1}}{\left(\frac{k}{n}\right)^{2+2\zeta_+}}+\frac{\left(\frac{s}{n}\right)^{-1}}{\left(\frac{k}{n}\right)^2}\right]s\\ &\le \frac{C}{k^2}\left[1+\left(\frac{s}{k}\right)^{2\zeta_+}\right],\quad \frac{\pi}{2} < s < c_{kn}\frac{k\pi}{\sqrt{2}}\,. \end{aligned} \tag{3.40}$$

Consequently,

$$|I_{4,k}^n| \le \frac{C}{k^2}\int_{\frac{\pi}{2}}^{k\pi}\left[1+\left(\frac{s}{k}\right)^{2\zeta_+}\right]\mathrm{d}s \le \frac{C}{k}\,. \tag{3.41}$$

On the other hand, we get

$$\begin{aligned}|\widetilde{G}(s,x_{kn}^{\varphi})| &\le \frac{C}{n^2}\,(1-x_{kn}^{\varphi})^{-\frac{3}{2}-\zeta_+}(1-y)^{\zeta_+}\frac{s}{n}\\ &\le \frac{C}{n^2}\left(\frac{k}{n}\right)^{-3-2\zeta_+}\left(\frac{s}{n}\right)^{2\zeta_++1}\\ &\le \frac{C}{k^{2\zeta_++3}}\,s^{2\zeta_++1},\quad 0 < s < c_{kn}\frac{k\pi}{\sqrt{2}}\,. \end{aligned} \tag{3.42}$$

Consequently,

$$|J_{4,k}^n| \le \frac{C}{k^{2\zeta_++3}}\int_0^{k\pi} s^{2\zeta_++1}\,\mathrm{d}s \le \frac{C}{k}\,. \tag{3.43}$$

In the second case $c_{kn}\frac{k\pi}{\sqrt{2}} < s < \min\left\{2k\pi, \frac{2\pi}{3}(n+1)\right\}$, i.e. in the case $\widetilde{x}_{2k,n}^{\varphi} < y = \cos\frac{s}{n+1} < \frac{1}{2}(1+x_{kn}^{\varphi})$, we have the estimates

$$|F(y,x_{kn}^{\varphi})| \le \frac{C}{n\,\zeta(x_{kn}^{\varphi})}\,\left[\varphi(y)|\zeta''(\xi_1)| + |\varphi'(y)|\,|\zeta'(\xi_2)|\right]$$

and

$$\begin{aligned}|\widetilde{F}(y,x_{kn}^{\varphi})| &\le C\,\frac{\varphi(x_{kn}^{\varphi})|\zeta'(\xi_3)|}{n\zeta(x_{kn}^{\varphi})}\left|\frac{\sin s-\sin k\pi}{\cos\frac{s}{n+1}-\cos\frac{k\pi}{n+1}}\right|\\ &= C\,\frac{\varphi(x_{kn}^{\varphi})|\zeta'(\xi_3)|}{n\zeta(x_{kn}^{\varphi})}\,\frac{\left|\int_0^1\cos\Big(k\pi+\lambda(s-k\pi)\Big)\mathrm{d}\lambda\right|}{\frac{1}{n+1}\int_0^1\sin\left(\frac{1}{n+1}[k\pi+\lambda(s-k\pi)]\right)\mathrm{d}\lambda}\end{aligned}$$

$$\leq C\,\frac{\varphi(x_{kn}^{\varphi})|\zeta'(\xi_3)|}{n\zeta(x_{kn}^{\varphi})}\;\frac{\min\left\{1,\left|\frac{1}{s-k\pi}\int_{k\pi}^{s}\cos u\mathrm{d}u\right|\right\}}{\frac{1}{n+1}\int_0^{\frac{1}{2}}\sin\left(\frac{1}{n+1}[k\pi+\lambda(s-k\pi)]\right)\mathrm{d}\lambda}$$
$$\leq C\,\frac{\varphi(x_{kn}^{\varphi})|\zeta'(\xi_3)|}{n\zeta(x_{kn}^{\varphi})}\;\frac{\min\{1,|s-k\pi|^{-1}\}}{\frac{k}{n^2}}$$

for some $\xi_1,\xi_2,\xi_3\in(x_{2k,n}^{\varphi},[1+x_{kn}^{\varphi}]/2)$ and $1-y\sim 1-x_{kn}^{\varphi}\sim 1-\xi_{1/2/3}$. This results in

$$|G(s,x_{kn}^{\varphi})|\leq\frac{C}{n^2}\,(1-x_{kn}^{\varphi})^{-\frac{3}{2}}\,\frac{s}{n}\leq\frac{C}{k^2}\,,\;c_{kn}\frac{k\pi}{\sqrt{2}}<s<\min\left\{2k\pi,\frac{2\pi}{3}(n+1)\right\}\,, \tag{3.44}$$

$$|I_{3,k}^n|\leq\frac{C}{k^2}\int_0^{2k\pi}\mathrm{d}s=\frac{C}{k}\,, \tag{3.45}$$

and

$$|\widetilde{G}(s,x_{kn}^{\varphi})|\leq\frac{C}{n^2\frac{k}{n}}\,\frac{\min\{1,|s-k\pi|^{-1}\}}{\frac{k}{n^2}}\,\frac{s}{n}\;=\;C\,\frac{s}{k^2}\min\{1,|s-k\pi|^{-1}\}\,, \tag{3.46}$$

$$|J_{3,k}^n|\leq\frac{C}{k^2}\int_0^{2k\pi}\min\left\{s,\frac{s}{|s-k\pi|}\right\}\mathrm{d}s=\frac{C\,(1+\log k)}{k}\,. \tag{3.47}$$

In the third case $2k\pi<s<\frac{2\pi}{3}(n+1)$, i.e. $n+1>3k$ and if y satisfies the restriction $-\frac{1}{2}<y<x_{2k,n}^{\varphi}$, then we obtain the relations

$$1-y>1-x_{2k,n}^{\varphi}=2\sin^2\frac{2k\pi}{2(n+1)}=2\left(1-\cos\frac{k\pi}{n+1}\right)\left(1+\cos\frac{k\pi}{n+1}\right)\geq 2(1-x_{kn}^{\varphi})$$

and

$$1-y>x_{kn}^{\varphi}-y=1-y-(1-x_{kn}^{\varphi})>\frac{1}{2}\,(1-y)\geq 1-x_{kn}^{\varphi}\,.$$

Consequently, we get

$$|F(y,x_{kn}^{\varphi})|\leq\frac{C}{n(1-x_{kn}^{\varphi})^{\zeta_+}}\left[(1-y)^{\zeta_+-\frac{3}{2}}+\frac{(1-x_{kn}^{\varphi})^{\zeta_+}}{(1-y)^{\frac{3}{2}}}\right]$$

and

$$|G(s,x_{kn}^{\varphi})|\leq C\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{1}{s^2}\,,\quad 2k\pi<s<\frac{2\pi}{3}(n+1). \tag{3.48}$$

Since $2(1-\zeta_+)>1$ (recall (3.27) and $\zeta_+=-\chi_+$), we obtain the estimate

$$|I_{2,k}^n|\leq C\int_{2k\pi}^{\infty}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{\mathrm{d}s}{s^2}\leq\frac{C}{k}\,. \tag{3.49}$$

On the other hand, we arrive at

$$|\widetilde{G}(s,x_{kn}^{\varphi})|\leq\frac{C}{n^2(1-x_{kn}^{\varphi})^{\zeta_+-\frac{1}{2}}}\,\frac{(1-y)^{\zeta_+}+(1-x_{kn}^{\varphi})^{\zeta_+}}{(1-y)^2}\,\frac{s}{n}$$
$$\leq\frac{C}{n^2}\left(\frac{k}{n}\right)^{1-2\zeta_+}\frac{\left(\frac{s}{n}\right)^{2\zeta_+}+\left(\frac{k}{n}\right)^{2\zeta_+}}{\left(\frac{s}{n}\right)^3}\,,$$

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such that

$$|\widetilde{G}(s,x^{\varphi}_{kn})| \le \frac{C\,k}{s^3}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]. \tag{3.50}$$

Consequently,

$$|J^n_{2,k}| \le C\,k\int_{2k\pi}^{\infty}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{\mathrm{d}s}{s^3} \le \frac{C}{k}. \tag{3.51}$$

From the upper estimates in the inequalities (3.40), (3.44), and (3.48) we conclude that the function

$$f(s) := C\left\{\begin{array}{lll} \max\left\{s^{2\zeta_+},1\right\} & \text{if} & \frac{\pi}{2}<s<2k\pi \\ (s^{2\zeta_+}+1)s^{-2} & \text{if} & 2k\pi<s<\infty \end{array}\right.$$

with the constant C depending on $\zeta_\pm$ and k, only, is an integrable majorant for the functions $G(s,x^{\varphi}_{kn})$, $n>3k-1$, in (3.37). Thus, we can change the order between the limit and the integration, and we obtain

$$\begin{aligned}
& d_k^{+,2} \\
&= \int_{\frac{\pi}{2}}^{\infty}\lim_{n\to\infty} G(s,x^{\varphi}_{kn})\,\mathrm{d}s \\
&= \sqrt{\frac{2}{\pi}}\int_{\frac{\pi}{2}}^{\infty}\lim_{n\to\infty}\frac{\cos s\,\sin\frac{s}{n+1}}{(n+1)^2\left(2\sin^2\frac{k\pi}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{k\pi}{2(n+1)}\right)^{\zeta_-}} \\
&\left\{\frac{\left[\left(2\sin^2\frac{s}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{s}{2(n+1)}\right)^{\zeta_-}-\left(2\sin^2\frac{k\pi}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{k\pi}{2(n+1)}^{\zeta_-}\right)\right]\sin\frac{s}{n+1}}{4\,\sin^2\frac{s+k\pi}{2(n+1)}\,\sin^2\frac{k\pi-s}{2(n+1)}}\right. \\
&\quad -\frac{\left(2\sin^2\frac{s}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{s}{2(n+1)}\right)^{\zeta_-}\left(\frac{\zeta_-}{2\cos^2\frac{s}{2(n+1)}}-\frac{\zeta_+}{2\sin^2\frac{s}{2(n+1)}}\right)\sin\frac{s}{n+1}}{2\,\sin\frac{s+k\pi}{2(n+1)}\,\sin\frac{k\pi-s}{2(n+1)}} \\
&+\frac{\cos\frac{s}{n+1}}{\sin\frac{s}{n+1}}\,* \\
&\quad \left. *\frac{\left(2\sin^2\frac{s}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{s}{2(n+1)}\right)^{\zeta_-}-\left(2\sin^2\frac{k\pi}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{k\pi}{2(n+1)}\right)^{\zeta_-}}{2\,\sin\frac{s+k\pi}{2(n+1)}\,\sin\frac{k\pi-s}{2(n+1)}}\right\}\mathrm{d}s \\
&= 4\sqrt{\frac{2}{\pi}}\int_{\frac{\pi}{2}}^{\infty}\cos s\left\{\frac{s^2\left[\left(\frac{s}{k\pi}\right)^{2\zeta_+}-1\right]}{\left[(k\pi)^2-s^2\right]^2}+\frac{\zeta_+\left(\frac{s}{k\pi}\right)^{2\zeta_+}+\frac{1}{2}\left[\left(\frac{s}{k\pi}\right)^{2\zeta_+}-1\right]}{(k\pi)^2-s^2}\right\}\mathrm{d}s\,.
\end{aligned}$$

Analogously, using (3.42), (3.46), and (3.50), we get that

$$\widetilde{f}(s) := C\left\{\begin{array}{lll} \max\left\{s^{2\zeta_++1},s\right\} & \text{if} & 0<s<2k\pi \\ (s^{2\zeta_+}+1)s^{-3} & \text{if} & 2k\pi<s<\infty \end{array}\right.$$

is an integrable majorant for the functions $\widetilde{G}(s,x_{kn}^{\varphi})$, $n>3k-1$, in (3.39). Hence

$$\begin{aligned} b_k^{+} &= \int_0^\infty \lim_{n\to\infty} \widetilde{G}(s,x_{kn}^{\varphi})\,\mathrm{d}s \\ &= \frac{(-1)^{k+1}}{\pi\mathrm{i}}\sqrt{\frac{2}{\pi}}\int_0^\infty \lim_{n\to\infty}\left\{\frac{\left(2\sin^2\frac{k\pi}{2(n+1)}\right)^{\frac12-\zeta_+}\left(2\cos^2\frac{k\pi}{2(n+1)}\right)^{\frac12-\zeta_-}}{(n+1)^2} *\right. \\ &\quad * \frac{\left(2\sin^2\frac{s}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{s}{2(n+1)}\right)^{\zeta_-}-\left(2\sin^2\frac{k\pi}{2(n+1)}\right)^{\zeta_+}\left(2\cos^2\frac{k\pi}{2(n+1)}\right)^{\zeta_-}}{4\,\sin^2\frac{s+k\pi}{2(n+1)}\,\sin^2\frac{k\pi-s}{2(n+1)}} * \\ &\quad \left. *\sin s\,\sin\frac{s}{n+1}\right\}\mathrm{d}s \\ &= \frac{4(-1)^{k+1}k}{\mathrm{i}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_+}-1}{[(k\pi)^2-s^2]^2}\,s\,\sin s\,\mathrm{d}s\,. \end{aligned}$$

The formulas (3.19) and (3.18) are shown.

Due to the estimates (3.33), (3.35), (3.36), (3.41), (3.45), and (3.49) we have $|d_k^n|\le C\,k^{-\varepsilon}$ for some $\varepsilon\in(0,1)$ and for $1\le k\le\frac{n+1}{2}$. Now, we consider $\frac{n+1}{2}\le k\le n$ and $j=n+1-k$. Then $1\le j\le\frac{n+1}{2}$ and, in view of $x_{n+1-j,n}^{\varphi}=-x_{jn}^{\varphi}$ and $U_n(-y)=(-1)^nU_n(y)$,

$$d_k^n=\frac{(-1)^{n+1}}{\widetilde{\zeta}(x_{jn}^{\varphi})}\int_{-1}^{1}\frac{\widetilde{\zeta}(y)-\widetilde{\zeta}(x_{jn}^{\varphi})}{y-x_{jn}^{\varphi}}\,\varphi(y)U_n(y)\,\mathrm{d}y\,,$$

where $\widetilde{\zeta}(y):=\zeta(-y)$. Hence, we get $|d_k^n|\le C\,j^{-\varepsilon}=C\,(n+1-k)^{-\varepsilon}$ for $\frac{n+1}{2}\le k\le n$ and

$$|d_k^n|\le\frac{C}{\min\{k,n+1-k\}^{\varepsilon}}\,,\quad k=1,\dots,n\,,\ n=1,2,\dots\ . \tag{3.52}$$

Analogously, from (3.38), (3.43), (3.47), and (3.51) we get

$$|b_k^n|\le\frac{C}{\min\{k,n+1-k\}^{\varepsilon}}\,,\quad k=1,\dots,n\,,\ n=1,2,\dots\ , \tag{3.53}$$

for some $\varepsilon\in(0,1)$.

