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A BOUNDARY AND FINITE ELEMENT COUPLING FOR A MAGNETICALLY NONLINEAR EDDY CURRENT PROBLEM*

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Abstract. The aim of this paper is to provide a mathematical and numerical analysis for a FEM-BEM coupling approximation of a magnetically nonlinear eddy current formulation by using FEM only on the conducting domain, and by imposing the integral conditions on its boundary. The nonlinear relationship between flux density and the magnetic field intensity is given by a physical parameter called magnetic reluctivity, which is assumed to depend on the Euclidean norm of the magnetic induction in the conducting domain. We use the nonlinear monotone operator theory for parabolic equations to show that the continuous formulation obtained for the coupling is a well-posed problem. Furthermore, we use Nédélec edge elements, standard nodal finite elements, and a backward-Euler time scheme, to obtain a fully discrete formulation and to prove quasi-optimal error estimates.

Key words. Time-dependent electromagnetic, eddy current model, nonlinear problems, boundary element method, finite element method.

AMS subject classifications. 65M60, 65M38, 78M10, 78M15.

1. Introduction. The eddy current model is obtained by dropping the displacement currents from Maxwell equations (see for instance [9, Chapter 8]) and it provides a reasonable approximation to the solution of the full Maxwell system in the low frequency range [5]. This model is commonly used in many physical problems such as induction heating, electromagnetic braking, electric generation, etc.; see [3, Chapter 9].

Among the numerical methods used to approximate eddy current equations, the finite element method (FEM) and methods combining the FEM and the boundary element method (FEM-BEM) are the most used; see for instance the recent book by Alonso and Valli [3] for a survey on this subject, including a large list of references. In applied mathematics, we can find many papers devoted to the numerical analysis of the linear time dependent eddy current model, in bounded domains as well as in unbounded domains, by using FEM and FEM-BEM methods: Meddahi and Selgas [20], Ma [17], Acevedo et al. [2], Kang and Kim [15], Prato et al. [26, 27], Acevedo and Meddahi [1], Bermudez et al. [10], Camaño and Rodríguez [11]. However, this is not the case for the eddy current problems involving ferromagnetic conducting materials, where the number of papers is considerably smaller. The computations and the analysis for the latter case are usually more complicated, because the relationship between flux density and the magnetic field intensity (which is given by a physical parameter called magnetic reluctivity, namely, the inverse of the permeability) is nonlinear.

If hysteresis effects and anisotropies are neglected, the reluctivity is a scalar function that typically has a nonlinear dependence on the absolute value of the magnetic induction. More precisely, we assume that in the conductor domain this dependence is given by the following material relationship [7, 8, 19]

$$\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B}, \tag{1.1}$$

where, as usual, $\boldsymbol{H}$ and $\boldsymbol{B}$ represent the magnetic field and the magnetic induction field, respectively, and ν is the reluctivity of the conductor. Among the papers dedicated to the study

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of this nonlinear eddy current problem, we cite the works by Bachinger et al. [7] and Bíró et al. [8], which are based on the so-called multiharmonic-approach, i.e., the solution of the problem is approximated by a truncation of the Fourier series. A finite element approach is presented in the recent paper by Kang and Chen [16], where the authors proposed a T-ψ finite element method for the eddy current model, considering some kind of inverse relationship to (1.1). Other important contributions in the context of the nonlinear eddy current model can be found in [14, 19].

On the other hand, Camaño and Rodríguez [11] studied a FEM-BEM coupling for a linear eddy current problem by introducing a time-primitive of the electric field as a main unknown (see [1, 2] for earlier formulations using the same main variable), where the reluctivity appears as a diffusion coefficient. However, in that work the authors did not consider the nonlinear case. Consequently, the study of this FEM-BEM coupling by considering the nonlinear case given by the relationship (1.1) is new. Therefore the analysis of the obtained formulation is the principal subject of this paper.

The paper is organized as follows: in Section 2, we summarize some results concerning functional spaces, tangential traces, and integral boundary operators. In Section 3 we introduce the nonlinear model problem to be studied, and deduce a symmetric FEM-BEM coupling for this problem. Next, in Section 4 we prove that it is uniquely solvable, by assuming some reasonable (physical) properties of the reluctivity [14], and using the nonlinear monotone operator theory for parabolic equations [28]. The construction of a fully discrete approximation for the problem, by using a backward Euler scheme for time and natural finite element (Nédélec and Lagrange) subspaces for the corresponding spatial variable, is reported in Section 5. Finally, error estimates that prove a quasi-optimal convergence are settled in Section 6.

