

## Contents

- 1 Shifted and extrapolated power methods for tensor  $\ell^p$ -eigenpairs.  
*Stefano Cipolla, Michela Redivo-Zaglia, and Francesco Tudisco.*

**Abstract.**

This work is concerned with the computation of  $\ell^p$ -eigenvalues and eigenvectors of square tensors with  $d$  modes. In the first part we propose two possible shifted variants of the popular (higher-order) power method, and, when the tensor is entry-wise nonnegative with a possibly reducible pattern and  $p$  is strictly larger than the number of modes, we prove convergence of both schemes to the Perron  $\ell^p$ -eigenvector and to the maximal corresponding  $\ell^p$ -eigenvalue of the tensor. Then, in the second part, motivated by the slow rate of convergence that the proposed methods achieve for certain real-world tensors when  $p \approx d$ , the number of modes, we introduce an extrapolation framework based on the simplified topological  $\varepsilon$ -algorithm to efficiently accelerate the shifted power sequences. Numerical results for synthetic and real world problems show the improvements gained by the introduction of the shifting parameter and the efficiency of the acceleration technique.

**Key Words.**

$\ell^p$ -eigenvalues, tensors, shifted higher-order power method, extrapolation methods, Shanks transformations,  $\varepsilon$ -algorithms

**AMS Subject Classifications.**

15A69, 15A18, 65B05, 65B10, 65B99, 65F15

- 28 Block generalized locally Toeplitz sequences: theory and applications in the unidimensional case.  
*Giovanni Barbarino, Carlo Garoni, and Stefano Serra-Capizzano.*

**Abstract.**

In computational mathematics, when dealing with a large linear discrete problem (e.g., a linear system) arising from the numerical discretization of a differential equation (DE), knowledge of the spectral distribution of the associated matrix has proved to be useful information for designing/analyzing appropriate solvers—especially, preconditioned Krylov and multigrid solvers—for the considered problem. Actually, this spectral information is of interest also in itself as long as the eigenvalues of the aforementioned matrix represent physical quantities of interest, which is the case for several problems from engineering and applied sciences (e.g., the study of natural vibration frequencies in an elastic material). The theory of generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the asymptotic spectral distribution of matrices  $A_n$  arising from virtually any kind of numerical discretization of DEs. Indeed, when the mesh-fineness parameter  $n$  tends to infinity, these matrices  $A_n$  give rise to a sequence  $\{A_n\}_n$ , which often turns out to be a GLT sequence or one of its “relatives”, i.e., a block GLT sequence or a reduced GLT sequence. In particular, block GLT sequences are encountered in the discretization of systems of

DEs as well as in the higher-order finite element or discontinuous Galerkin approximation of scalar/vectorial DEs. This work is a review, refinement, extension, and systematic exposition of the theory of block GLT sequences. It also includes several emblematic applications of this theory in the context of DE discretizations.

**Key Words.**

asymptotic distribution of singular values and eigenvalues, block Toeplitz matrices, block generalized locally Toeplitz matrices, numerical discretization of differential equations, finite differences, finite elements, isogeometric analysis, discontinuous Galerkin methods, tensor products, B-splines.