Using the estimates (3.28) and (3.29) together with (3.52) and Remark 3.7, we get, for each fixed $m=1,2,\dots$, the ℓ^2 limit relations

$$\mathbf{V}_n\mathbf{A}_n\mathbf{W}_ne_{m-1}\longrightarrow\mathbf{V}_+\mathbf{A}\mathbf{W}e_{m-1} \tag{3.54}$$

$$\mathbf{D}_n\mathbf{A}_n\mathbf{W}_n\mathbf{V}_n\mathbf{D}_n^{-1}e_{m-1}\longrightarrow\mathbf{D}_+\mathbf{A}\mathbf{W}\mathbf{V}_+\mathbf{D}_+^{-1}e_{m-1} \tag{3.55}$$

and the corresponding limit relations for the adjoint operators, where the operators $\mathbf{V}_+$ and $\mathbf{W}$ are defined by (3.17). □

4. The Subalgebra $\mathcal{A}$ of the Algebra $\mathcal{F}$. In this section we prove that further sequences of approximate operators[2] belong to the algebra $\mathcal{F}$. Using these and the operator sequences of

[2] We conjecture that these sequences can be generated by the sequences of Section 3.

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the collocation method, we shall form a C^*-algebra which is the basic algebra for the stability analysis of the collocation method.

We consider the C^* algebra $\mathcal{L}(\ell^2)$ of continuous operators in ℓ^2. By $alg\,\mathcal{T}(\mathbf{PC})$ we denote the closed subalgebra generated by the Toeplitz matrices $(\hat{g}_{k-j})_{k,j=0}^{\infty}$ with piecewise continuous symbols $g(t) := \sum_{l\in\mathbb{Z}} \hat{g}_l t^l$ defined on

$$\mathbb{T} := \{t \in \mathbb{C} : \ |t| = 1\}$$

and continuous on $\mathbb{T} \setminus \{\pm 1\}$. Note that, for any $\varepsilon > 0$, the operator $R \in alg\,\mathcal{T}(\mathbf{PC})$ admits the representation

$$R \;=\; (\hat{g}_{k-j})_{k,j=0}^{\infty} + M + M' + R_c + R_\varepsilon, \tag{4.1}$$

$$M \;:=\; \left(m\left(\frac{k+1}{j+1}\right) \frac{1}{j+1} \right)_{k,j=0}^{\infty}, \quad M' \;:=\; \left((-1)^{k-j} m'\left(\frac{k+1}{j+1}\right) \frac{1}{j+1} \right)_{k,j=0}^{\infty},$$

where the ℓ^2 operator norm of R_ε is less than ε, where $R_c \in \mathcal{L}(\ell^2)$ is a compact operator, where the generating function g of the Toeplitz matrix is piecewise continuous and continuous on $\mathbb{T} \setminus \{\pm 1\}$, and where m, m' are suitably chosen functions from $\mathbf{C}^\infty(0,\infty)$ (for more details cf. part iv) of Lemma 7.1). The existence of the representation (4.1) is a simple consequence of the Gohberg-Krupnik symbol calculus (cf. the subsequent Lemma 7.1 or [20] and [29, 16, 28]). For $R \in alg\,\mathcal{T}(\mathbf{PC})$, we use the projections P_n from (2.1) and define the finite sections $R_n := P_n R|_{\operatorname{im} P_n} \in \mathcal{L}(\operatorname{im} P_n)$. Furthermore, using the notation from the beginning of Section 2, we form the operators

$$R_n^\omega := (E_n^{(\omega)})^{-1} R_n E_n^{(\omega)}, \quad \omega \in \{3,4\},$$

mapping $\operatorname{im} L_n$ into $\operatorname{im} L_n$. We get

LEMMA 4.1. **i)** *Suppose χ^s and χ^b are smooth functions over $[-1,1]$ such that their values are in $[0,1]$, such that χ^s has a small support with $\operatorname{supp}[\chi^s \circ \cos] \subseteq [t-\varepsilon^s, t+\varepsilon^s]$, where $\cos$ is considered as a function defined on $[0,\pi]$, and such that χ^b has a support with $\operatorname{supp}[\chi^b \circ \cos] \cap [t-\varepsilon^b, t+\varepsilon^b] = \emptyset$ and $\varepsilon^b > \varepsilon^s$. Then, for any $R \in alg\,\mathcal{T}(\mathbf{PC})$ and for any $\varepsilon > 0$, there is a constant C such that $\varepsilon^b/\varepsilon^s > C$ implies the locality property*

$$\left\| \left(\chi^b(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} R_n \left(\chi^s(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon,$$
$$\left\| \left(\chi^s(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} R_n \left(\chi^b(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon.$$

Moreover, if the support of χ^s satisfies $\operatorname{supp}[\chi^s \circ \cos] \subseteq [t-\varepsilon^s, t+\varepsilon^s] \subseteq [0, \pi - \varepsilon^b]$, then we get

$$\left\| (I-P_n) R P_n \left(\chi^s(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon,$$
$$\left\| \left(\chi^s(x^\varphi_{(j+1)n}) \delta_{j,k} \right)_{j,k=0}^{n-1} P_n R (I-P_n) \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon.$$

ii) *For any $R \in alg\,\mathcal{T}(\mathbf{PC})$, the sequences $\{R_n^\omega\}_{n=0}^\infty$, $\omega \in \{3,4\}$, belong to $\mathcal{F}$. If R is the Toeplitz operator $(\hat{g}_{k-j})_{k,j=0}^\infty$, then*

$$W_3\left\{R_n^3\right\} = R,\ W_4\left\{R_n^3\right\} = \widetilde{R},\ W_3\left\{R_n^4\right\} = \widetilde{R},\ W_4\left\{R_n^4\right\} = R,\ \widetilde{R} := (\hat{g}_{j-k})_{k,j=0}^\infty.$$

Proof. **i)** The first assertion is a simple consequence of the more general estimates

$$\left\| \left(d^1_k \delta_{k,j}\right)_{k,j=0}^{\infty} R \left(d^2_j \delta_{k,j}\right)_{k,j=0}^{\infty} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon \,, \tag{4.2}$$

$$\left\| \left(d^2_k \delta_{k,j}\right)_{k,j=0}^{\infty} R \left(d^1_j \delta_{k,j}\right)_{k,j=0}^{\infty} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon \tag{4.3}$$

which hold for any two sequences d^1_k and d^2_j with $|d^i_k| \le 1$, with $d^1_k = 0$ for each k in $\{k : \; |t - k/n| \ge \varepsilon^s\}$, and with $d^2_k = 0$ for each k in $\{k : \; |t - k/n| \le \varepsilon^b\}$. Here t is a fixed non-negative real and, like in the lemma, we suppose $\varepsilon^b/\varepsilon^s > C$ for a sufficiently large constant C depending on R and ε, only. Of course, it suffices to prove (4.3) since (4.2) follows by passing to the adjoint matrices.

It is not hard to see that, if the assertion of (4.3) is true for two operators R and R', then it is true for the linear combination and for the product of R and R'. Moreover, if it is true for a sequence of operators, then it holds for the (operator norm) limit operator as well. Hence, it is sufficient to verify assertion (4.3) for the generating Toeplitz matrices $R = (\hat{g}_{k-j})_{k,j=0}^{\infty}$ with $\hat{g}_k$ the Fourier coefficients of a piecewise smooth function g. Now, without loss of generality, we assume $t = 0$. From Young's inequality for discrete convolution operators we conclude

$$\begin{aligned}
&\left\| \left(d^2_j \delta_{j,k}\right)_{j,k=0}^{n-1} R \left(d^1_k \delta_{j,k}\right)_{j,k=0}^{n-1} (\xi_k)_{k=0}^{n-1} \right\|_{\ell^2} \\
&\le C \left(\sum_{k:\; k<\varepsilon^s n} |\xi_k| \right) \sqrt{\sum_{l:\; l=k-j,\; j<\varepsilon^s n,\; k>\varepsilon^b n} |\hat{g}_l|^2} \\
&\le C \sqrt{\sum_{k:\; k<\varepsilon^s n} 1} \sqrt{\sum_k |\xi_k|^2} \sqrt{\sum_{l:\; l>(\varepsilon^b-\varepsilon^s)n} |\hat{g}_l|^2} \;\le\; C \sqrt{\frac{\varepsilon^s}{\varepsilon^b - \varepsilon^s}} \, \|(\xi_k)_{k=0}^{\infty}\|_{\ell^2} \,.
\end{aligned}$$

In the last steps we have used the fact that the Fourier coefficients $\hat{g}_k$ of a piecewise smooth function g satisfy the estimate $|\hat{g}_k| < C/k$ for $k \ne 0$. Obviously, the last right-hand side is less than $\varepsilon \|(\xi_k)_k\|$ if $\varepsilon^b > C\varepsilon^s$ for sufficiently large C.

ii-a) Now we turn to the proof of ii). Without loss of generality, we restrict ourselves to the sequence R^3_n. First, we restrict our proof to the case of generating Toeplitz matrices $R = (\hat{g}_{k-j})_{k,j=0}^{\infty}$. The general case follows in part ii-b). From the definitions of R^3_n we conclude

$$\begin{aligned}
E^{(3)}_n R^3_n (E^{(3)}_n)^{-1} L^{(3)}_n &= R_n P_n \;=\; P_n R P_n \longrightarrow R, \\
E^{(4)}_n R^3_n (E^{(4)}_n)^{-1} L^{(4)}_n &= \widetilde{V}_n V_n^{-1} R_n V_n \widetilde{V}_n^{-1} P_n \;=\; \left(\hat{g}_{[n-1-k]-[n-1-j]}\right)_{k,j=0}^{n-1} P_n \longrightarrow \widetilde{R} \,.
\end{aligned}$$

Similarly, the convergence of the adjoint operators can be derived. For the proof of the existence of $W_\omega\{R^3_n\}$ with $\omega = 1, 2$, we remark that

$$E^{(2)}_n R^3_n (E^{(2)}_n)^{-1} = E^{(1)}_n \widetilde{R}^3_n (E^{(1)}_n)^{-1}$$

with the finite sections $\widetilde{R}_n$ of the Toeplitz matrix $\widetilde{R} := ((-1)^{k-j} \hat{g}_{k-j})_{k,j=0}^{\infty}$ corresponding to the piecewise smooth symbol $\breve{g}(t) = g(-t)$. Hence, we only have to analyze the limit $W_1\{R^3_n\}$. Similarly, we get the identity (cf. (2.3))

$$\left(E^{(1)}_n R^3_n (E^{(1)}_n)^{-1}\right)^* = E^{(1)}_n [R^*]^3_n (E^{(1)}_n)^{-1}$$

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with the finite sections $[R^*]_n$ of the Toeplitz matrix $R^* = (\overline{\hat{g}_{j-k}})_{k,j=0}^{\infty}$ corresponding to the piecewise smooth symbol $\overline{g(t)}$. Hence, it remains to analyze the limit $W_1\{R_n^3\}$ for the sequence $\left\{E_n^{(1)}R_n^3(E_n^{(1)})^{-1}L_n\right\}$, and the limit $W_1\{R_n^3\}^*$ for $(E_n^{(1)}R_n^3(E_n^{(1)})^{-1})^*L_n$ follows analogously. Due to the uniform boundedness of these sequences, it remains to show the convergence of $E_n^{(1)}R_n^3(E_n^{(1)})^{-1}L_nf$ for smooth functions f with support $\operatorname{supp} f$ contained in the open interval $(-1,1)$. Moreover, due to the $\mathbf{L}^2_\sigma$ convergence $M_nf - L_nf \longrightarrow 0$ (cf. Lemma 2.5), we only need to prove the convergence of $E_n^{(1)}R_n^3(E_n^{(1)})^{-1}M_nf$, and, in view of part i) of the present lemma, we need to show this convergence only for the interval $[-1+\varepsilon^s, 1-\varepsilon^s]$ with a small ε^s. However, since the Jacobi weight functions ρ and ρ^{-1} are bounded in $[-1+\varepsilon^s, 1-\varepsilon^s]$ and since the $\sqrt{\frac{n+1}{\pi}}(\omega_{kn}\delta_{k,j})_{k,j=1}^n$ and $\sqrt{\frac{\pi}{n+1}}(\omega_{kn}^{-1}\delta_{k,j})_{k,j=1}^n$ are strongly convergent discretizations of the operators of multiplication by these weights (cf. Lemma 3.8), we can suppose $\rho \equiv 1$ and $\omega_{kn} = 1$ without loss of generality. We are going to prove the convergence to $W_1\{R_n^3\}$ for Toeplitz matrices with smooth generating functions g_C (functions in the Wiener class) and for a special generating function g_{PC} with jumps at ± 1. The case of general piecewise continuous generating functions will follow from the combination of the particular convergence results and a density argument.

If the generating function $g(t)$ is equal to t^l for a fixed l, then the application of R_n to a vector is nothing else than a shift in the indices. Consequently, $E_n^{(1)}R_n^3(E_n^{(1)})^{-1}M_nf = M_n\widetilde{f}$ with a shifted smooth function $\widetilde{f}$ defined by $[\widetilde{f}\circ\cos](s) = [f\circ\cos](s - l/[n+1])$. In view of the convergence of the interpolation M_n (cf. Lemma 2.5), which is uniform on the compact set of shifted functions, we conclude $E_n^{(1)}R_n^3(E_n^{(1)})^{-1}M_nf \longrightarrow f$. This result for a fixed l, however, implies

$$E_n^{(1)}R_n^3(E_n^{(1)})^{-1}M_nf \longrightarrow g(1)f$$

for all $R = (\hat{g}_{k-j})_{k,j=0}^{\infty}$ with generators $g(t) = \sum_l \hat{g}_l t^l$ from the Wiener class, i.e. with generators such that $\sum_l |\hat{g}_l| < \infty$.