2. Preliminaries. In this paper, Ω_c is an open, bounded and connected subset of $\mathbb{R}^3$ which represents the domain occupied by a conductive material, with a Lipschitz continuous boundary $\Gamma := \partial\Omega_c$, and $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_c}$ is a connected set occupied by insulating materials. We denote by $\boldsymbol{n}_C$ (respectively $\boldsymbol{n}_I$, where $\boldsymbol{n}_I = -\boldsymbol{n}_C$) the exterior unit normal vector on Γ, i.e., from Ω_c to Ω_I (respectively, from Ω_I to Ω_c).

Let $(f, g)_{0,\Omega_j}$ be the usual inner product on $L^2(\Omega_j)$ and $\|\cdot\|_{0,\Omega_j}$ the corresponding induced norm with $j \in \{C, I\}$. As usual, for each $s > 0$, $\|\cdot\|_{s,\Omega_c}$ is the standard norm of the Sobolev space $H^s(\Omega_c)$ and $|\cdot|_{s,\Omega_c}$ represents the corresponding semi-norm; see [13, Section 1.1].

The space $H^{1/2}(\Gamma)$ is defined by localization on the Lipschitz surface Γ; see [22, Definition 3.8]. We denote by $\|\cdot\|_{1/2,\Gamma}$ the norm in $H^{1/2}(\Gamma)$ and, for simplicity, we use the integration symbol on Γ to denote the duality pairing between $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$. Furthermore, in what follows boldface symbols will be used to denote a vectorial space which is a Cartesian product of three of the corresponding component Sobolev spaces. More precisely, $\boldsymbol{L}^2(\Omega_c) := (L^2(\Omega_c))^3$ and, analogously, $\boldsymbol{H}^s(\Omega_c) := (H^s(\Omega_c))^3$, and $\boldsymbol{H}^{1/2}(\Gamma) := (H^{1/2}(\Gamma))^3$.

Let $\gamma : H^1(\Omega_c) \longrightarrow H^{1/2}(\Gamma)$ be the standard trace and $\boldsymbol{\gamma} : \boldsymbol{H}^1(\Omega_c) \longrightarrow \boldsymbol{H}^{1/2}(\Gamma)$ the vectorial trace acting by components. We use $u|_\Gamma$ and $\boldsymbol{u}|_\Gamma$ in order to denote $\gamma(u)$ and $\boldsymbol{\gamma}(\boldsymbol{u})$ for all $u \in H^1(\Omega_c)$ and $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_c)$ respectively.

We recall that

$$\mathbf{H}(\mathrm{div}; \Omega_c) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c) : \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega_c)\},$$
$$\mathbf{H}(\mathbf{curl}; \Omega_c) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c) : \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)\},$$

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are Hilbert spaces with the following inner products, respectively,

$$(\boldsymbol{u},\boldsymbol{v})_{\mathbf{H}(\mathbf{curl};\Omega_c)} := (\boldsymbol{u},\boldsymbol{v})_{0,\Omega_c} + (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_{0,\Omega_c},$$
$$(\boldsymbol{u},\boldsymbol{v})_{\mathbf{H}(\mathrm{div};\Omega_c)} := (\boldsymbol{u},\boldsymbol{v})_{0,\Omega_c} + (\mathrm{div}\,\boldsymbol{u},\mathrm{div}\,\boldsymbol{v})_{0,\Omega_c}.$$

The following result about the normal trace of functions in $\mathbf{H}(\mathrm{div};\Omega_c)$ can be found in [6, Lemma 3.10].