**AMS Subject Classifications.**

15A18, 15B05, 47B06, 65N06, 65N30, 65N25, 15A60, 15A69, 65D07

**113** Block Generalized Locally Toeplitz Sequences: Theory and Applications in the Multidimensional Case.

*Giovanni Barbarino, Carlo Garoni, and Stefano Serra-Capizzano.*

**Abstract.**

In computational mathematics, when dealing with a large linear discrete problem (e.g., a linear system) arising from the numerical discretization of a partial differential equation (PDE), knowledge of the spectral distribution of the associated matrix has proved to be useful information for designing/analyzing appropriate solvers—especially, preconditioned Krylov and multigrid solvers—for the considered problem. Actually, this spectral information is of interest also in itself as long as the eigenvalues of the aforementioned matrix represent physical quantities of interest, which is the case for several problems from engineering and applied sciences (e.g., the study of natural vibration frequencies in an elastic material). The theory of multilevel generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the asymptotic spectral distribution of matrices  $A_n$  arising from virtually any kind of numerical discretization of PDEs. Indeed, when the mesh-fineness parameter  $n$  tends to infinity, these matrices  $A_n$  give rise to a sequence  $\{A_n\}_n$ , which often turns out to be a multilevel GLT sequence or one of its “relatives”, i.e., a multilevel block GLT sequence or a (multilevel) reduced GLT sequence. In particular, multilevel block GLT sequences are encountered in the discretization of systems of PDEs as well as in the higher-order finite element or discontinuous Galerkin approximation of scalar/vectorial PDEs. In this work, we systematically develop the theory of multilevel block GLT sequences as an extension of the theories of (unilevel) GLT sequences [Garoni and Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications. Vol. I.*, Springer, Cham, 2017], multilevel GLT sequences [Garoni and Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications. Vol. II.*, Springer, Cham, 2018], and block GLT sequences [Barbarino, Garoni, and Serra-Capizzano, *Electron. Trans. Numer. Anal.*, 53 (2020), pp. 28–112]. We also present several emblematic applications of this theory in the context of PDE discretizations.

**Key Words.**

asymptotic distribution of singular values and eigenvalues, multilevel block Toeplitz matrices, multilevel block generalized locally Toeplitz matrices, numerical discretization of partial differential equations, finite differences, finite elements, isogeometric analysis, discontinuous Galerkin methods, tensor products, B-splines

**AMS Subject Classifications.**

15A18, 15B05, 47B06, 65N06, 65N30, 65N25, 15A60, 15A69, 65D07

**217** A simplified L-curve method as error estimator.

*Stefan Kindermann and Kemal Raik.*

**Abstract.**

The L-curve method is a well known heuristic method for choosing the regularization parameter for ill-posed problems by selecting it according to the maximal curvature of the L-curve. In this article, we propose a simplified version that replaces the curvature essentially by the derivative of the parameterization on the  $y$ -axis. This method shows a similar behaviour to the original L-curve method, but unlike the latter, it may serve as an error estimator under typical conditions. Thus, we can accordingly prove convergence for the simplified L-curve method.

**Key Words.**

heuristic parameter choice, L-curve method, regularization

**AMS Subject Classifications.**

47A52, 65F22, 65J20, 65J22

**239** Transformed rank-1 lattices for high-dimensional approximation.

*Robert Nasdala and Daniel Potts.*

**Abstract.**

This paper describes an extension of Fourier approximation methods for multivariate functions defined on the torus  $\mathbb{T}^d$  to functions in a weighted Hilbert space  $L_2(\mathbb{R}^d, \omega)$  via a multivariate change of variables  $\psi : \left(-\frac{1}{2}, \frac{1}{2}\right)^d \rightarrow \mathbb{R}^d$ . We establish sufficient conditions for  $\psi$  and  $\omega$  such that the composition of a function in such a weighted Hilbert space with  $\psi$  yields a function in the Sobolev space  $H_{\text{mix}}^m(\mathbb{T}^d)$  of functions on the torus with mixed smoothness of natural order  $m \in \mathbb{N}_0$ . In this approach we adapt algorithms for the evaluation and reconstruction of multivariate trigonometric polynomials on the torus  $\mathbb{T}^d$  based on single and multiple reconstructing rank-1 lattices. Since in applications it may be difficult to choose a related function space, we make use of dimension incremental construction methods for sparse frequency sets. Various numerical tests confirm the obtained theoretical results for the transformed methods.