Now we consider the discretized operator of $\rho^{-1}S\rho I$. The corresponding matrix takes the form (cf. Equation (3.10))

$$\left(\mathrm{i}\frac{\cos'\left(\frac{(k+1)\pi}{n+1}\right)\frac{\widetilde{\delta}_{jk}}{n+1}}{\cos\left(\frac{(k+1)\pi}{n+1}\right) - \cos\left(\frac{(j+1)\pi}{n+1}\right)}\right)_{k,j=0}^{n-1} = \left(\mathrm{i}\frac{\widetilde{\delta}_{jk}}{k-j}\right)_{k,j=0}^{n-1} + \left(r_{(k+1)(j+1)}\right)_{k,j=0}^{n-1}, \tag{4.4}$$

where

$$r_{kj} := \frac{\widetilde{\delta}_{jk}}{n+1}\ k^\dagger\left(\frac{k\pi}{n+1}, \frac{j\pi}{n+1}\right), \qquad k^\dagger(t,s) := \mathrm{i}\left[\frac{\cos'(t)}{\cos(t) - \cos(s)} - \frac{1}{t-s}\right].$$

Clearly, the kernel $k^\dagger(t,s)$ is smooth. For a cut off function χ with $\operatorname{supp}\chi \subset (-1,1)$, the integral operator $K^\dagger$ corresponding to the kernel $\chi(\cos(t))k^\dagger(t,s)\chi(\cos(s))$ can be approximated by a quadrature method $K_n^\dagger \in \mathcal{L}(\operatorname{im} L_n)$ such that the matrix with respect to $\{\widetilde{\ell}_{kn}^{\varphi}\}_{k=1}^n$ is

$$\begin{aligned} &E_n^{(3)}K_n^\dagger(E_n^{(3)})^{-1} \\ &= \left(\chi(x^\varphi_{(k+1)n})\delta_{k,j}\right)_{k,j=0}^{n-1}\left(\frac{\pi}{n+1}k^\dagger\left(\frac{(k+1)\pi}{n+1}, \frac{(j+1)\pi}{n+1}\right)\right)_{k,j=0}^{n-1}\left(\chi(x^\varphi_{(k+1)n})\delta_{k,j}\right)_{k,j=0}^{n-1} \end{aligned}$$

and that the sequence $K_n^\dagger$ belongs to $\mathcal{J}$ (cf. the proof of Lemma 2.4). In view of (2.2) the matrix $(r_{(k+1)(j+1)})_{k,j=0}^{n-1}$ coincides with the matrix $E_n^{(3)}[K_n^\dagger - W_nK_n^\dagger W_n](E_n^{(3)})^{-1}$, which

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implies that $\{M_n\chi L_n(E_n^{(3)})^{-1}(r_{(k+1)(j+1)})_{k,j=0}^{n-1}E_n^{(3)}M_n\chi L_n\}$ is included in $\mathcal{J}$, and we get the strong convergence for the operator whose matrix representation with respect to the basis $\{\widetilde{\ell}_{kn}^{\varphi}\}_{k=1}^{n}$ is

$$\left(\chi(x_{(k+1)n}^{\varphi})\delta_{k,j}\right)_{k,j=0}^{n-1}\left(r_{(k+1)(j+1)}\right)_{k,j=0}^{n-1}\left(\chi(x_{(k+1)n}^{\varphi})\delta_{k,j}\right)_{k,j=0}^{n-1}.$$

Due to the strong convergence toward $W_1\{A_n\}$ in Lemma 3.9 and due to (4.4), we get the strong convergence for the operator whose matrix representation with respect to the basis $\{\widetilde{\ell}_{kn}^{\varphi}\}_{k=1}^{n}$ is the finite section $(\mathrm{i}\widetilde{\delta}_{jk}/[j-k])_{k,j=0}^{n-1}$ of the Toeplitz matrix

$$R_{PC} := (\mathrm{i}\widetilde{\delta}_{jk}/[j-k])_{k,j=0}^{\infty}\,.$$

We denote the generating function of the last Toeplitz matrix by g_{PC}. Note that $g_{PC}(t) = -\pi\,\mathrm{sgn}\,\Im\mathrm{m}\,t$ is a piecewise constant function with jumps at $\pm 1 \in \mathbb{T}$.

Suppose now that the generating function g of the Toeplitz matrix R takes the form $g(t) = \lambda g_+(t)g_{PC}(t) + \mu g_-(t)g_{PC}(t) + g_C(t)$ with fixed numbers λ and μ, with $g_\pm(t) = t^{-1} \mp 1$, and with g_C from the Wiener class. Then we get the representation

$$R = \lambda R_+ R_{PC}(t) + \mu R_- R_{PC}(t) + R_C \tag{4.5}$$

with Toeplitz matrices R_+ and R_- generated by the functions g_+ and g_-, respectively. Though $P_nR_\pm R_{PC}|_{\mathrm{im}\,P_n}$ is different from $[P_nR_\pm|_{\mathrm{im}\,P_n}][P_nR_{PC}|_{\mathrm{im}\,P_n}]$, we get

$$\begin{aligned}&\left(\chi^\dagger(x_{(j+1)n}^{\varphi})\delta_{j,k}\right)_{j,k=0}^{n-1}P_nR_\pm R_{PC}|_{\mathrm{im}\,P_n}\\ &= \left(\chi^\dagger(x_{(j+1)n}^{\varphi})\delta_{j,k}\right)_{j,k=0}^{n-1}[P_nR_\pm|_{\mathrm{im}\,P_n}]\left(\chi(x_{(j+1)n}^{\varphi})\delta_{j,k}\right)_{j,k=0}^{n-1}[P_nR_{PC}|_{\mathrm{im}\,P_n}],\end{aligned} \tag{4.6}$$

where χ and $\chi^\dagger$ stand for functions such that $\mathrm{supp}\,\chi \subset (-1,1)$, $\mathrm{supp}\,\chi^\dagger \subset (-1,1)$, and $\mathrm{supp}\,\chi^\dagger \subset \{t \in [-1,1]:\ \chi(t) = 1\}$, and such that there is an x_{jn}^{φ} with $\chi(x_{jn}^{\varphi}) = 1$ and $\mathrm{supp}\,\chi^\dagger \cap [-1, x_{jn}^{\varphi}] = \emptyset$. Indeed, the matrices $R_\pm$ consist of two non-zero diagonals, namely the main diagonal and the one above the main diagonal. Therefore, the difference between $P_nR_\pm$ and $P_nR_\pm P_n$ is a matrix with exactly one non-zero entry. However, if the finite section from the left is replaced by P_{n-1}, then $P_{n-1}R_\pm = P_{n-1}R_\pm P_n$. The assumptions on χ and $\chi^\dagger$ ensure that the finite section matrices $(\chi^\dagger(x_{(j+1)n}^{\varphi})\delta_{j,k})_{j,k=0}^{n-1}$ and $(\chi(x_{(j+1)n}^{\varphi})\delta_{j,k})_{j,k=0}^{n-1}$ act similarly to P_{n-1} and P_n when applied to $R_\pm$. Thus (4.6) holds true, and, together with the just proved strong convergences of $E_n^{(1)}[R_{PC}]_n^3(E_n^{(1)})^{-1}L_n$ and $E_n^{(1)}[R_\pm]_n^3(E_n^{(1)})^{-1}L_n$, we arrive at the $\mathbf{L}_\sigma^2$ convergence of $E_n^{(1)}[R_\pm R_{PC}]_n^3(E_n^{(1)})^{-1}L_nf$ over the interior of the interval $[-1,1]$. Consequently, we obtain the strong convergence of $E_n^{(1)}R_n^3(E_n^{(1)})^{-1}L_n$ for R^3 from the representation (4.5).

Finally, a general generating function g which is piecewise continuous over $\mathbb{T}$ and continuous over $\mathbb{T}\setminus\{\pm 1\}$ can be approximated in the supremum norm by a function of the form from the previous paragraph. Hence, the strong convergence property extends to Toeplitz matrices with generating function g piecewise continuous over $\mathbb{T}$ and continuous over $\mathbb{T}\setminus\{\pm 1\}$.

ii-b) Now we consider the case of arbitrary $R \in alg\,\mathcal{T}(\mathbf{PC})$. Because of the relations $E_n^{(3)}R_n^3(E_n^{(3)})^{-1}L_n^{(3)} = P_nRP_n$ and $E_n^{(4)}R_n^3(E_n^{(4)})^{-1}L_n^{(4)} = \widetilde{W}_nR\widetilde{W}_n$ with

$$\widetilde{W}_n\,\{\xi_0,\xi_1,\dots\} = \{\xi_{n-1},\dots,\xi_0,0,\dots\}\ ,$$

the existence of $W_\omega\left\{R_n^3\right\}$ for $\omega = 3,4$ is well known (see, for example, [3, Cor. 7.14]). It remains to derive the convergence to the limits $W_\omega\left\{R_n^3\right\}$ for $\omega = 1,2$.

In view of (4.1), we have to consider the two cases $R = M$ and $R = M'$. For similarity sake (use (2.2) for M'), we may restrict our proof to the case $R = M$. Since R_n^3 is uniformly bounded, we have to show the limits for a dense subset only. Hence, it suffices to consider the limits for $R_n^3 M_n \chi L_n$ with χ a smooth function vanishing in neighbourhoods of the end points ± 1. Moreover, in accordance with part i) of Lemma 4.1, we need to show the strong limits for $M_n \chi' L_n R_n^3 M_n \chi L_n$ only, where χ' is another smooth function vanishing in neighbourhoods of ± 1. Setting

$$k(x,y) := \frac{\chi'(x)\chi(y)}{\varrho(x)\vartheta(y)} m\left(\frac{\arccos(x)}{\arccos(y)}\right)\frac{1}{\arccos(y)}, \tag{4.7}$$

the operator $M_n \chi' L_n R_n^3 M_n \chi L_n$ takes the form K_n of Lemma 2.4. Now Lemma 2.4 and Corollary 2.2, imply the strong convergences. □

By $\mathcal{A}$ we denote the smallest C^*-subalgebra of $\mathcal{F}$ generated by all sequences of the ideal $\mathcal{J}$, by all sequences $\{R_n^\omega\}$ with $\omega \in \{3,4\}$ and $R \in alg\,\mathcal{T}(\mathbf{PC})$, and by all sequences of the form

$$\{M_n(aI + b\mu^{-1}S\mu I)L_n\}\,, \quad a,b \in \mathbf{PC}\,,$$

where $\mu := v^{\gamma,\delta}$ satisfies (1.5) and (1.6). We shall prove the missing invertibility of the collocation sequence in the quotient algebra $\mathcal{F}/\mathcal{J}$ (cf. Theorem 2.3) by showing the invertibility in the quotient algebra $\mathcal{A}/\mathcal{J}$. For $\{A_n\} \in \mathcal{F}$, we write $\{A_n\}^o$ for the coset $\{A_n\} + \mathcal{J}$ of $\mathcal{F}/\mathcal{J}$.

5. A Subalgebra in the Center of the Quotient Algebra $\mathcal{A}/\mathcal{J}$ and the Local Principle of Allan and Douglas. In this section we show that a set of discretized multiplication operators forms a subalgebra contained in the center of the quotient algebra $\mathcal{A}/\mathcal{J}$. However, to a general Banach algebra and to a general central subalgebra we can apply the local principle of Allan and Douglas in order to analyze the invertibility of an element. We formulate the corresponding assertions for our specific setting.

LEMMA 5.1. *The cosets $\{M_n f L_n\}^o$, where $f \in \mathbf{C}[-1,1]$, belong to the center of $\mathcal{A}/\mathcal{J}$.*

Proof. We have to show that the commutator of $\{M_n f L_n\}$ with the generating elements of $\mathcal{A}$ are contained in $\mathcal{J}$. First we observe $M_n f L_n M_n a L_n = M_n a f L_n$ (cf. (3.5)), which implies $\{M_n f L_n\}^o\{M_n a L_n\}^o = \{M_n a L_n\}^o\{M_n f L_n\}^o$.

Next we turn to the commutators of the discretized multiplication operators with the discretized Cauchy singular integral operator. We first suppose $f = p$ is a polynomial of degree not greater than m. Then we get $M_n p L_{n-m} = p L_{n-m}$ for $n > m$. Consequently,

$$\begin{aligned} &M_n p L_n M_n \mu^{-1} S \mu L_n - M_n \mu^{-1} S \mu L_n M_n p L_n \\ &\quad = M_n p \mu^{-1} S \mu L_n - M_n \mu^{-1} S \mu M_n p L_n \\ &\quad = M_n \mu^{-1}(pS - Sp)\mu L_n + M_n \mu^{-1} S \mu (I - M_n) p (L_n - L_{n-m})\,. \end{aligned}$$

Obviously, the sequence $\{M_n \mu^{-1}(pS - Sp)\mu L_n\}$ belongs to $\mathcal{J}$. Moreover, we observe the identity $L_n - L_{n-m} = W_n L_m W_n$, and

$$M_n \mu^{-1} S \mu (I - M_n) p (L_n - L_{n-m})$$

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$$= M_n\mu^{-1}S\mu(I - M_n)pL_nW_nL_mW_n$$

$$= \left[M_n\mu^{-1}(Sp - pS)\mu L_n + M_np\mu^{-1}S\mu L_n - M_n\mu^{-1}S\mu L_nM_npL_n\right]W_nL_mW_n\,,$$

which shows that the sequence $\left\{M_n\mu^{-1}S\mu(I - M_n)p(L_n - L_{n-m})\right\}$ belongs to $\mathcal{J}$, too. Taking into account the closedness of $\mathcal{J}$ and (3.7), we arrive at the relation $\{M_nfL_n\}^o$ $\left\{M\mu^{-1}S\mu L_n\right\}^o = \left\{M_n\mu^{-1}S\mu L_n\right\}^o\{M_nfL_n\}^o$ valid for all $f \in \mathbf{C}[-1,1]$.