LEMMA 2.1. *If $\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div};\Omega_c)$, then the restriction of $\boldsymbol{\psi}\cdot\boldsymbol{n}$ to each Σ_j belongs to $H^{-1/2}(\Sigma_j)$. Furthermore, if $\Omega_c^0 := \bigcup_{j=1}^{L}\Sigma_j$, following Green's formula, it holds for all $\chi \in H^1(\Omega_c^0)$ that*

$$\int_\Gamma (\boldsymbol{\psi}\cdot\boldsymbol{n})\chi\,d\zeta + \sum_{j=1}^{L}\int_{\Sigma_j}(\boldsymbol{\psi}\cdot\boldsymbol{n})[\chi]_j\,d\zeta = \int_{\Omega_c^0}\boldsymbol{\psi}\cdot\nabla\chi\,d\boldsymbol{x} + \int_{\Omega_c^0}(\nabla\cdot\boldsymbol{\psi})\chi\,d\boldsymbol{x}, \tag{2.1}$$

where $[\chi]_j$ denotes the jump of χ through Σ_j.

On the other hand, in order to deal with the functions defined on the unbounded set Ω_I, we need to introduce the Beppo-Levi space

$$W^1(\Omega_I) := \left\{\varphi \in L^2_{\mathrm{loc}}(\Omega_I) : \frac{\varphi}{\sqrt{1+|x|^2}} \in L^2(\Omega_I), \nabla\varphi \in L^2(\Omega_I)\right\}, \tag{2.2}$$

which is a Banach space with the norm $\varphi \mapsto \|\nabla\varphi\|_{0,\Omega_I}$; see [23, Theorem 2.5.11]. Another important space for the analysis is the space of harmonic Neumann vector-fields

$$\mathbb{H}(\Omega_I) := \left\{\boldsymbol{v} \in \boldsymbol{L^2}(\Omega_I) : \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \mathrm{div}\,\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}_I = 0 \text{ on } \Gamma\right\}. \tag{2.3}$$

To define a basis of $\mathbb{H}(\Omega_I)$, we assume that there exist open connected surfaces $\Sigma_j^{\mathrm{ext}} \subset \Omega_I$, $j = 1,\dots,L$, such that $\Omega_I^0 := \Omega_I \setminus \cup_{j=1}^{L}\Sigma_j^{\mathrm{ext}}$ is simply connected, and the boundary curves $\partial\Sigma_j^{\mathrm{ext}}$ are in Γ; see [6, Hypothesis 3.3].

Now, we fix a unit normal vector $\boldsymbol{n}_j^{\mathrm{ext}}$. Then, for each $j = 1, \dots, L$, consider the following problem, which admits a unique solution:

Find $z_j \in W^1(\Omega_I \setminus \Sigma_j^{\mathrm{ext}})$ such that

$$\begin{aligned} &\Delta z_j = 0 \ \text{ in } \ \Omega_I\setminus\Sigma_j^{\mathrm{ext}}, && \nabla z_j\cdot\boldsymbol{n}_I = 0 \ \text{ on } \ \Gamma,\\ &[\![\nabla z_j\cdot\boldsymbol{n}_j^{\mathrm{ext}}]\!]_{\Sigma_j^{\mathrm{ext}}} = \mathbf{0}, && [\![z_j]\!]_{\Sigma_j^{\mathrm{ext}}} = 1. \end{aligned} \tag{2.4}$$

The set $\{\widetilde{\nabla} z_j \in L^2(\Omega_I) : j = 1,\dots,L\}$, where $\widetilde{\nabla} z_j$ are the extensions to $(L^2(\Omega_I))^3$ of ∇z_j, is a basis of $\mathbb{H}(\Omega_I)$; see [6, Prop. 3.14].

Next, we introduce on Γ the single and double layer potentials, which are formally defined by

$$\mathcal{S} : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad \mathcal{S}(\xi)(\boldsymbol{x}) := \int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\cdot\xi(\boldsymbol{y})\,d\zeta_{\boldsymbol{y}},$$
$$\mathcal{D} : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad \mathcal{D}(\eta)(\boldsymbol{x}) := \int_\Gamma \frac{\boldsymbol{x}-\boldsymbol{y}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\cdot\eta(\boldsymbol{y})\boldsymbol{n}_C(\boldsymbol{y})\,d\zeta_{\boldsymbol{y}},$$

respectively, and the hyper-singular operator $\mathcal{H} : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$, formally defined as the normal derivative

$$\mathcal{H}(\eta)(\boldsymbol{x}) := -\nabla_{\boldsymbol{x}}\left(\int_\Gamma \frac{\boldsymbol{x}-\boldsymbol{y}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\cdot\eta(\boldsymbol{y})\boldsymbol{n}_C(\boldsymbol{y})\,d\zeta_{\boldsymbol{y}}\right)\cdot\boldsymbol{n}_C(\boldsymbol{x}).$$