**Key Words.**

approximation on unbounded domains, change of variables, sparse multivariate trigonometric polynomials, lattice rule, multiple rank-1 lattice, fast Fourier transform

**AMS Subject Classifications.**

65T, 42B05

**283** A multigrid frame based method for image deblurring.

*Alessandro Buccini and Marco Donatelli.*

**Abstract.**

Iterative soft thresholding algorithms combine one step of a Landweber method (or accelerated variants) with one step of thresholding of the wavelet (framelet) coefficients. In this paper, we improve these methods by using the framelet multilevel

decomposition for defining a multigrid deconvolution with grid transfer operators given by the low-pass filter of the frame. Assuming that an estimate of the noise level is available, we combine a recently proposed iterative method for  $\ell_2$ -regularization with linear framelet denoising by soft-thresholding. This combination allows a fast frequency filtering in the Fourier domain and produces a sparse reconstruction in the wavelet domain. Moreover, its employment in a multigrid scheme ensures stable convergence and a reduced noise amplification. The proposed multigrid method is independent of the imposed boundary conditions, and the iterations can be easily projected onto a closed and convex set, e.g., the nonnegative cone. We study the convergence of the proposed algorithm and prove that it is a regularization method. Several numerical results prove that this approach is able to provide highly accurate reconstructions in several different scenarios without requiring the setting of any parameter.

**Key Words.**

image deblurring, multigrid methods, iterative regularization methods

**AMS Subject Classifications.**

65F22, 65N55, 65F10, 15B05

- 313** Convergence results and low-order rates for nonlinear Tikhonov regularization with oversmoothing penalty term.

*Bernd Hofmann and Robert Plato.*

**Abstract.**

For Tikhonov regularization of ill-posed nonlinear operator equations, convergence is studied in a Hilbert scale setting. We include the case of oversmoothing penalty terms, which means that the exact solution does not belong to the domain of definition of the considered penalty functional. In this case, we try to close a gap in the present theory, where Hölder-type convergence rates results have been proven under corresponding source conditions, but assertions on norm convergence for regularized solutions without source conditions are completely missing. A result of the present work is to provide sufficient conditions for convergence under a priori and a posteriori regularization parameter choice strategies without any additional smoothness assumption on the solution. The obtained error estimates moreover allow us to prove low-order convergence rates under associated (for example logarithmic) source conditions. Some numerical illustrations are also given.

**Key Words.**

ill-posed problem, inverse problem, Tikhonov regularization, oversmoothing penalty, a priori parameter choice strategy, discrepancy principle, logarithmic source condition

**AMS Subject Classifications.**

65J20, 65J15, 65J22, 47J06, 47J05

- 329** Residual whiteness principle for parameter-free image restoration.

*Alessandro Lanza, Monica Pragliola, and Fiorella Sgallari.*

**Abstract.**

Selecting the regularization parameter in the image restoration variational framework is of crucial importance, since it can highly influence the quality of the final restoration. In this paper, we propose a parameter-free approach for automatically

selecting the regularization parameter when the blur is space-invariant and known and the noise is additive white Gaussian with unknown standard deviation, based on the so-called *residual whiteness principle*. More precisely, the regularization parameter is required to minimize the *residual whiteness function*, namely the normalized auto-correlation of the residual image of the restoration. The proposed method can be applied to a wide class of variational models, such as those including in their formulation regularizers of Tikhonov and Total Variation type. For non-quadratic regularizers, the residual whiteness principle is nested in an iterative optimization scheme based on the alternating direction method of multipliers. The effectiveness of the proposed approach is verified by solving some test examples and performing a comparison with other parameter estimation state-of-the-art strategies, such as the discrepancy principle.

**Key Words.**

image restoration, variational methods, regularization parameter, additive white Gaussian noise, alternating direction method of multipliers

**AMS Subject Classifications.**

68U10, 94A08, 65K10

- 352** Error estimates of Gaussian-type quadrature formulae for analytic functions on ellipses—a survey of recent results.

*D. Lj. Djukić, R. M. Mutavdžić Djukić, A. V. Pejčev, and M. M. Spalević.*

**Abstract.**

This paper presents a survey of recent results on error estimates of Gaussian-type quadrature formulas for analytic functions on confocal ellipses.

**Key Words.**

error estimate, remainder term, analytic function, contour integral representation

**AMS Subject Classifications.**

65D32, 65D30, 41A55

- 383** Optimized Surface Parameterizations with Applications to Chinese Virtual Broadcasting.