Next, we have to consider the commutators of the discretized multiplication operators $\{M_nfL_n\}$ with the sequences $\{R_n^3\}$ and $\{R_n^4\}$ for matrices $R \in alg\,\mathcal{T}(\mathbf{PC})$. For similarity reasons, we only treat $\{R_n^3\}$. In view of (4.1), we have to consider the cases $R = (\hat{g}_{k-j})_{k,j=0}^\infty$ and $R = M, M'$. We start with $R = (\hat{g}_{k-j})_{k,j=0}^\infty$. If the function f is Lipschitz and if the generating function g of R is a trigonometric polynomial, then we get

$$\left(f(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1}R_n - R_n\left(f(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1}$$

$$= \left([f(x^\varphi_{(j+1)n}) - f(x^\varphi_{(k+1)n})]\hat{g}_{k-j}\right)_{j,k=0}^{n-1}$$

and

$$\left|[f(x^\varphi_{(j+1)n}) - f(x^\varphi_{(k+1)n})]\hat{g}_{k-j}\right| \le \frac{C}{n}|j-k||\hat{g}_{k-j}|\,,$$

where $\hat{g}_k = 0$ for all sufficiently large $|k|$. Hence, the norm of the commutator

$$\{M_nfL_n\}\{R_n^3\} - \{R_n^3\}\{M_nfL_n\}$$

tends to zero. Consequently, due to (3.7) and the closedness of $\mathcal{J}$, for continuous f and g, we get $\{M_nfL_n\}\{R_n^3\} - \{R_n^3\}\{M_nfL_n\} \in \mathcal{J}$.

For piecewise twice continuously differentiable functions g and Lipschitz continuous f, we only get the estimate

$$\left|[f(x^\varphi_{(j+1)n}) - f(x^\varphi_{(k+1)n})]\hat{g}_{k-j}\right| \le \frac{C}{n}\,\frac{|j-k|}{1+|j-k|} \le \frac{C}{n}. \tag{5.1}$$

This, however, allows us to replace R_n by $(\chi(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}R_n(\chi(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}$ with a smooth and bounded function χ which is identically equal to one except in two small neighborhoods of the two interval end-points. Indeed, $\mathcal{J}$ is closed and the difference of R_n and the modified matrix $(\chi(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}R_n(\chi(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}$ is small by (5.1) and a simple Frobenius (Hilbert-Schmidt) norm estimate. Now we suppose that χ vanishes identically in a small neighborhood of the interval end-points. If we consider the function g_{PC} from the proof to part ii) of Lemma 4.1, then we have the representation (cf. (4.4))

$$\left(\chi(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1}R_n\left(\chi(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1} =$$

$$\left(\chi(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1}E_n^{(3)}M_n\rho^{-1}S\rho(E_n^{(3)})^{-1}\left(\chi(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1} - E_n^{(3)}K_n^\dagger(E_n^{(3)})^{-1}$$

with the quadrature discretization $K_n^\dagger$ to the compact integral operator $K^\dagger$. However, since $K_n^\dagger$ is in $\mathcal{J}$ (cf. Lemma 2.4), since the discretized operators $(\chi(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}$ and $(f(x^\varphi_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}$

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commute, and since the commutator of the discretized singular operator commutes with the discretized multiplication operators by the previous parts of the present proof, we conclude that the commutator of $E_n^{(3)}(\chi(x^{\varphi}_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}$
$R_n(\chi(x^{\varphi}_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1}(E_n^{(3)})^{-1}$ and M_nfL_n is in $\mathcal{J}$ for R generated by g_{PC}. In other words, we get $\{M_nfL_nR_n^3 - R_n^3M_nfL_n\} \in \mathcal{J}$.

Now a general generator function, piecewise continuous on $\mathbb{T}$ and continuous on $\mathbb{T} \setminus \{\pm 1\}$, can be represented in the form $g(t) = \lambda g_+(t)g_{PC}(t) + \mu g_-(t)g_{PC}(t) + g_C(t)$ with g_C continuous (cf. the proof to part ii) of Lemma 4.1). Hence, the just proved relation $\{M_nfL_nR_n^3 - R_n^3M_nfL_n\} \in \mathcal{J}$ for $g = g_C$, $g_\pm$, and $g = g_{PC}$ show that $\{M_nfL_nR_n^3 - R_n^3M_nfL_n\} \in \mathcal{J}$ holds for general generators g, too.

Finally, we consider the commutator for the case $R = M, M'$. For similarity reason, we restrict our proof to $R = M$. Since the commutator is linear with respect to f, we may suppose that f is identically zero in the neighbourhood of one end point of the interval. Clearly,

$$M_nfL_nR_n^3 - R_n^3M_nfL_n = M_nfL_nR_n^3M_n(1-f)L_n + M_n(f-1)L_nR_n^3M_nfL_n\,.$$

Due to this, we only have to show $\{M_n\chi'L_nR_n^3M_n\chi L_n\} \in \mathcal{J}$ for smooth functions χ' and χ such that one of the two vanishes in a small neighbourhood of 1 and the other in a small neighbourhood of -1. In view of part i) of Lemma 4.1, we even may suppose that both functions χ and χ' vanish in small neighbourhoods of ± 1. In part ii-b) of the proof of Lemma 4.1 we have seen that $K_n = M_n\chi'L_nR_n^3M_n\chi|_{\operatorname{im} L_n}$ is a small perturbation of $L_nK|_{\operatorname{im} L_n}$ with K the compact integral operator corresponding to the kernel function (4.7). Thus $\{M_n\chi'L_nR_n^3M_n\chi L_n\} \in \mathcal{J}$, and the proof is completed. □

Now we formulate the local principle of Allan and Douglas applied to the algebra $\mathcal{A}/\mathcal{J}$ and to a central subalgebra $\mathcal{C}$. Due to Lemma 5.1 the set

$$\mathcal{C} := \left\{ \{M_nfL_n\}^o : f \in \mathbf{C}[-1,1] \right\}$$

forms a C^*-subalgebra of the center of $\mathcal{A}/\mathcal{J}$. This is $*$-isomorphic to $\mathbf{C}[-1,1]$ via the isomorphism $\{M_nfL_n\}^o \mapsto f$, and, consequently, the maximal ideal space of $\mathcal{C}$ is equal to $\{\mathcal{I}_\tau : \tau \in [-1,1]\}$ with $\mathcal{I}_\tau := \left\{ \{M_nfL_n\}^o : f \in \mathbf{C}[-1,1],\ f(\tau) = 0 \right\}$. By $\mathcal{J}_\tau$ we denote the smallest closed ideal of $\mathcal{A}/\mathcal{J}$ which contains $\mathcal{I}_\tau$, i.e.

$$\mathcal{J}_\tau := \tag{5.2}$$
$$\operatorname{clos}_{\mathcal{A}/\mathcal{J}} \left\{ \sum_{j=1}^m \{A_n^jM_nf_jL_n\}^o : \{A_n^j\} \in \mathcal{A},\ f_j \in \mathbf{C}[-1,1],\ f_j(\tau) = 0,\ m = 1,2,\ldots \right\}.$$

The local principle of Allan and Douglas claims

THEOREM 5.2 ([5] and [28], Theorem 1.21). *The ideal $\mathcal{J}_\tau$ is a proper ideal in $\mathcal{A}/\mathcal{J}$ for all $\tau \in [-1,1]$. Suppose $\{A_n\}^o$ is an arbitrary element of $\mathcal{A}/\mathcal{J}$. Then $\{A_n\}^o$ is invertible if and only if $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ for all $\tau \in [-1,1]$.*

6. The Local Invertibility at Points τ with $-1 < \tau < 1$. In this section we analyze the invertibility of $\{A_n\}^o + \mathcal{J}_\tau$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ for τ in the interior of the interval $[-1,1]$ (cf. Theorem 5.2). We fix a τ with $-1 < \tau < 1$ and set

$$h_\tau(t) := \begin{cases} 0 & \text{if } -1 \le t \le \tau\,, \\ 1 & \text{if } \tau < t \le 1\,. \end{cases}$$

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Then, for $a, b \in \mathbf{PC}$, we get

$$\{M_n a L_n\}^o + \mathcal{J}_\tau = a(\tau+0)\,\{M_n h_\tau L_n\}^o + a(\tau)\,\{M_n(1-h_\tau)L_n\}^o + \mathcal{J}_\tau$$

and $\left\{M_n \mu^{-1} S \mu L_n\right\}^o + \mathcal{J}_\tau = \left\{M_n \rho^{-1} S \rho L_n\right\}^o + \mathcal{J}_\tau$. Consequently, the subalgebra of $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ containing all sequences $\left\{M_n[aI + b\mu^{-1}S\mu]L_n\right\}^o + \mathcal{J}_\tau$ is generated by $\{L_n\}^o + \mathcal{J}_\tau$,

$$p := \frac{1}{2}\left(\{L_n\}^o + \left\{M_n \rho^{-1} S \rho L_n\right\}^o\right) + \mathcal{J}_\tau\,, \tag{6.1}$$

$$q := \{M_n h_\tau L_n\}^o + \mathcal{J}_\tau\,. \tag{6.2}$$

To analyze the invertibility in this C^* subalgebra, we utilize the following two-projections lemma.

LEMMA 6.1 (cf. e.g. [27] or [28], Section 1.16). *Suppose that $\mathcal{B}$ is a unital C^*-algebra, and that $p, q \in \mathcal{B}$ are projections (i.e. self-adjoint idempotent elements) such that the spectrum $\sigma_{\mathcal{B}}(pqp)$ coincides with the interval $[0,1]$. Then the smallest closed subalgebra of $\mathcal{B}$, which contains p, q, and the unit element e, is *-isomorphic to the C^*-algebra of all continuous 2×2 matrix functions on $[0,1]$, which are diagonal at 0 and 1. The isomorphism can be chosen in such a way that it maps e, p, q into the functions*

$$\mu \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{pmatrix}, \tag{6.3}$$

respectively.

Next we verify that our projections p and q from (6.1) and (6.2) satisfy the assumptions of the lemma, i.e. that p and q are projections and that $\sigma_{(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau}(pqp) = [0,1]$. If this is done, then we can apply Lemma 6.1 and we see that the local algebra $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ is *-isomorphic to a C^*-algebra of continuous 2×2 matrix functions on $[0,1]$ which are diagonal at 0 and 1. The isomorphism can be chosen in such a way that it maps $\{L_n\}^o + \mathcal{J}_\tau$, $\frac{1}{2}(\{L_n\}^o + \{M_n\rho^{-1}S\rho L_n\}^o) + \mathcal{J}_\tau$, and $\{M_n h_\tau P_n\}^o + \mathcal{J}_\tau$ into the functions given in (6.3), respectively. In particular, $\{M_n[aI + b\mu^{-1}S\mu I + K]L_n\}^o$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ (recall that $\{M_n K L_n\} \in \mathcal{J}$ due to Lemma 2.4) if the corresponding matrix symbol function

$$\mu \mapsto \begin{pmatrix} (1-\mu)c(\tau-0) + \mu c(\tau+0) & \sqrt{\mu(1-\mu)}[d(\tau+0) - d(\tau-0)] \\ \sqrt{\mu(1-\mu)}[c(\tau+0) - c(\tau-0)] & \mu d(\tau-0) + (1-\mu)d(\tau+0) \end{pmatrix},$$

$$c(\tau\pm 0) \;:=\; a(\tau\pm 0) + b(\tau\pm 0), \quad d(\tau\pm 0) \;:=\; a(\tau\pm 0) - b(\tau\pm 0)$$

is invertible. This, however, is satisfied if the operator $A = aI + b\mu^{-1}S\mu I + K$ is invertible in $\mathbf{L}^2_\sigma$ by the invertibility criteria of singular operators (cf. [11]). In other words, the invertibility condition of the coset $\{M_n A L_n\}^o + \mathcal{J}_\tau$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ does not impose a new stability condition on the operator equation.

Now we turn to the operators p and q and show that these are selfadjoint projections. Obviously, q is a selfadjoint projection. We prove that the same is true for p. In view of (3.1), (3.2), and

$$T_{n+2} - T_n = -2\varphi^2 U_n\,,\; n = 1, 2, \ldots\,, \quad T_2 - \sqrt{2}T_0 = -2\varphi^2 U_0\,,$$

we get

$$\rho^{-1}S\rho\,\varphi\,\rho^{-1}S\rho\widetilde{u}_n = \rho^{-1}S\varphi S\varphi U_n = \mathrm{i}\rho^{-1}S\varphi T_{n+1}$$

$$= \begin{cases} \frac{1}{2}\,\rho^{-1}(T_n - T_{n+2}) & \text{if } n = 1, 2, \ldots\,, \\ -\frac{1}{2}\,\rho^{-1}T_2 & \text{if } n = 0\,, \end{cases}$$

$$= \begin{cases} \varphi\widetilde{u}_n & \text{if } n = 1, 2, \ldots\,, \\ \varphi\widetilde{u}_0 - \frac{1}{\sqrt{2}}\,\rho^{-1}T_0 & \text{if } n = 0\,. \end{cases}$$

Thus, by the continuity of $\rho^{-1}S\rho : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ we have

(6.4) $$\rho^{-1}S\rho\,\varphi\,\rho^{-1}S\rho I = \varphi I + K_0\,, \quad K_0 u = -\frac{1}{\sqrt{2}}\,\langle u, \widetilde{u}_0\rangle_\sigma\,\rho^{-1}T_0\,.$$

We recall the relation

$$S\rho L_n u = \mathrm{i}\sum_{k=0}^{n-1}\langle u, \widetilde{u}\rangle_\sigma\,T_{k+1} = \frac{\mathrm{i}}{2}\sum_{k=0}^{n-1}\langle u, \widetilde{u}_k\rangle_\sigma\,(U_{k+1} - U_{k-1})\,,$$

which implies

$$M_n\varphi\,\rho^{-1}S\rho L_n u = M_n\vartheta S\rho L_n u = \vartheta S\rho L_n u - \frac{\mathrm{i}}{2}\,\langle u, \widetilde{u}_{n-1}\rangle_\sigma\,\widetilde{u}_n$$

$$= \vartheta S\rho L_n u - \frac{\mathrm{i}}{2}\,V W_n L_1 W_n u\,,$$

where $V : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ denotes the shift operator $V\,u = \sum_{k=0}^\infty \langle u, \widetilde{u}_k\rangle_\sigma\,\widetilde{u}_{k+1}$. Consequently, due to (6.4), we have the identity

$$M_n\rho^{-1}S\rho L_n M_n\varphi\,\rho^{-1}S\rho L_n = M_n(\varphi I + K_0)L_n - \frac{\mathrm{i}}{2}\,M_n\rho^{-1}S\rho V L_n W_n L_1 W_n$$

and

$$\begin{aligned} &\left\{M_n\rho^{-1}S\rho L_n\right\}^o \left\{M_n\rho^{-1}S\rho L_n\right\}^o + \mathcal{J}_\tau \\ &\quad = \frac{1}{\varphi(\tau)}\,\left\{M_n\rho^{-1}S\rho L_n\right\}^o \left\{M_n\varphi\,\rho^{-1}S\rho L_n\right\}^o + \mathcal{J}_\tau \\ &\quad = \frac{1}{\varphi(\tau)}\,\left\{M_n\varphi L_n\right\}^o + \mathcal{J}_\tau \\ &\quad = \left\{L_n\right\}^o + \mathcal{J}_\tau\,, \quad -1 < \tau < 1\,. \end{aligned}$$