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Notice that the restrictions to the boundary, as well as the normal derivative above, have to be understood in a weak sense; for rigorous definitions see, for instance, [18]. These operators are linear and bounded. Denote by $\mathcal{D}' : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ the adjoint operator of $\mathcal{D}$, i.e.,

$$\int_\Gamma (\mathcal{D}'\xi)\eta \, d\zeta := \int_\Gamma \xi(\mathcal{D}\eta) \, d\zeta \qquad \forall \xi \in H^{-1/2}(\Gamma), \quad \forall \eta \in H^{1/2}(\Gamma).$$

The operators above satisfy the following properties; see, e.g., [18, 23] for the corresponding proofs.

THEOREM 2.2. *Let $\phi \in W^1(\Omega_{\mathrm{I}})$ be a harmonic function, i.e., $\triangle\phi = 0$ in Ω_{I}. Then, the following identities hold on Γ:*

$$\left(\frac{1}{2}\mathcal{I} - \mathcal{D}\right)(\phi|_\Gamma) - \mathcal{S}\left(\frac{\partial \phi}{\partial \boldsymbol{n}_{\mathrm{I}}}\right) = 0, \qquad -\left(\frac{1}{2}\mathcal{I} + \mathcal{D}'\right)\left(\frac{\partial \phi}{\partial \boldsymbol{n}_{\mathrm{I}}}\right) + \mathcal{H}\left(\phi|_\Gamma\right) = 0.$$

LEMMA 2.3. *There exist $k_1, k_2 > 0$ such that*

$$\int_\Gamma \mathcal{S}(\eta)\eta \, d\zeta \geq k_1 \|\eta\|^2_{-1/2,\Gamma}, \qquad \forall \eta \in H^{-1/2}(\Gamma),$$
$$\int_\Gamma \mathcal{H}(\phi)\phi \, d\zeta \geq k_2 \|\phi\|^2_{1/2,\Gamma}, \qquad \forall \phi \in H_0^{1/2}(\Gamma),$$

where

$$H_0^{1/2}(\Gamma) := \left\{ \phi \in H^{1/2}(\Gamma) : \int_\Gamma \phi \, d\zeta = 0 \right\}.$$

Furthermore, the operators $\mathcal{H}$ and $\mathcal{S}$ are related by the following identity

$$\int_\Gamma (\mathcal{H}\psi)\phi \, d\zeta = \int_\Gamma (\mathbf{curl}_\Gamma \, \phi)\boldsymbol{\pi}_\tau \widetilde{\mathcal{S}}(\mathbf{curl}_\Gamma \, \psi) \, d\zeta, \qquad \forall \psi, \phi \in H^{1/2}(\Gamma),$$

where $\boldsymbol{\pi}_\tau \boldsymbol{u}$ represents the tangential component trace of $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}; \Omega_{\mathrm{c}})$, $\widetilde{\mathcal{S}}$ is the continuous linear operator defined by $\widetilde{\mathcal{S}} := \mathcal{S} \circ \boldsymbol{i}_{\boldsymbol{\pi}}$, and $\boldsymbol{i}_{\boldsymbol{\pi}}$ is the adjoint operator of $\boldsymbol{\pi}_{\boldsymbol{\tau}}$.

LEMMA 2.4. *$\mathcal{H}(1) = 0$, $\mathcal{D}(1) = -1/2$, and*

$$\int_\Gamma \mathcal{H}(\eta) \, d\zeta = 0, \quad \forall \eta \in H^{1/2}(\Gamma).$$

THEOREM 2.5. *The linear operator $\mathcal{H} : H^{1/2}(\Gamma)/\mathbb{R} \longrightarrow H_0^{-1/2}(\Gamma)$, where*

$$H_0^{-1/2}(\Gamma) := \left\{ \eta \in H^{-1/2}(\Gamma) : \int_\Gamma \eta \, d\zeta = 0 \right\},$$

defines an isomorphism.