*Mei-Heng Yueh, Hsiao-Han Huang, Tiexiang Li, Wen-Wei Lin, and Shing-Tung Yau.*

**Abstract.**

Surface parameterizations have been widely applied in computer-aided design for the geometric processing tasks of surface registration, remeshing, texture mapping, and so on. In this paper, we present an efficient balanced energy minimization algorithm for the computation of simply connected open surface parameterizations with balanced angle and area distortions. The existence of a nontrivial accumulation function of the proposed algorithm is guaranteed under some mild conditions, and the limiting function is shown to be one-to-one. Comparisons of the proposed algorithm with angle- and area-preserving parameterizations show that the angular distortion is close to that of an angle-preserving parameterization while the area distortion is significantly improved. An application of the proposed algorithm involving surface remeshing, registration, and morphing to the Chinese virtual broadcasting technique is demonstrated.

**Key Words.**

surface parameterization, simply connected open surface, balanced energy minimization, virtual broadcasting

**AMS Subject Classifications.**

15B48, 52C26, 65F05, 68U05, 65D18

- 406** A subspace-accelerated split Bregman method for sparse data recovery with joint  $\ell_1$ -type regularizers.

*Valentina De Simone, Daniela di Serafino, and Marco Viola.*

**Abstract.**

We propose a subspace-accelerated Bregman method for the linearly constrained minimization of functions of the form  $f(\mathbf{u}) + \tau_1 \|\mathbf{u}\|_1 + \tau_2 \|D\mathbf{u}\|_1$ , where  $f$  is a smooth convex function and  $D$  represents a linear operator, e.g., a finite difference operator, as in anisotropic total variation and fused lasso regularizations. Problems of this type arise in a wide variety of applications, including portfolio optimization, learning of predictive models from functional magnetic resonance imaging (fMRI) data, and source detection problems in electroencephalography. The use of  $\|D\mathbf{u}\|_1$  is aimed at encouraging structured sparsity in the solution. The subspaces where the acceleration is performed are selected so that the restriction of the objective function is a smooth function in a neighborhood of the current iterate. Numerical experiments for multi-period portfolio selection problems using real data sets show the effectiveness of the proposed method.

**Key Words.**

split Bregman method, subspace acceleration, joint  $\ell_1$ -type regularizers, multi-period portfolio optimization

**AMS Subject Classifications.**

65K05, 90C25

- 426** Cubature formulae for the Gaussian weight. Some old and new rules.

*Ramón Orive, Juan C. Santos-León, and Miodrag M. Spalević.*

**Abstract.**

In this paper we review some of the main known facts about cubature rules to approximate integrals over domains in  $\mathbb{R}^n$ , in particular with respect to the Gaussian weight  $w(\mathbf{x}) = e^{-\mathbf{x}^T \mathbf{x}}$ , where  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Some new rules are also presented. Taking into account the well-known issue of the “curse of dimensionality”, our aim is at providing rules with a certain degree of algebraic precision and a reasonably small number of nodes as well as an acceptable stability. We think that the methods used to construct these new rules are of further applicability in the field of cubature formulas. The efficiency of new and old rules are compared by means of several numerical experiments.

**Key Words.**

cubature formulas, Gaussian weight

**AMS Subject Classifications.**

65D32

- 439** Performance and stability of direct methods for computing generalized inverses of the graph Laplacian.

*Michele Benzi, Paraskevi Fika, and Marilena Mitrouli.*

**Abstract.**

We consider the computation of generalized inverses of the graph Laplacian for both

undirected and directed graphs, with a focus on the group inverse and the closely related absorption inverse. These generalized inverses encode valuable information about the underlying graph as well as the regular Markov process generated by the graph Laplacian. In [Benzi et al., *Linear Algebra Appl.*, 574 (2019), pp. 123–152], both direct and iterative numerical methods have been developed for the efficient computation of the absorption inverse and related quantities. In this work, we present two direct algorithms for the computation of the group inverse and the absorption inverse. The first is based on the Gauss-Jordan elimination and the reduced row echelon form of the Laplacian matrix and the second on the *bottleneck matrix*, the inverse of a submatrix of the Laplacian matrix. These algorithms can be faster than the direct algorithms proposed in [Benzi et al., *Linear Algebra Appl.*, 574 (2019), pp. 123–152].