Hence, we conclude $p^2 = p$. From (3.1), (3.2), and the three-term-recurrence relation

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)\,, \quad k = 1, 2, \ldots\,,$$

we find

$$V = \psi I - \mathrm{i}\vartheta S\rho I\,, \quad V^* = \psi I + \mathrm{i}\vartheta S\rho I\,, \quad \psi(x) = x\,.$$

This implies

$$\begin{aligned} \left\{M_n\rho^{-1}S\rho L_n\right\}^o + \mathcal{J}_\tau &= -\frac{\mathrm{i}}{\varphi(\tau)}\,\left\{M_n\mathrm{i}\vartheta S\rho L_n\right\}^o + \mathcal{J}_\tau \\ &= -\frac{\mathrm{i}}{\varphi(\tau)}\,(\left\{V^*L_n\right\}^o - \left\{M_n\psi L_n\right\}^o) + \mathcal{J}_\tau \end{aligned}$$

and, consequently,

$$\left(\left\{M_n\rho^{-1}S\rho L_n\right\}^o + \mathcal{J}_\tau\right)^* = \frac{\mathrm{i}}{\varphi(\tau)}\left(\{L_nVL_n\}^o - \{M_n\psi L_n\}^o\right) + \mathcal{J}_\tau$$
$$= \frac{\mathrm{i}}{\varphi(\tau)}\left\{M_n(-\mathrm{i}\vartheta)S\rho L_n\right\}^o + \mathcal{J}_\tau$$
$$= \left\{M_n\rho^{-1}S\rho L_n\right\}^o + \mathcal{J}_\tau\,.$$

Thus, we get $p^* = p$.

Now we turn to the spectrum. It remains to prove $\sigma_{(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau}(pqp) = [0,1]$. To this end, we introduce $\mathcal{G}$ as the smallest C^*-subalgebra of $\mathcal{L}(\mathbf{L}^2_\sigma)$ which contains all operators $aI + b\mu^{-1}S\mu I$ with $a, b \in \mathbf{PC}[-1,1]$ and the ideal $\mathcal{K} = \mathcal{K}(\mathbf{L}^2_\sigma)$ of all compact operators in $\mathbf{L}^2_\sigma$. By $\mathcal{J}^{\mathcal{G}}_\tau$, $\tau \in [-1,1]$, we denote the smallest closed ideal of $\mathcal{G}/\mathcal{K}$, which contains all cosets $fI + \mathcal{K}$ with $f \in \mathbf{C}[-1,1]$ and $f(\tau) = 0$. We need the following

LEMMA 6.2. *If $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$, then $(W_1\{A_n\} + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau$ is invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}^{\mathcal{G}}_\tau$.*

Proof. Take $\{A_n\} \in \mathcal{A}$, and assume that there exists a sequence $\{B_n\} \in \mathcal{A}$ such that $\{B_n\}^o\{A_n\}^o + \mathcal{J}_\tau = \{L_n\}^o + \mathcal{J}_\tau$. Then

$$B_nA_n = L_n + J_n + \sum_{\omega=1}^{4}(E_n^{(\omega)})^{-1}L_n^{(\omega)}T_\omega E_n^{(\omega)} + C_n$$

with some $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$ and some $\{J_n\}^o \in \mathcal{J}_\tau$, $\{C_n\} \in \mathcal{N}$. For each $\vartheta > 0$ there exist sequences $\{A_n^{(j)}\} \in \mathcal{A}$ and functions $f_j \in \mathbf{C}[-1,1]$ with $f_j(\tau) = 0$ such that

$$\|\{J_n\}^o - \{D_n\}^o\|_{\mathcal{A}/\mathcal{J}} < \vartheta$$

for $D_n := \sum_{j=1}^{m_\vartheta} A_n^{(j)}M_nf_jL_n$. Hence, there are $T_{\omega,\vartheta} \in \mathcal{K}(\mathbf{X}_\omega)$ and $\{C_n^\vartheta\} \in \mathcal{N}$ such that

$$\left\|J_nL_n - \sum_{j=1}^{m_\vartheta}A_n^{(j)}M_nf_jL_n - \sum_{\omega=1}^{4}(E_n^{(\omega)})^{-1}L_n^{(\omega)}T_{\omega,\vartheta}E_n^{(\omega)} - C_n^\vartheta L_n\right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} < \vartheta\,,$$

$n = 1, 2, \ldots$. We conclude $\|W_1\{J_n\} - \sum_{j=1}^{m_\vartheta} W_1\{A_n^{(j)}\}f_jI - T_{1,\vartheta}\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \le \vartheta$, which implies $W_1\{J_n\} + \mathcal{K} \in \mathcal{J}^{\mathcal{G}}_\tau$. Thus, because of $W_1\{B_n\}W_1\{A_n\} = I + W_1\{J_n\} + T_1$, the coset $(W_1\{A_n\} + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau$ is invertible from the left in $(\mathcal{G}/\mathcal{K})/\mathcal{J}^{\mathcal{G}}_\tau$. The invertibility from the right can be shown analogously. □

The product pqp is a selfadjoint non-negative element of $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$, which implies that the spectrum $\sigma_{(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau}(pqp)$ is a subset of $[0,1]$. We prove that the spectrum of pqp coincides with the whole interval. For this, assume that there exists a $\lambda \in (0,1)$ such that $pqp - \lambda e$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$. This is equivalent to the invertibility of

$$(q-\lambda)p - \lambda(e-p) = \frac{1}{2}\{M_n(h_\tau - \lambda)L_n\}^o\left(\{L_n\}^o + \{M_n\rho^{-1}S\rho L_n\}^o\right)$$
$$-\frac{\lambda}{2}\left(\{L_n\}^o - \{M_n\rho^{-1}S\rho L_n\}^o\right) + \mathcal{J}_\tau\,.$$

From Lemma 6.2 we conclude that

$$(A+\mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau := [(h_\tau - \lambda)(I + \rho^{-1}S\rho I) - \lambda(I - \rho^{-1}S\rho I) + \mathcal{K}] + J^{\mathcal{G}}_\tau$$

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is invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}_\tau^\mathcal{G}$. If $-1 \le x < \tau$, then we have

$$(A+\mathcal{K}) + \mathcal{J}_x^\mathcal{G} = (-2\lambda I + \mathcal{K}) + \mathcal{J}_x^\mathcal{G} \,,$$

and $-2\lambda I + \mathcal{K}$ is invertible in $\mathcal{G}/\mathcal{K}$. If $\tau < x \le 1$, then

$$(A+\mathcal{K}) + \mathcal{J}_x^\mathcal{G} = [(1-2\lambda)I + \rho^{-1}S\rho I + \mathcal{K}] + \mathcal{J}_x^\mathcal{G} \,,$$

which is also invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}_x^\mathcal{G}$. From the local principle of Allan and Douglas we conclude the Fredholmness of $(h_\tau - \lambda)(I + \rho^{-1}S\rho I) - \lambda(I-S)$ in $\mathbf{L}^2_\sigma$. But this is a contradiction since $0 \in \left[1-\frac{1}{\lambda},1\right] = \left[\frac{\lambda - h_\tau(\tau+0)}{\lambda}, \frac{\lambda - h_\tau(\tau-0)}{\lambda}\right]$.

7. The Local Invertibility for $\tau = \pm 1$. In this section we analyze the invertibility of $\{A_n\}^o + \mathcal{J}_{\pm 1}$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_{\pm 1}$ (cf. Theorem 5.2) and show that the invertibility of the operators $W_3\{A_n\}$ and $W_4\{A_n\}$ imply the invertibility of $\{A_n\}^o + \mathcal{J}_{+1}$ and $\{A_n\}^o + \mathcal{J}_{-1}$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_{\pm 1}$, respectively.

First we recall a lemma on the Gohberg-Krupnik symbol for operators from $alg\,\mathcal{T}(\mathbf{PC})$ and on special elements of the algebra $alg\,\mathcal{T}(\mathbf{PC})$ which is useful for the proof that the limit operators belong to $alg\,\mathcal{T}(\mathbf{PC})$.

LEMMA 7.1 ([20] and [29, 16] or Lemma 11.4 of [28]). **i)** *There is a continuous mapping* $\mathbf{Symb}$ *from* $alg\,\mathcal{T}(\mathbf{PC})$ *to a set of functions defined over* $\mathbb{T}\times[0,1]$. *For each* $R \in alg\,\mathcal{T}(\mathbf{PC})$, *the corresponding function* $\mathbf{Symb}_R(t,\mu)$ *will be called the symbol of* R. *This symbol satisfies:*

1) *For any* $t \ne \pm 1$, *the value* $\mathbf{Symb}_R(t,\mu)$ *is independent of* μ, *and the function* $t \mapsto \mathbf{Symb}_R(t,0)$ *is continuous on* $\{t \in \mathbb{T} : \ \Im\mathrm{m}\, t \ge 0\}$ *and on* $\{t \in \mathbb{T} : \ \Im\mathrm{m}\, t \le 0\}$ *with the limits*

$$\begin{aligned}
\mathbf{Symb}_R(1+0,0) &:= \lim_{t\longrightarrow +1,\ \Im\mathrm{m}\, t>0} \mathbf{Symb}_R(t,0) &= \mathbf{Symb}_R(1,1)\,,\\
\mathbf{Symb}_R(1-0,0) &:= \lim_{t\longrightarrow +1,\ \Im\mathrm{m}\, t<0} \mathbf{Symb}_R(t,0) &= \mathbf{Symb}_R(1,0),\\
\mathbf{Symb}_R(-1+0,0) &:= \lim_{t\longrightarrow -1,\ \Im\mathrm{m}\, t<0} \mathbf{Symb}_R(t,0) &= \mathbf{Symb}_R(-1,1),\\
\mathbf{Symb}_R(-1-0,0) &:= \lim_{t\longrightarrow -1,\ \Im\mathrm{m}\, t>0} \mathbf{Symb}_R(t,0) &= \mathbf{Symb}_R(-1,0).
\end{aligned}$$

Moreover, the function $\mu \mapsto \mathbf{Symb}_R(\pm 1,\mu)$ *is continuous on* $[0,1]$.

2) *For any* $R \in alg\,\mathcal{T}(\mathbf{PC})$, *the operator* R *is Fredholm if and only if the symbol* $\mathbf{Symb}_R$ *does not vanish over* $\mathbb{T}\times[0,1]$.

3) *For any Fredholm operator* $R \in alg\,\mathcal{T}(\mathbf{PC})$, *the index of* R *is the negative winding number of the closed curve*

$$\begin{aligned}
\Gamma := &\left\{\mathbf{Symb}_R(e^{\mathrm{i}s},0) : \ 0<s<\pi\right\} \cup \left\{\mathbf{Symb}_R(-1,s) : \ 0 \le s \le 1\right\}\\
&\cup \left\{\mathbf{Symb}_R(-e^{\mathrm{i}s},0) : \ 0<s<\pi\right\} \cup \left\{\mathbf{Symb}_R(1,s) : \ 0 \le s \le 1\right\}
\end{aligned}$$

with respect to the point 0, *where the direction of the curve* Γ *is determined by the parametrizations of its definition.*

4) *An operator* $R \in alg\,\mathcal{T}(\mathbf{PC})$ *is compact if and only if* $\mathbf{Symb}_R(t,\mu)$ *vanishes over* $\mathbb{T}\times[0,1]$.

ii) *Suppose the generating function* $g(t) = \sum_l \hat{g}_l t^l$ *of the Toeplitz matrix* $(\hat{g}_{k-j})_{k,j=0}^\infty$ *is piecewise continuous on* $\mathbb{T}$ *and continuous on* $\mathbb{T}\setminus\{\pm 1\}$, *and take a complex* z *with* $|\Re\mathrm{e}\, z| < 1/2$. *Then the matrix operator*

$$R := ([k+1]^{-z}\delta_{k,j})_{k,j=0}^\infty (\hat{g}_{k-j})_{k,j=0}^\infty ([k+1]^{z}\delta_{k,j})_{k,j=0}^\infty$$

belongs to $alg\,\mathcal{T}(\mathbf{PC})$*, and its symbol is given by*

$$\mathbf{Symb}_R(t,\mu) = \begin{cases} g(t) & \text{if } t \in \mathbb{T}\setminus\{\pm 1\} \\ \dfrac{\mu g(\pm 1+0) + (1-\mu) g(\pm 1 - 0) e^{-\mathrm{i}2\pi z}}{\mu + (1-\mu)e^{-\mathrm{i}2\pi z}} & \text{if } t = \pm 1. \end{cases}$$

iii) *For any fixed Toeplitz matrix* $R = (\hat{g}_{k-j})_{k,j=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$ *with a generating function which is piecewise twice continuously differentiable, the operator function* $\{z \in \mathbb{C} : \ |\Re\mathrm{e}\, z| < 1/2\} \ni z \mapsto ([k+1]^{-z}\delta_{k,j})_{k,j=0}^{\infty}\ R\ ([k+1]^{z}\delta_{k,j})_{k,j=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$ *is continuous in the operator norm.*
iv) *Suppose the Mellin transform* $\widehat{m}(z) := \int_0^\infty m(\sigma)\sigma^{z-1}\mathrm{d}\sigma$ *of the univariate function* $m : (0,\infty) \longrightarrow \mathbb{C}$ *is analytic in the strip* $1/2-\varepsilon < \Re\mathrm{e}\, z < 1/2+\varepsilon$ *for a small* $\varepsilon > 0$*. Moreover, suppose*

$$\sup_{z:\,1/2-\varepsilon<\Re\mathrm{e}\, z<1/2+\varepsilon} \left| \frac{\mathrm{d}^k}{\mathrm{d}z^k}\widehat{m}(z)(1+|z|)^k \right| < \infty, \quad k = 0,1,\ldots\ .$$

Then m *is infinitely differentiable on* $(0,\infty)$*. The operators* $M, M' \in \mathcal{L}(\ell^2)$ *defined by*

$$M := \left(m\left(\frac{k+1}{j+1}\right) \frac{1}{j+1}\right)_{k,j=0}^{\infty}, \quad M' := \left((-1)^{k-j} m\left(\frac{k+1}{j+1}\right) \frac{1}{j+1}\right)_{k,j=0}^{\infty}$$

belong to the algebra $alg\,\mathcal{T}(\mathbf{PC})$ *and their symbols are given by*

$$\mathbf{Symb}_M(t,\mu) = \begin{cases} \widehat{m}\left(\frac{1}{2} + \frac{\mathrm{i}}{2\pi}\log\frac{\mu}{1-\mu}\right) & \text{if } t = 1 \\ 0 & \text{else}\ , \end{cases}$$

$$\mathbf{Symb}_{M'}(t,\mu) = \begin{cases} \widehat{m}\left(\frac{1}{2} + \frac{\mathrm{i}}{2\pi}\log\frac{\mu}{1-\mu}\right) & \text{if } t = -1 \\ 0 & \text{else}\ . \end{cases}$$

Now we turn to the local invertibility. For symmetry reasons, we may restrict our consideration to the invertibility of $\{A_n\}^o + \mathcal{J}_1$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_1$. The following lemma shows that this invertibility follows from the invertibility of $W_3\{A_n\}$.