3. A FEM-BEM coupling formulation for the nonlinear eddy current problem.

The eddy current problem is obtained by neglecting the displacement of the Ampere-Maxwell Law from the full Maxwell system of equations, more precisely, the eddy current model consists of the following set of equations (see [5]):

$$\partial_t \boldsymbol{B} + \mathbf{curl}\, \boldsymbol{E} \;=\; \mathbf{0}, \qquad \text{in } \mathbb{R}^3 \times (0, T), \tag{3.1a}$$

$$\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{\sigma}\boldsymbol{E} = \boldsymbol{J}, \qquad \text{in } \mathbb{R}^3 \times [0,T], \tag{3.1b}$$

$$\boldsymbol{B}(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3, \tag{3.1c}$$

$$\boldsymbol{H}(\boldsymbol{x},t), \boldsymbol{E}(\boldsymbol{x},t) = \mathcal{O}(|\boldsymbol{x}|^{-2}), \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{3.1d}$$

$$\operatorname{div}(\epsilon_0 \boldsymbol{E}) = 0, \qquad \text{in } \Omega_I \times [0,T], \tag{3.1e}$$

$$\int_\Gamma \epsilon_0 \boldsymbol{E} \cdot \boldsymbol{n} = 0, \qquad \text{in } [0,T], \tag{3.1f}$$

where, $\boldsymbol{E} : \mathbb{R}^3 \times [0,T] \longrightarrow \mathbb{R}^3$ is the electric field, $\boldsymbol{H} : \mathbb{R}^3 \times [0,T] \longrightarrow \mathbb{R}^3$ is the magnetic field, and $\boldsymbol{B} : \mathbb{R}^3 \times [0,T] \longrightarrow \mathbb{R}^3$ is the magnetic induction. The electric conductivity $\boldsymbol{\sigma}$ in the conductor is assumed to be a symmetric and positive definite matrix with bounded entries, but it vanishes inside the insulator. Furthermore, the coefficient ϵ_0 is the electric permittivity in the vacuum and $\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{x},t)$ represents the current density applied to the system, whose compact support is included in Ω_c

$$\operatorname{supp} \boldsymbol{J}(\cdot,t) \subseteq \Omega_\mathrm{c}, \qquad \forall t \in [0,T].$$

The initial condition $\boldsymbol{B}_0$ must satisfy[1]

$$\operatorname{div} \boldsymbol{B}_0 = 0, \quad \text{in } \mathbb{R}^3.$$

From now on, we will use the notation $\boldsymbol{H}_\mathrm{C} := \boldsymbol{H}|_{\Omega_\mathrm{c}}$, $\boldsymbol{H}_\mathrm{I} := \boldsymbol{H}|_{\Omega_\mathrm{I}}$, $\boldsymbol{E}_\mathrm{C} := \boldsymbol{E}|_{\Omega_\mathrm{c}}$, $\boldsymbol{E}_\mathrm{I} := \boldsymbol{E}|_{\Omega_\mathrm{I}}$. Analogously, $\boldsymbol{B}_{\mathrm{C},0} := \boldsymbol{B}_0|_{\Omega_\mathrm{c}}$, $\boldsymbol{B}_{\mathrm{I},0} := \boldsymbol{B}_0|_{\Omega_\mathrm{I}}$, etc. Moreover, to simplify notation, in some cases the spatial variable $\boldsymbol{x}$ and the time variable t are omitted.

Finally, we suppose that the conductive material of the problem is magnetically nonlinear, that is, the relationship between the magnetic induction and the intensity of the magnetic field is nonlinear. More precisely, in a similar way as in [7, 14], we assume that

$$\boldsymbol{H}(\boldsymbol{x},t) = \begin{cases} \nu_\mathrm{C}(|\boldsymbol{B}_\mathrm{C}(\boldsymbol{x},t)|)\boldsymbol{B}_\mathrm{C}(\boldsymbol{x},t), & \forall (\boldsymbol{x},t) \in \Omega_\mathrm{c} \times [0,T], \\ \nu_1 \boldsymbol{B}_\mathrm{I} & \forall (\boldsymbol{x},t) \in \Omega_\mathrm{I} \times [0,T], \end{cases} \tag{3.2}$$

where ν_1 is a positive constant and $\nu_\mathrm{C} : \mathbb{R}_0^+ \to \mathbb{R}^+$ represents the magnetic reluctivity (inverse to magnetic permeability), which characterizes, in a certain way, the resistance exerted to the passage of magnetic flux when the material is under the influence of a magnetic field. Furthermore, we assume that ν_C satisfies

$$0 < \nu_{\min} \le \nu_\mathrm{C}(s) \le \nu_{\max}, \qquad \forall s \in \mathbb{R}_0^+.$$