**Key Words.**

Laplacian matrix, group inverse, absorption inverse, Gauss-Jordan method

**AMS Subject Classifications.**

15A09, 05C50, 65F50

- 459 The minimal-norm Gauss-Newton method and some of its regularized variants.  
*Federica Pes and Giuseppe Rodriguez.*

**Abstract.**

Nonlinear least-squares problems appear in many real-world applications. When a nonlinear model is used to reproduce the behavior of a physical system, the unknown parameters of the model can be estimated by fitting experimental observations by a least-squares approach. It is common to solve such problems by Newton's method or one of its variants such as the Gauss-Newton algorithm. In this paper, we study the computation of the minimal-norm solution to a nonlinear least-squares problem, as well as the case where the solution minimizes a suitable semi-norm. Since many important applications lead to severely ill-conditioned least-squares problems, we also consider some regularization techniques for their solution. Numerical experiments, both artificial and derived from an application in applied geophysics, illustrate the performance of the different approaches.

**Key Words.**

nonlinear least-squares, nonlinear inverse problem, regularization, Gauss-Newton method

**AMS Subject Classifications.**

65F22, 65H10, 65F20

- 481 Matrix completion for matrices with low-rank displacement.  
*Damiana Lazzaro and Serena Morigi.*

**Abstract.**

The matrix completion problem consists in the recovery of a low-rank or approximately low-rank matrix from a sampling of its entries. The solution rank is typically unknown, and this makes the problem even more challenging. However, for a broad class of interesting matrices with so-called displacement structure, the originally ill-posed completion problem can find an acceptable solution by exploiting the knowledge of the associated displacement rank. The goal of this paper is to propose a variational non-convex formulation for the low-rank matrix completion problem

with low-rank displacement and to apply it to important classes of medium-large scale structured matrices. Experimental results show the effectiveness and efficiency of the proposed approach for Toeplitz and Hankel matrix completion problems.

**Key Words.**

matrix completion, low-rank matrices, displacement rank, non-convex optimization

**AMS Subject Classifications.**

65K10, 65F22, 15A29

**500** Substitution algorithms for rational matrix equations.

*Massimiliano Fasi and Bruno Iannazzo.*

**Abstract.**

We study equations of the form  $r(X) = A$ , where  $r$  is a rational function and  $A$  and  $X$  are square matrices of the same size. We develop two techniques for solving these equations by inverting (through a substitution strategy) two schemes for the evaluation of rational functions of matrices. For triangular matrices, the new methods yield the same computational cost as the evaluation schemes from which they are obtained. A general equation can be reduced to upper triangular form by exploiting the Schur decomposition of the given matrix. For real data, the algorithms rely on the real Schur decomposition in order to compute real solutions using only real arithmetic. Numerical experiments show that our implementations are faster than existing alternatives without sacrificing accuracy.

**Key Words.**

rational matrix equation, Paterson–Stockmeyer scheme, powering technique, rational function evaluation, primary matrix function

**AMS Subject Classifications.**

15A24, 65F60

**522** Finite element discretization of semilinear acoustic wave equations with kinetic boundary conditions.

*Marlis Hochbruck and Jan Leibold.*

**Abstract.**

We consider isoparametric finite element discretizations of semilinear acoustic wave equations with kinetic boundary conditions and derive a corresponding error bound as our main result. The difficulty is that such problems are stated on domains with curved boundaries and this renders the discretizations nonconforming. Our approach is to provide a unified error analysis for nonconforming space discretizations for semilinear wave equations. In particular, we introduce a general, abstract framework for nonconforming space discretizations in which we derive a-priori error bounds in terms of interpolation, data, and conformity errors. The theory applies to a large class of problems and discretizations that fit into the abstract framework. The error bound for wave equations with kinetic boundary conditions is obtained from the general theory by inserting known interpolation and geometric error bounds into the abstract error result of the unified error analysis.