LEMMA 7.2. **i)** *Suppose* $R \in alg\,\mathcal{T}(\mathbf{PC})$ *is invertible and consider the sequence* R_n^3*, then the coset* $\{[R^{-1}]_n^3\}^o + \mathcal{J}_1$ *is the inverse of* $\{R_n^3\}^o + \mathcal{J}_1$ *in* $(\mathcal{A}/\mathcal{J})/\mathcal{J}_1$*.*
ii) *Suppose (1.5) and (1.6), and consider* $A_n = M_n[aI + b\mu^{-1}S\mu I + K]L_n$ *and* $R := W_3\{A_n\}$*. Then* R *is in* $alg\,\mathcal{T}(\mathbf{PC})$*.*
iii) *Under the assumptions of assertion ii), the cosets* $\{R_n^3\}^o + \mathcal{J}_1$ *and* $\{A_n\}^o + \mathcal{J}_1$ *coincide. In particular,* $\{A_n\}^o + \mathcal{J}_1$ *is invertible if* R *is invertible.*

Proof. **i)** To show the assertion i), we only have to prove that, for any $R, R^\dagger \in alg\,\mathcal{T}(\mathbf{PC})$,

$$R_n^3[R^\dagger]_n^3 - [RR^\dagger]_n^3 = (E_n^{(3)})^{-1}\Big[R_nR_n^\dagger - [RR^\dagger]_n\Big]E_n^{(3)} \in \mathcal{J}_1. \tag{7.1}$$

We choose a smooth and bounded function χ on $[-1,1]$ such that χ is identically equal to one in a neighborhood of 1 and such that $\operatorname{supp}\chi \subset [1-\varepsilon, 1]$ for a small prescribed $\varepsilon > 0$. Then $\{M_n\chi L_n\}^o + \mathcal{J}_1$ is the unit element $\{L_n\}^o + \mathcal{J}_1$, and

$$\begin{aligned} M_n\chi L_n(E_n^{(3)})^{-1}&\Big[R_nR_n^\dagger - [RR^\dagger]_n\Big]E_n^{(3)} \\ &= (E_n^{(3)})^{-1}\left[\left(\chi(x^\varphi_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1} R(P_n - I)R^\dagger\right]P_nE_n^{(3)} \end{aligned}$$

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However, for an arbitrarily prescribed small positive threshold and for sufficiently small ε in the restriction of the support of χ, the norm of the matrix

$$(\chi(x^{\varphi}_{(j+1)n})\delta_{j,k})_{j,k=0}^{n-1} R(P_n - I)$$

is less than this threshold by the proof to part i) of Lemma 4.1. Consequently, (7.1) is true.

ii) Now we prove $R = W_3\{A_n\} \in alg\,\mathcal{T}(\mathbf{PC})$. For the limits of the discretized multiplication operators (cf. Lemma 3.8), this is obvious. It remains to consider the limit operators A_+ and $\mathbf{A}^{\mu}_{+}$ (cf. the Lemmata 3.9 and 3.10). Moreover, since the diagonal entries in the diagonal matrices $\mathbf{B}_+$ and $\mathbf{V}_+$ tend to zero (cf. (3.31) and (3.32)) and since the compact operators belong to $alg\,\mathcal{T}(\mathbf{PC})$, we only have to show that A_+, $\mathbf{A}$, and $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$ belong to $alg\,\mathcal{T}(\mathbf{PC})$ (cf. (3.13) and (3.14)). These three operators can be treated in the same manner. Hence, we consider only one of them. For definiteness, we take $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$.

We start with a well-known formula for the Mellin transform (cf. e.g. [10, 11])

$$\frac{1}{\pi \mathrm{i}}\frac{1}{1-x} = \frac{1}{2\pi\mathrm{i}}\int_{\{z:\, \Re\mathrm{e}\, z=1/2\}} x^{-z}\{-\mathrm{i}\cot(\pi z)\}\mathrm{d}z\,, \quad x>0\,,$$

and, by straightforward transformations and by the residue theorem (cf. (3.27) for the analyticity of the integrand), we conclude

$$\begin{aligned}\frac{1}{\pi\mathrm{i}}\frac{2x^{2\chi_+}}{1-x^2} &= \frac{1}{2\pi\mathrm{i}}\int_{\{z:\, \Re\mathrm{e}\, z=1/2\}} x^{-(2z-2\chi_+)}\{-\mathrm{i}\cot(\pi z)\}2\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\{\zeta:\, \Re\mathrm{e}\, \zeta=1-2\chi_+\}} x^{-\zeta}\left\{-\mathrm{i}\cot\left(\pi\left(\frac{\zeta}{2}+\chi_+\right)\right)\right\}\mathrm{d}\zeta\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\{\zeta:\, \Re\mathrm{e}\, \zeta=1/2\}} x^{-\zeta}\left\{-\mathrm{i}\cot\left(\pi\left(\frac{\zeta}{2}+\chi_+\right)\right)\right\}\mathrm{d}\zeta.\end{aligned}$$

Subtracting the similar formula for $\frac{1}{\pi\mathrm{i}}\frac{x^{[\chi_+-1/4]}}{1-x}$, we obtain

$$\begin{aligned}\frac{1}{\pi\mathrm{i}}\frac{2x^{2\chi_+}}{1-x^2} - \frac{1}{\pi\mathrm{i}}\frac{x^{[\chi_+-1/4]}}{1-x} &= \frac{1}{2\pi\mathrm{i}}\int_{\{\zeta:\, \Re\mathrm{e}\, \zeta=1/2\}} x^{-\zeta}\mathcal{B}(\zeta)\mathrm{d}\zeta,\\ \mathcal{B}(\zeta) &:= -\mathrm{i}\cot\left(\pi\left(\frac{\zeta}{2}+\chi_+\right)\right) + \mathrm{i}\cot\left(\pi\left(\zeta+\left[\chi_+-\frac{1}{4}\right]\right)\right).\end{aligned} \tag{7.2}$$

Note that (7.2) holds with the integral defined in the principal value sense. For $x<1$ resp. $x>1$, formula (7.2) can be derived rigorously by simply applying the residue theorem over $\{\zeta : \Re\mathrm{e}\,\zeta < 1/2\}$ resp. $\{\zeta : \Re\mathrm{e}\,\zeta > 1/2\}$ and by taking into account that $\mathcal{B}(\zeta) = O(e^{-|\Im\mathrm{m}\,\zeta|})$ for $|\Im\mathrm{m}\,\zeta| \longrightarrow \infty$. Choosing a $\psi\in\mathbb{R}$ with $\max\{-1/2, -2\chi_+\} < \psi < 1/4-\chi_+$ and applying, again, a simple transformation and the residue theorem, we arrive at

$$\begin{aligned}\kappa(x) &:= (1-x)\left\{\frac{2x^{1/2}}{1-x^2}x^{2[\chi_+-1/4]} - \frac{1}{1-x}x^{[\chi_+-1/4]}\right\}\\ &= \frac{1}{2}\int_{\{\zeta:\, \Re\mathrm{e}\, \zeta=\psi\}} x^{-\zeta}\Big\{\mathcal{B}(\zeta)-\mathcal{B}(\zeta+1)\Big\}\mathrm{d}\zeta - x^{[\chi_+-1/4]}.\end{aligned}$$

Consequently, we get

$$\left(\frac{1}{\pi\mathrm{i}}\frac{\left(\frac{j+1}{k+1}\right)^{2\chi_+}2(k+1)(1-\delta_{k,j})}{(k+1)^2-(j+1)^2}\right)_{j,k=0}^{\infty} - \left(\frac{1}{\pi\mathrm{i}}\frac{\left(\frac{j+1}{k+1}\right)^{[\chi_+-1/4]}(1-\delta_{k,j})}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty} =$$

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$$\frac{1}{\pi \mathrm{i}}\left(\frac{\left(\frac{j+1}{k+1}\right)^{2[\chi_+-1/4]}2\left(\frac{j+1}{k+1}\right)^{1/2}(1-\delta_{k,j})}{1-\left(\frac{j+1}{k+1}\right)^2}\frac{1}{k+1}-\frac{\left(\frac{j+1}{k+1}\right)^{[\chi_+-1/4]}(1-\delta_{k,j})}{1-\frac{j+1}{k+1}}\frac{1}{k+1}\right)_{j,k=0}^{\infty}$$

$$=\left(\frac{1}{\pi \mathrm{i}}\,\kappa\left(\frac{j+1}{k+1}\right)\frac{1-\delta_{k,j}}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty}$$

$$=\frac{1}{2}\int_{\{\zeta:\,\Re\mathrm{e}\,\zeta=\psi\}}\left(\frac{1}{\pi \mathrm{i}}\,\frac{\left(\frac{j+1}{k+1}\right)^{-\zeta}(1-\delta_{k,j})}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty}\Big\{\mathcal{B}(\zeta)-\mathcal{B}(\zeta+1)\Big\}\mathrm{d}\zeta$$

$$-\left(\frac{1}{\pi \mathrm{i}}\,\frac{\left(\frac{j+1}{k+1}\right)^{[\chi_+-1/4]}(1-\delta_{k,j})}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty},$$

$$\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$$

$$=-\frac{1}{2}\int_{\{\zeta:\,\Re\mathrm{e}\,\zeta=\psi\}}\left(\frac{1}{\pi \mathrm{i}}\,\frac{\left(\frac{j+1}{k+1}\right)^{-\zeta}(1-\delta_{k,j})}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty}\Big\{\mathcal{B}(\zeta)-\mathcal{B}(\zeta+1)\Big\}\mathrm{d}\zeta\,. \tag{7.3}$$

Note that the last integral is to be understood in the sense of Bochner (cf. [32]). This is possible since the operator function $\{\zeta:\ \Re\mathrm{e}\,\zeta=\psi\}\ni\zeta\mapsto\big(\frac{1}{\pi \mathrm{i}}\frac{[(j+1)/(k+1)]^{-\zeta}(1-\delta_{k,j})}{(k+1)-(j+1)}\big)_{j,k}$ is continuous and uniformly bounded and since $\{\zeta:\ \Re\mathrm{e}\,\zeta=\psi\}\ni\zeta\mapsto[\mathcal{B}(\zeta)-\mathcal{B}(\zeta+1)]$ is a continuous and absolutely integrable function. Obviously, the matrix $\big(\frac{1}{\pi \mathrm{i}}\,\frac{(1-\delta_{k,j})}{(k+1)-(j+1)}\big)_{j,k}$ is a Toeplitz matrix and its generating function $g(e^{\mathrm{i}2\pi s})=-\sum_{l\neq 0}\frac{1}{\pi \mathrm{i}}\frac{1}{l}e^{\mathrm{i}2\pi l s}=2s-1,\ 0\le s<1$ is piecewise continuous and continuous on $\mathbb{T}\setminus\{1\}$. Thus, in view of Lemma 7.1, the integral representation (7.3) proves that the operator $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$ is in $alg\,\mathcal{T}(\mathbf{PC})$.

iii) It remains to show the local equivalence of A_n and R_n^3, i.e. that $\{R_n^3-A_n\}^o\in\mathcal{J}_1$. We show this result separately for (cf. the Lemmata 3.8, 3.9, and 3.10)

$$\Big\{[a(1)I]_n^3-M_naL_n\Big\}^o\in\mathcal{J}_1, \tag{7.4}$$

$$\Big\{[A_+]_n^3-M_n\rho^{-1}S\rho L_n\Big\}^o\in\mathcal{J}_1, \tag{7.5}$$

$$\Big\{[\mathbf{A}_+^\mu]_n^3-(E_n^{(3)})^{-1}\mathbf{A}_n^+E_n^{(3)}\Big\}^o\in\mathcal{J}_1, \tag{7.6}$$

where $\mathbf{A}_+^\mu$ and $\mathbf{A}_n^+$ are defined in (3.14) and (3.25), respectively. That is

$$\big[\mathbf{A}_+^\mu\big]_n^3-(E_n^{(3)})^{-1}\mathbf{A}_n^+E_n^{(3)}$$

$$=(E_n^{(3)})^{-1}\big[P_n(\mathbf{B}_++\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}-\mathbf{A}-\mathbf{D}_+\mathbf{A}\mathbf{W}\mathbf{V}_+\mathbf{D}_+^{-1}+\mathbf{V}_+\mathbf{A}\mathbf{W})$$

$$-\mathbf{B}_n-\mathbf{D}_n\mathbf{A}_n\mathbf{D}_n+\mathbf{A}_n+\mathbf{D}_n\mathbf{A}_n\mathbf{W}_n\mathbf{V}_n\mathbf{D}_n^{-1}-\mathbf{V}_n\mathbf{A}_n\mathbf{W}_n\big]\,E_n^{(3)}\,.$$

Let $\mathbf{C}_1$ denote the class of continuous functions $f:[-1,1]\longrightarrow[0,1]$ with $f(1)=1$. Then, the first inclusion (7.4) is an immediate consequence of the limit $a(\tau)\longrightarrow a(1)$ for $\tau\longrightarrow 1$