To obtain a suitable formulation for problem (3.1), we proceed as in [11] by introducing the variable

$$\boldsymbol{A}_\mathrm{C}(\boldsymbol{x},t) := -\int_0^T \boldsymbol{E}_\mathrm{C}(\boldsymbol{x},s)\,ds + \boldsymbol{A}_{\mathrm{C},0}(\boldsymbol{x}), \tag{3.3}$$

where $\boldsymbol{A}_{\mathrm{C},0}$ is a vector potential of $\boldsymbol{B}_{\mathrm{C},0}$, namely, a vector field (which exists because $\operatorname{div}(\boldsymbol{B}_{\mathrm{C},0}) = 0$ in Ω_c and $\Gamma := \partial\Omega_\mathrm{c}$ is connected and Lipschitz, see for instance [6, Lemma 3.5]) such that

$$\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C},0} = \boldsymbol{B}_{\mathrm{C},0}, \qquad \text{in } \Omega_\mathrm{c}. \tag{3.4}$$

[1] From this condition and (3.1a), we deduce that $\operatorname{div} \boldsymbol{B} = 0$ in $\mathbb{R}^3 \times [0,T]$. Then $\boldsymbol{B}$ satisfies Gauss's Law for magnetism.

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We can notice that the new variable $\boldsymbol{A}_{\rm C}$ is defined on $\Omega_{\rm c}$. Moreover, by applying **curl** to (3.3), using (3.1a) and (3.4), we obtain

$$\boldsymbol{B}_{\rm C} = \mathbf{curl}\,\boldsymbol{A}_{\rm C}, \qquad \text{in } \Omega_{\rm c} \times [0,T].$$

Thus, by replacing this last relationship in (3.1b), we have

$$\boldsymbol{\sigma}\partial_t \boldsymbol{A}_{\rm C} + \mathbf{curl}\,(\nu_{\rm C}(|\,\mathbf{curl}\,\boldsymbol{A}_{\rm C}|)\,\mathbf{curl}\,\boldsymbol{A}_{\rm C}) = \boldsymbol{J}, \qquad \text{in } \Omega_{\rm c} \times (0,T).$$

Now, in order to rewrite the equations for $\boldsymbol{B}_{\rm I}$, recalling that $\operatorname{supp} \boldsymbol{J} \subseteq \Omega_{\rm c}$ and $\boldsymbol{\sigma} = \mathbf{0}$ in $\Omega_{\rm I}$, we notice equations (3.1b) and (3.2) imply

$$\mathbf{curl}\,\boldsymbol{B}_{\rm I} = \mathbf{0}, \qquad \text{in } \Omega_{\rm I} \times [0,T].$$

Consequently, we can consider the following representation of curl-free vector-fields in $\Omega_{\rm I}$ (see [12, Remark 7]), given by using the Beppo Levi space $W^1(\Omega_{\rm I})$ defined in (2.2) and the space of the harmonic Neumann vector-fields $\mathbb{H}(\Omega_{\rm I})$.

LEMMA 3.1. *The following $L^2(\Omega_{\rm I})$-orthogonal decomposition holds*

$$\{u \in \boldsymbol{L}^{\mathbf{2}}(\Omega_{\rm I}) : \mathbf{curl}\, u = \mathbf{0} \;\; in \;\; \Omega_{\rm I}\} = \nabla(W^1(\Omega_{\rm I})) \oplus \mathbb{H}(\Omega_{\rm I}).$$

The previous lemma establishes the existence, at each time $t \in [0,T]$, of a function $\psi_{\rm I}(t)$ in $W^1(\Omega_{\rm I})$ and real constants $\{\alpha_j(t)\}_{j=1}^L$ such that

$$\boldsymbol{B}_{\rm I}(\boldsymbol{x},t) = \nabla\psi_{\rm I}(\boldsymbol{x},t) + \sum_{j=1}^{L} \alpha_j(t)\widetilde{\nabla} z_j(\boldsymbol{x}), \qquad \text{in } \Omega_{\rm I} \times [0,T]. \tag{3.5}$$