**Key Words.**

wave equation, dynamic boundary conditions, nonconforming space discretization, error analysis, a-priori error bounds, semilinear evolution equations, operator semi-groups, isoparametric finite elements

**AMS Subject Classifications.**  
65M12, 65M15, 65J08, 65M60

**541** Krylov type methods for linear systems exploiting properties of the quadratic numerical range.

*Andreas Frommer, Birgit Jacob, Karsten Kahl, Christian Wyss, and Ian Zwaan.*

**Abstract.**

The quadratic numerical range  $W^2(A)$  is a subset of the standard numerical range of a linear operator, which still contains its spectrum. It arises naturally in operators that have a  $2 \times 2$  block structure, and it consists of at most two connected components, none of which necessarily convex. The quadratic numerical range can thus reveal spectral gaps, and it can in particular indicate that the spectrum of an operator is bounded away from 0.

We exploit this property in the finite-dimensional setting to derive Krylov subspace-type methods to solve the system  $Ax = b$ , in which the iterates arise as solutions of low-dimensional models of the operator whose quadratic numerical range is contained in  $W^2(A)$ . This implies that the iterates are always well-defined and that, as opposed to standard FOM, large variations in the approximation quality of consecutive iterates are avoided, although 0 lies within the convex hull of the spectrum. We also consider GMRES variants that are obtained in a similar spirit. We derive theoretical results on basic properties of these methods, review methods on how to compute the required bases in a stable manner, and present results of several numerical experiments illustrating improvements over standard FOM and GMRES.

**Key Words.**

quadratic numerical range, full orthogonalization method (FOM), generalized minimal residual method (GMRES), linear systems, projection methods, two-level orthogonal Arnoldi method

**AMS Subject Classifications.**  
15A60, 47A12, 65F10, 65N55

**562** A frugal FETI-DP and BDDC coarse space for heterogeneous problems.

*Alexander Heinlein, Axel Klawonn, Martin Lanser, and Janine Weber.*

**Abstract.**

The convergence rate of domain decomposition methods is generally determined by the eigenvalues of the preconditioned system. For second-order elliptic partial differential equations, coefficient discontinuities with a large contrast can lead to a deterioration of the convergence rate. Only by implementing an appropriate coarse space, or second level, a robust domain decomposition method can be obtained. In this article, a new frugal coarse space for FETI-DP (Finite Element Tearing and Interconnecting-Dual Primal) and BDDC (Balancing Domain Decomposition by Constraints) methods is presented, which has a lower set-up cost than competing adaptive coarse spaces. In particular, in contrast to adaptive coarse spaces, it does not require the solution of any local generalized eigenvalue problems. The approach considered here aims at a low-dimensional approximation of the adaptive coarse space by using appropriate weighted averages, and it is robust for a broad range of coefficient distributions for diffusion and elasticity problems. However, in general, for completely arbitrary coefficient distributions with high contrast, some additional,

adaptively chosen constraints are necessary in order to guarantee robustness. In this article, the robustness is heuristically justified as well as numerically shown for several coefficient distributions. The new coarse space is compared to adaptive coarse spaces, and parallel scalability up to 262 144 parallel cores for a parallel BDDC implementation with the new coarse space is shown. The superiority of the new coarse space over classic coarse spaces with respect to parallel weak scalability and time-to-solution is confirmed by numerical experiments. Since the new frugal coarse space is computationally inexpensive, it could serve as a new default coarse space, which, for very challenging coefficient distributions, could then still be enhanced by adaptively chosen constraints.

**Key Words.**

FETI-DP, BDDC, robust coarse spaces, adaptive domain decomposition methods

**AMS Subject Classifications.**

68W10, 65N22, 65N55, 65F08, 65F10, 65Y05