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and the resulting relation (cf. the definition of the local ideal $\mathcal{J}_1$ in (5.2))

$$\inf_{f\in\mathbf{C}_1} \left\| \left\{ M_n f L_n \right\}^o \left\{ [a(1)I]_n^3 - M_n a L_n \right\}^o \right\| \le \inf_{f\in\mathbf{C}_1} \sup_{n=1,2,\dots} \left\| \left(f(x^\varphi_{(k+1)n}) \left[a(1) - a(x^\varphi_{(k+1)n}) \right] \delta_{k,j} \right)_{k,j=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} = 0 .$$

Next we turn to (7.5). We introduce the function $\Phi(s) = \cos\sqrt{s}\,,\ s \in [0, \pi^2/4]$. Then

$$h(s', r') := \frac{\Phi'(s')}{\Phi(s') - \Phi(r')} - \frac{1}{s' - r'}$$

is bounded for $s', r' \in [0, \pi^2/4]$ and, for $s, r \in [0, \pi/2]$,

$$\frac{\sin s}{\cos r - \cos s} - \frac{2s}{s^2 - r^2} = \frac{2s\Phi'(s^2)}{\Phi(s^2) - \Phi(r^2)} - \frac{2s}{s^2 - r^2} = 2s\, h(s^2, r^2) .$$

Hence, due to the definition of A_+ and due to (3.10), the entries of

$$E_n^{(3)} \left([A_+]_n^3 - M_n \rho^{-1} S \rho L_n \right) (E_n^{(3)})^{-1}$$

can be estimated by

$$\left| \frac{\varphi(x^\varphi_{kn})}{n+1} \frac{1}{x^\varphi_{kn} - x^\varphi_{jn}} - \frac{2k}{\pi(j^2 - k^2)} \right|$$

$$= \left| \frac{\sin\frac{k\pi}{n+1}}{(n+1)\left(\cos\frac{k\pi}{n+1} - \cos\frac{j\pi}{n+1}\right)} - \frac{2\frac{k\pi}{n+1}}{(n+1)\left[\left(\frac{j\pi}{n+1}\right)^2 - \left(\frac{k\pi}{n+1}\right)^2\right]} \right|$$

$$= \left| \frac{1}{n+1} \frac{2k\pi}{n+1}\, h\left(\left(\frac{k\pi}{n+1}\right)^2, \left(\frac{j\pi}{n+1}\right)^2 \right) \right| \le \frac{C\,k}{(n+1)^2}\,,\ 1 \le k \le \frac{n+1}{2} . \tag{7.7}$$

Consequently, we get

$$\inf_{f\in\mathbf{C}_1} \left\| \{M_n f L_n\}^o \left\{ [A_+]_n^3 - M_n\rho^{-1}S\rho L_n \right\}^o \right\|$$
$$\le \inf_{f\in\mathbf{C}_1} \sup_{n=1,2,\dots} \left\| \left(f(x^\varphi_{(k+1)n})\delta_{k,j} \right)_{j,k=0}^{n-1} E_n^{(3)} \left([A_+]_n^3 - M_n\rho^{-1}S\rho L_n \right) (E_n^{(3)})^{-1} \right\|_{\mathcal{L}(\ell^2)}$$
$$\le \inf_{f\in\mathbf{C}_1} \sup_{n=1,2,\dots} \left\| \left(f(x^\varphi_{(k+1)n}) \frac{Ck}{(n+1)^2} \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} .$$

Using a Frobenius norm estimate and choosing f with a support $\mathrm{supp}(f\circ\cos) \subset [0,\varepsilon]$ with an arbitrarily prescribed small ε, we get a bound less than $C\varepsilon^{3/2}$, and the inclusion (7.5) follows.

For a fixed k_0, the projection $P_{k_0} \in \mathcal{L}(\ell^2)$ is compact. Hence,

$$\{(E_n^{(3)})^{-1} P_n P_{k_0} E_n^{(3)}\} \in \mathcal{J} ,$$

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and, for $f \in \mathbf{C}_1$ with $\mathrm{supp}(f \circ \cos) \subset [0, \varepsilon]$, we arrive at (cf. (3.32))

$$\begin{aligned} &\left\| \left\{ M_n f L_n \right\}^o (E_n^{(3)})^{-1} \mathbf{V}_n E_n^{(3)} \right\}^o \right\| \\ \text{(7.8)} \quad &\leq \sup_{n=1,2,\ldots} \left\| P_n \left(I - P_{k_0} \right) \left(f(x^\varphi_{(k+1)n}) |d^n_k| \delta_{k,j} \right)_{k,j=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \leq \sup_{k_0 \leq k \leq \frac{\varepsilon}{\pi}(n+1)} \left| \frac{C}{k^{\varepsilon_0}} \right| \leq \frac{C}{k_0^{\varepsilon_0}} \end{aligned}$$

for some $\varepsilon_0 > 0$. Replacing $\mathbf{V}_n$ by $[\mathbf{V}_+]^3_n$, $\mathbf{B}_n$, and $[\mathbf{B}_+]^3_n$, respectively, we obtain the bounds

$$\left\| \left\{ M_n f L_n \right\}^o \left\{ [\mathbf{V}_+]^3_n \right\}^o \right\| \leq \frac{C}{k_0^{\varepsilon_0}} \,, \tag{7.9}$$

$$\left\| \left\{ M_n f L_n \right\}^o \left\{ (E_n^{(3)})^{-1} \mathbf{B}_n E_n^{(3)} \right\}^o \right\| \leq \frac{C}{k_0^{\varepsilon_0}} \,, \tag{7.10}$$

$$\left\| \left\{ M_n f L_n \right\}^o \left\{ [\mathbf{B}_+]^3_n \right\}^o \right\| \leq \frac{C}{k_0^{\varepsilon_0}} \,. \tag{7.11}$$

which are analogous to that in (7.8).

The entries of $P_n \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} P_n - \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}$ can be written in the form

$$\begin{aligned} &\left(\frac{j}{k} \right)^{2\chi_+} \frac{2k(1-\delta_{j,k})}{\pi \mathrm{i}(j^2-k^2)} - \frac{\chi(x^\varphi_{jn})}{\chi(x^\varphi_{kn})} \frac{\varphi(x^\varphi_{kn})}{\mathrm{i}(n+1)} \frac{1-\delta_{j,k}}{x^\varphi_{kn} - x^\varphi_{jn}} \\ &= \frac{\chi(x^\varphi_{jn})}{\chi(x^\varphi_{kn})} \left[\frac{2k}{\pi \mathrm{i}(j^2-k^2)} - \frac{\varphi(x^\varphi_{kn})}{\mathrm{i}(n+1)} \frac{1}{x^\varphi_{kn} - x^\varphi_{jn}} \right] (1-\delta_{j,k}) \\ &\quad + \left[1 - \frac{\chi(x^\varphi_{jn})}{\left(\frac{j\pi}{n+1} \right)^{2\chi_+} 2^{\chi_- - \chi_+}} \right] \left(\frac{j}{k} \right)^{2\chi_+} \frac{2k(1-\delta_{j,k})}{j^2-k^2} \frac{\left(\frac{k\pi}{n+1} \right)^{2\chi_+} 2^{\chi_- - \chi_+}}{\chi(x^\varphi_{kn})} \\ &\quad + \left[1 - \frac{\left(\frac{k\pi}{n+1} \right)^{2\chi_+} 2^{\chi_- - \chi_+}}{\chi(x^\varphi_{kn})} \right] \left(\frac{j}{k} \right)^{2\chi_+} \frac{2k(1-\delta_{j,k})}{j^2-k^2} \,. \end{aligned}$$

Denoting the first addend on the right-hand side by r_{jk}, using (7.7), and taking into account (3.27), we get the Frobenius norm estimate

$$\begin{aligned} &\sup_{n=1,2,\ldots} \left\| \left(f(x^\varphi_{(j+1)n}) r_{(j+1)(k+1)} f(x^\varphi_{(k+1)n}) \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \\ &\qquad \leq \sup_{n=1,2,\ldots} \frac{C}{n^2} \sqrt{\sum_{1 \leq j \leq \frac{\varepsilon}{\pi}(n+1)} j^{4\chi_+}} \sqrt{\sum_{1 \leq k \leq \frac{\varepsilon}{\pi}(n+1)} k^{2-4\chi_+}} \\ &\qquad \leq \sup_{n=1,2,\ldots} \frac{C\sqrt{(n\varepsilon)^{4\chi_+ + 1}} \sqrt{(n\varepsilon)^{3-4\chi_+}}}{n^2} = C\,\varepsilon^2 \end{aligned}$$

for any $f \in \mathbf{C}_1$ with $\mathrm{supp}(f \circ \cos) \subset [0, \varepsilon]$. This results in

$$\left\| \{M_n f L_n\} \left\{ (E_n^{(3)})^{-1} (P_n \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} - \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}) E_n^{(3)} \right\} \{M_n f L_n\} \right\| \leq C\varepsilon \,. \tag{7.12}$$

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In particular, the choice $\chi_+ = \chi_- = 0$ gives the same bound $C\varepsilon$ for the sequence $\left\{P_n \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} - \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}\right\}$ replaced by the sequence $\{P_n \mathbf{A} - \mathbf{A}_n\}$. This and the estimates (7.8)–(7.12) lead us to

$$\left\| \{M_n f L_n\}^o \left\{ [\mathbf{A}_+^\mu]_n^3 - (E_n^{(3)})^{-1} \mathbf{A}_n^+ E_n^{(3)} \right\}^o \{M_n f L_n\}^o \right\| \le C\varepsilon + C \frac{1}{k_0^{\varepsilon_0}} ,$$

and (7.6) follows. ∎

8. The Conditions in the Main Theorem. Due to the Sections 6 and 7, the necessary and sufficient condition for the convergence of the collocation method is the invertibility of the four limit operators $W_\omega\{A_n\}$, $\omega \in T$ which are defined in the Lemmata 3.8 and 3.10. The first $W_1\{A_n\} := A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ is the operator of the original equation and the second is $W_2\{A_n\} := aI - b[\sigma\varphi]^{-1/2} S [\sigma\varphi]^{1/2} I \in \mathcal{L}(\mathbf{L}^2_\sigma)$. The third and fourth operators $W_3\{A_n\}$ and $W_4\{A_n\}$ are operators in the discrete ℓ^2 space. In this section we show that the invertibility of the operators $W_3\{A_n\}$ and $W_4\{A_n\}$ are equivalent to the conditions ii) and iii) in Theorem 1.1. Moreover, we show that the conditions i) and ii) imply the invertibility of $W_2\{A_n\}$. This completes the proof to Theorem 1.1.

First we turn to the invertibility of $W_3\{A_n\}$ and $W_4\{A_n\}$. Due to part ii) of Lemma 7.2, they belong to the algebra generated by Toeplitz matrices with piecewise continuous generating function. Therefore, their Fredholm property and their index can be expressed in terms of the symbol due to Gohberg and Krupnik (cf. Lemma 7.1). We apply part i) of Lemma 7.1 to reformulate the condition on the invertibility of $W_3\{A_n\}$ and $W_4\{A_n\}$. Again, for symmetry reasons we concentrate on $W_3\{A_n\}$. Clearly, the operator $W_3\{A_n\}$ is invertible if and only if the null space $\ker W_3\{A_n\}$ is $\{0\}$ and if $W_3\{A_n\}$ is a Fredholm operator with index zero. Since $\ker W_3\{A_n\} = \{0\}$ is contained in the conditions of Theorem 1.1, we have to analyze the Fredholm property and the index. Thus, in view of part i) of Lemma 7.1, we need the symbol of $W_3\{A_n\}$. The symbols of the compact operators $\mathbf{B}_+$, $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}\mathbf{W}\mathbf{V}_+$, and $\mathbf{V}_+\mathbf{A}\mathbf{W}$ are zero. For $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$, we infer from (7.3) that

$$\mathbf{Symb}_{\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}}(t,\mu) = -\frac{1}{2} \int_{\{\zeta:\, \Re e\, \zeta = \psi\}} \mathbf{Symb}_{T_\zeta}(t,\mu) \Big\{ \mathcal{B}(\zeta) - \mathcal{B}(\zeta+1) \Big\} \mathrm{d}\zeta,$$

$$T_\zeta := \left(\frac{1}{\pi \mathrm{i}} \, \frac{\left(\frac{j+1}{k+1}\right)^{-\zeta} (1 - \delta_{k,j})}{(k+1) - (j+1)} \right)_{j,k=0}^{\infty} ,$$

$$\mathbf{Symb}_{T_\zeta}(t,\mu) = \begin{cases} 2s - 1 & \text{if } t = e^{\mathrm{i}2\pi s} \in \mathbb{T} \setminus \{1\} \\ \dfrac{-\mu \ + (1-\mu)\ e^{-\mathrm{i}2\pi\zeta}}{\mu + (1-\mu) e^{-\mathrm{i}2\pi\zeta}} & \text{if } t = 1 \end{cases}$$

$$= \begin{cases} 2s - 1 & \text{if } t = e^{\mathrm{i}2\pi s},\ 0 < s < 1 \\ -(-\mathrm{i}) \cot \left(\pi \left(\frac{1}{2} + \zeta + \frac{1}{2\pi \mathrm{i}} \log \left(\frac{\mu}{1-\mu} \right) \right) \right) & \text{if } t = 1. \end{cases}$$

We observe that $\mathbf{Symb}_{T_\zeta}$ is 1-periodic with respect to variable ζ, and, applying the residue theorem, we arrive at

$$\mathbf{Symb}_{\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}}(t,\mu)$$
$$= -\frac{1}{2} \left\{ \int_{\{\zeta:\, \Re e\, \zeta = \psi\}} \mathbf{Symb}_{T_\zeta}(t,\mu) \mathcal{B}(\zeta) \mathrm{d}\zeta - \int_{\{\zeta:\, \Re e\, \zeta = \psi+1\}} \mathbf{Symb}_{T_\zeta}(t,\mu) \mathcal{B}(\zeta) \mathrm{d}\zeta \right\}$$

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$$= -\mathbf{Symb}_{T_{[1/4-\chi_+]}}(t,\mu) - \begin{cases} 0 & \text{if } t \in \mathbb{T}\setminus\{1\} \\ \mathcal{B}\left(\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\log\frac{\mu}{1-\mu}\right) & \text{if } t=1. \end{cases}$$