Moreover, by taking divergence in this last equation and recalling that $\Delta z_j = 0$ in $\Omega_{\rm I}$, we obtain

$$\triangle \psi_{\rm I} = 0, \qquad \text{in } \Omega_{\rm I} \times [0,T].$$

On the other hand, multiplying (3.1a) by $\widetilde{\nabla} z_j$, using Green's formula and (3.5), we obtain

$$\int_\Gamma (\partial_t \boldsymbol{A}_{\rm C}) \times \boldsymbol{n}_{\rm C} \cdot \widetilde{\nabla} z_j \, d\zeta = \int_{\Omega_{\rm I}} \partial_t \boldsymbol{B}_{\rm I} \cdot \widetilde{\nabla} z_j \, d\boldsymbol{x} = \sum_{i=1}^{L} \alpha_i'(t) \int_{\Omega_{\rm I}} \widetilde{\nabla} z_i \cdot \widetilde{\nabla} z_j \, d\boldsymbol{x}.$$

Next, by integrating between 0 and s ($0 < s < T$), we deduce

$$\begin{aligned} &\sum_{i=1}^{L} \alpha_i(s) \int_{\Omega_{\rm I}} \widetilde{\nabla} z_i \cdot \widetilde{\nabla} z_j \, d\boldsymbol{x} - \int_\Gamma \boldsymbol{A}_{\rm C}(s) \times \boldsymbol{n}_{\rm C} \cdot \widetilde{\nabla} z_j \, d\zeta \\ &\qquad = \sum_{i=1}^{L} \alpha_i(0) \int_{\Omega_{\rm I}} \widetilde{\nabla} z_i \cdot \widetilde{\nabla} z_j \, d\boldsymbol{x} - \int_\Gamma \boldsymbol{A}_{{\rm C},0} \times \boldsymbol{n}_{\rm C} \cdot \widetilde{\nabla} z_j \, d\zeta, \end{aligned} \tag{3.6}$$

for all $j = 1, \ldots, L$.

Denoting by $\boldsymbol{n}_j$ is the unitary outward normal vector to the cut $\Sigma_j^{\rm ext}$, we define

$$\boldsymbol{N} = \left[\int_{\Sigma_j^{ext}} \frac{\partial z_i}{\partial \boldsymbol{n}_j} \, d\zeta \right]_{1 \le i,j \le L}, \tag{3.7}$$

$$\boldsymbol{Z} = [\widetilde{\nabla} z_1, \ldots, \widetilde{\nabla} z_L]^T, \tag{3.8}$$

$$\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_L]^T.$$

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Thus, we can use the Green's formula (2.1) to write (3.6) as

$$\boldsymbol{N}\boldsymbol{\alpha} - \int_{\Gamma} \boldsymbol{Z}(\boldsymbol{A}_{\mathrm{C}} \times \boldsymbol{n}_{\mathrm{C}})\, d\zeta = \boldsymbol{N}\boldsymbol{\alpha}_0 - \int_{\Gamma} \boldsymbol{Z}(\boldsymbol{A}_{\mathrm{C},0} \times \boldsymbol{n}_{\mathrm{C}})\, d\zeta,$$

where $\boldsymbol{\alpha}_0 := \boldsymbol{\alpha}(0)$ can be computed by using (3.5), the orthogonality of $\nabla(W^1(\Omega_{\mathrm{I}}))$ with $\mathbb{H}(\Omega_{\mathrm{I}})$, and considering that $\boldsymbol{B}_{\mathrm{I}}(\boldsymbol{x}, 0)$ is known. Observe that $\boldsymbol{N}$ is symmetric and positive definite.