Completely analogous derivations lead to the formulas

$$\mathbf{Symb}_{\mathbf{A}}(t,\mu) = -\mathbf{Symb}_{T_{1/4}}(t,\mu) - \begin{cases} 0 & \text{if } t \in \mathbb{T}\setminus\{1\} \\ \mathcal{B}^\dagger\left(\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\log\frac{\mu}{1-\mu}\right) & \text{if } t=1, \end{cases}$$

$$\mathbf{Symb}_{A_+}(t,\mu) = -\mathbf{Symb}_{T^\dagger_{1/4}}(t,\mu) - \begin{cases} 0 & \text{if } t \in \mathbb{T}\setminus\{\pm 1\} \\ \pm\mathcal{B}^\dagger\left(\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\log\frac{\mu}{1-\mu}\right) & \text{if } t=\pm 1, \end{cases}$$

$$\mathcal{B}^\dagger(\zeta) := -\mathrm{i}\cot\left(\pi\left(\frac{\zeta}{2}\right)\right) + \mathrm{i}\cot\left(\pi\left(\zeta-\frac{1}{4}\right)\right).$$

Here the operator $T^\dagger$ is the Toeplitz matrix $\left(\frac{1}{\pi\mathrm{i}}\frac{\widetilde{\delta}_{j,k}}{(k+1)-(j+1)}\right)_{j,k=0}^{\infty}$ with the generating function defined by $g^\dagger(e^{\mathrm{i}2\pi s}) = -\sum_l \frac{1}{\pi\mathrm{i}}\frac{\widetilde{\delta}_{l,0}}{l}e^{\mathrm{i}2\pi ls} = \operatorname{sign}(s-1/2),\ 0 \le s < 1$, and $T^\dagger_\zeta \in alg\,\mathcal{T}(\mathbf{PC})$ is the matrix operator given by

$$T^\dagger_\zeta := \left((j+1)^{-\zeta}\delta_{j,k}\right)_{j,k=0}^{\infty}\ T^\dagger\ \left((k+1)^{\zeta}\delta_{j,k}\right)_{j,k=0}^{\infty}$$

and the symbol (cf. part ii) of Lemma 7.1)

$$\mathbf{Symb}_{T^\dagger_\zeta}(t,\mu) = \begin{cases} g^\dagger(t) & \text{if } t\in\mathbb{T}\setminus\{1\} \\ \dfrac{\mu g^\dagger(1\pm 0)+(1-\mu)g^\dagger(1\mp 0)e^{-\mathrm{i}2\pi\zeta}}{\mu+(1-\mu)e^{-\mathrm{i}2\pi\zeta}} & \text{if } t=\pm 1 \end{cases}$$

$$= \begin{cases} -1 & \text{if } t\in\mathbb{T},\ \Im\mathrm{m}\, t>0 \\ +1 & \text{if } t\in\mathbb{T},\ \Im\mathrm{m}\, t<0 \\ \mp(-\mathrm{i})\cot\left(\pi\left(\frac{1}{2}+\zeta+\frac{1}{2\pi\mathrm{i}}\log\left(\frac{\mu}{1-\mu}\right)\right)\right) & \text{if } t=\pm 1. \end{cases}$$

Putting the symbols for all ingredients together (cf. the Lemmata 3.8, 3.9, and 3.10), we finally obtain

$$\mathbf{Symb}_{W_3\{A_n\}}(t,\mu)$$

$$= a(1)-b(1)\begin{cases} -1 & \text{if } t\in\mathbb{T},\ \Im\mathrm{m}\, t>0 \\ +1 & \text{if } t\in\mathbb{T},\ \Im\mathrm{m}\, t<0 \\ (-\mathrm{i})\cot\left(\pi\left(\frac{1}{4}+\chi_+ +\frac{\mathrm{i}}{4\pi}\log\left(\frac{\mu}{1-\mu}\right)\right)\right) & \text{if } t=1 \\ \mathrm{i}\cot\left(\pi\left(\frac{1}{4}+\frac{\mathrm{i}}{4\pi}\log\left(\frac{\mu}{1-\mu}\right)\right)\right) & \text{if } t=-1. \end{cases}$$

Hence (cf. Lemma 7.1), the limit operator $W_3\{A_n\}$ is a Fredholm operator with index zero if the curve

$$\text{(8.1)}\qquad \Gamma_3 := \left\{a(1)-b(1)(-\mathrm{i})\cot\left(\pi\left(\frac{1}{2}+\frac{\alpha}{2}-\gamma+\lambda\mathrm{i}\right)\right):\ -\infty\le\lambda\le\infty\right\}$$
$$\cup\left\{a(1)+b(1)(-\mathrm{i})\cot\left(\pi\left(\frac{1}{2}-\frac{1}{4}+\lambda\mathrm{i}\right)\right):\ -\infty\le\lambda\le\infty\right\}$$

determined by the symbol function $\mathbf{Symb}_{W_3\{A_n\}}$ does not run through the zero point and if the winding number $\mathrm{wind}\,\Gamma_3$ with respect to zero vanishes.

Similarly, for the limit operator $W_4\{A_n\}$, we get that $W_4\{A_n\}$ is a Fredholm operator with index zero if the curve

$$\text{(8.2)}\quad \Gamma_4 := \left\{a(-1)+b(-1)(-\mathrm{i})\cot\left(\pi\Big(\frac{1}{2}+\frac{\beta}{2}-\delta+\lambda\mathrm{i}\Big)\right) :\ -\infty\le\lambda\le\infty\right\}$$
$$\cup\left\{a(-1)-b(-1)(-\mathrm{i})\cot\left(\pi\Big(\frac{1}{2}-\frac{1}{4}+\lambda\mathrm{i}\Big)\right) :\ -\infty\le\lambda\le\infty\right\}$$

does not run through the zero point and if the winding number $\mathrm{wind}\,\Gamma_4$ with respect to zero vanishes.

Hence, it turns out that the operators $W_3\{A_n\}$ and $W_4\{A_n\}$ are invertible if and only if their null spaces $\ker W_3\{A_n\}$ and $\ker W_4\{A_n\}$ are trivial and if the winding numbers $\mathrm{wind}\,\Gamma_3$ and $\mathrm{wind}\,\Gamma_4$ of the curves Γ_3 and Γ_4 surrounding the essential spectra vanish. We observe that each of the two curves Γ_3 and Γ_4 is the union of two circular arcs, which, of course, may degenerate to a straight line segment. Indeed, setting $e^{2\pi\lambda}=\mu/(1-\mu),\ \mu\in[0,1]$ and choosing κ with $|\kappa|<1/2$, we get

$$(-\mathrm{i})\cot\left(\pi(\frac{1}{2}+\kappa+\mathrm{i}\lambda)\right) = \frac{(1-\mu)-\mu e^{-\mathrm{i}2\pi\kappa}}{(1-\mu)+\mu e^{-\mathrm{i}2\pi\kappa}},$$
$$\frac{a(-1)+b(-1)(-\mathrm{i})\cot\left(\pi(\frac{1}{2}+\kappa+\mathrm{i}\lambda)\right)}{a(-1)-b(-1)} = \left\{f_\kappa(\mu)1+\frac{a(-1)+b(-1)}{a(-1)-b(-1)}(1-f_\kappa(\mu))\right\},$$
$$f_\kappa(\mu) := \frac{\mu}{\mu+(1-\mu)e^{\mathrm{i}2\pi\kappa}}.$$

Clearly, $1-f_\kappa(\mu)=f_{-\kappa}(1-\mu)$ and the linear rational function $[0,1]\ni\mu\mapsto f_\kappa(\mu)$ describes the circular arc connecting the points zero and one such that the straight line segment $[0,1]$ is seen from the points of the arc under an angle of $\pi(1-2\kappa)$. The point zero is not in the closed convex hull of the circular arc

$$\{a(-1)+b(-1)(-\mathrm{i})\cot(\pi(\frac{1}{2}+\kappa+\mathrm{i}\lambda)) :\ \lambda\in\mathbb{R}\}$$

if and only if

$$\nu 1+(1-\nu)\left\{f_\kappa(\mu)1+\frac{a(-1)+b(-1)}{a(-1)-b(-1)}(1-f_\kappa(\mu))\right\}\neq 0,\quad 0\le\nu,\mu\le 1,$$

i.e., if and only if

$$\nu 1+(1-\nu)\frac{a(-1)+b(-1)}{a(-1)-b(-1)}\neq -\frac{f_\kappa(\mu)}{1-f_\kappa(\mu)},\quad 0\le\nu,\mu\le 1,$$

This holds if and only if (cf. (1.7))

$$\nu 1+(1-\nu)\frac{a(-1)+b(-1)}{a(-1)-b(-1)} = \nu 1+(1-\nu)\left|\frac{a(-1)+b(-1)}{a(-1)-b(-1)}\right| e^{\mathrm{i}2\pi\kappa_-}\notin e^{-\mathrm{i}2\pi\kappa}[-\infty,0],$$
$$0\le\nu\le 1,$$

which is, in case of $-1/2-\kappa<\kappa_-\le 1/2-\kappa$, equivalent to $-1/2<\kappa_-<1/2-\kappa$ for $\kappa>0$ and to $-\kappa-1/2<\kappa_-<1/2$ for $\kappa<0$. Furthermore, zero is not at the

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curve Γ_4 or in its interior if and only if either zero is not contained in the convex hulls of $\{a(-1)+b(-1)(-\mathrm{i})\cot(\pi(\frac{1}{2}+\frac{\beta}{2}-\delta+\mathrm{i}\lambda)) :\ \lambda\in\mathbb{R}\}$ and of $\{a(-1)-b(-1)(-\mathrm{i})\cot(\pi(\frac{1}{2}-\frac{1}{4}+\mathrm{i}\lambda)) :\ \lambda\in\mathbb{R}\}$, or zero is contained in the interior of both convex hulls, or, if $\beta/2-\delta>0$, zero is on the straight line from $a(-1)-b(-1)$ to $a(-1)+b(-1)$. In other words, zero is not at the curve Γ_4 or in its interior if and only if (1.8) is satisfied for κ_-. Similarly, zero is not at the curve Γ_3 or in its interior if and only if (1.8) is satisfied for κ_+. The invertibility of $W_3\{A_n\}$ and $W_4\{A_n\}$ is equivalent to the conditions ii) and iii) in Theorem 1.1.

Now we consider the invertibility of $W_2\{A_n\} := \{aI - b[\sigma\varphi]^{-1/2}S[\sigma\varphi]^{1/2}\} \in \mathcal{L}(\mathbf{L}^2_\sigma)$. From the general theory of one-dimensional singular integral equations (cf. [11], Theorem 9.4.1) we infer that $W_2\{A_n\}$ is invertible if and only if its symbol

$$\mathbf{Symb}_{W_2\{A_n\}}(t,\mu)$$

$$:= \begin{cases} \dfrac{a(t)+b(t)}{a(t)-b(t)} & \text{if } a,b \text{ are continuous at } t\in(-1,1) \\ (1-\mu)\dfrac{a(t-0)+b(t-0)}{a(t-0)-b(t-0)} + \mu\dfrac{a(t+0)+b(t+0)}{a(t+0)-b(t+0)} & \text{if } a \text{ or } b \text{ not continuous at } t\in(-1,1) \\ (1-f_{-1/4}(\mu))1 + f_{-1/4}(\mu)\dfrac{a(-1)+b(-1)}{a(-1)-b(-1)} & \text{if } t=-1 \\ (1-f_{-1/4}(\mu))\dfrac{a(1)+b(1)}{a(1)-b(1)} + f_{-1/4}(\mu)1 & \text{if } t=1 \end{cases}$$

does not vanish for $(t,\mu)\in[-1,1]\times[0,1]$ and if the winding number is zero. Note that, if $-1<t_1<t_2<\ldots<t_k<1$ is the grid of discontinuity points, then the winding number of the symbol is the winding number of the closed curve

$$\begin{aligned} &\{\mathbf{Symb}_{W_2\{A_n\}}(-1,\mu) :\ 0\le\mu\le 1\} \cup \{\mathbf{Symb}_{W_2\{A_n\}}(t,0) :\ -1<t<t_1\} \\ &\cup\{\mathbf{Symb}_{W_2\{A_n\}}(t_1,\mu) :\ 0\le\mu\le 1\} \cup \{\mathbf{Symb}_{W_2\{A_n\}}(t,0) :\ t_1<t<t_2\} \\ &\cup\{\mathbf{Symb}_{W_2\{A_n\}}(t_2,\mu) :\ 0\le\mu\le 1\} \cup \{\mathbf{Symb}_{W_2\{A_n\}}(t,0) :\ t_2<t<t_3\} \cup \\ &\qquad\qquad\ldots \\ &\cup\{\mathbf{Symb}_{W_2\{A_n\}}(t,0) :\ t_k<t<1\} \cup \{\mathbf{Symb}_{W_2\{A_n\}}(1,\mu) :\ 0\le\mu\le 1\} \end{aligned}$$

with respect to zero. Now suppose condition ii) of Theorem 1.1 is satisfied. Hence, zero is not contained in the domains enclosed by the curves Γ_3 and Γ_4 (cf. (8.1) and (8.2)), and the non singularity and the vanishing winding number for the symbol $\mathbf{Symb}_{W_2\{A_n\}}$ is equivalent to the non singularity and the vanishing winding number for the symbol function

$$\mathbf{Symb}_A(t,\mu)$$

$$:= \begin{cases} \dfrac{a(t)+b(t)}{a(t)-b(t)} & \text{if } a,b \text{ are continuous at } t\in(-1,1) \\ (1-\mu)\dfrac{a(t-0)+b(t-0)}{a(t-0)-b(t-0)} + \mu\dfrac{a(t+0)+b(t+0)}{a(t+0)-b(t+0)} & \text{if } a \text{ or } b \text{ not continuous at } t\in(-1,1) \\ (1-f_{[\delta-\beta/2]}(\mu))1 + f_{[\delta-\beta/2]}(\mu)\dfrac{a(-1)+b(-1)}{a(-1)-b(-1)} & \text{if } t=-1 \\ (1-f_{[\gamma-\alpha/2]}(\mu))\dfrac{a(1)+b(1)}{a(1)-b(1)} + f_{[\gamma-\alpha/2]}(\mu)1 & \text{if } t=1 \end{cases}$$

corresponding to the singular integral operator A. Since A is invertible by condition i) of Theorem 1.1, the symbol does not vanish and the winding number is zero. Hence, the conditions i) and ii) imply the invertibility of $W_2\{A_n\}$.

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