In summary, we have the following formulation for the problem (3.1) in terms of the new variables $\boldsymbol{A}_{\mathrm{C}}$, ψ_{I} and $\boldsymbol{\alpha}$:

Find $\boldsymbol{A}_{\mathrm{C}} : \Omega_{\mathrm{c}} \times [0, T] \to \mathbb{R}^3$, $\psi_{\mathrm{I}} : \Omega_{\mathrm{I}} \times [0, T] \to \mathbb{R}^3$ and $\boldsymbol{\alpha} : [0, T] \to \mathbb{R}^L$ such that

$$\boldsymbol{\sigma}\partial_t \boldsymbol{A}_{\mathrm{C}} + \mathbf{curl}\,(\nu_{\mathrm{C}}(|\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}}|)\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}}) \;=\; \boldsymbol{J}, \qquad \text{in } \Omega_{\mathrm{c}} \times (0, T), \tag{3.9a}$$

$$\boldsymbol{N}\boldsymbol{\alpha} - \int_{\Gamma} \boldsymbol{Z}(\boldsymbol{A}_{\mathrm{C}} \times \boldsymbol{n}_{\mathrm{C}})\, d\zeta \;=\; \boldsymbol{N}\boldsymbol{\alpha}_0 - \int_{\Gamma} \boldsymbol{Z}(\boldsymbol{A}_{\mathrm{C},0} \times \boldsymbol{n}_{\mathrm{C}})\, d\zeta, \qquad \text{in } [0, T], \tag{3.9b}$$

$$\Delta\psi_{\mathrm{I}} \;=\; 0, \qquad \text{in } \Omega_{\mathrm{I}} \times [0, T], \tag{3.9c}$$

$$\nu_{\mathrm{C}}(|\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}}|)\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}} \times \boldsymbol{n}_{\mathrm{C}} + \nu_1(\nabla\psi_{\mathrm{I}} + \boldsymbol{Z}^t\boldsymbol{\alpha}) \times \boldsymbol{n}_{\mathrm{I}} \;=\; \mathbf{0}, \qquad \text{in } \Gamma \times [0, T], \tag{3.9d}$$

$$\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}} \cdot \boldsymbol{n}_{\mathrm{C}} + \nabla\psi_{\mathrm{I}} \cdot \boldsymbol{n}_{\mathrm{I}} \;=\; 0, \qquad \text{on } \Gamma \times [0, T], \tag{3.9e}$$

$$\boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}, 0) \;=\; \boldsymbol{A}_{\mathrm{C},0}, \qquad \text{in } \Omega_{\mathrm{c}}. \tag{3.9f}$$

It is important to notice that equations (3.9d) and (3.9e) arise from the fact that $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl}; \mathbb{R}^3)$ and $\boldsymbol{B} \in \mathbf{H}(\mathrm{div}; \mathbb{R}^3)$.

REMARK 3.2. Equations (3.9a) and (3.9d) include the magnetic reluctivity of the media, therefore these equations give the nonlinear behavior of the eddy current model. Consequently, the techniques used to obtain well-posedness as well as theoretical convergence analysis (see Theorem 4.1, Theorem 5.3 and Section 6 below) differ considerably from the ones used in the linear case; see [11].

In what follows, we show a FEM-BEM coupling formulation for the previous problem (3.9). To do this, we assume that $(\boldsymbol{A}_{\mathrm{C}}, \psi_{\mathrm{I}}, \boldsymbol{\alpha})$ is a solution of (3.9) and introduce the new variable $\psi(t) := \psi_{\mathrm{I}}|_{\Gamma}(t) - c(t)$, where $c : [0, T] \longrightarrow \mathbb{R}$ is a function such that

$$\psi(t) \in \mathrm{H}_0^{1/2}(\Gamma) := \left\{ \varphi \in H^{1/2}(\Gamma) : \int_{\Gamma} \varphi\, d\zeta = 0 \right\},$$

for any $t \in [0, T]$. Then, by proceeding as in [11, Section 4], we can deduce the following FEM-BEM coupling weak formulation for (3.9) using the integral operators defined in Section 2:

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Find $\boldsymbol{A}_{\rm C} \in L^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\rm c})) \cap \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\rm c})')$, $\psi \in L^2(0,T;H_0^{1/2}(\Gamma))$, and $\boldsymbol{\alpha} \in L^2(0,T;\mathbb{R}^L)$ such that

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_{\rm c}} \boldsymbol{\sigma}\boldsymbol{A}_{\rm C}\cdot\boldsymbol{w}_{\rm C}\,d\boldsymbol{x} + \int_{\Omega_{\rm c}} (\nu_{\rm C}(|\,\mathbf{curl}\,\boldsymbol{A}_{\rm C}|)\,\mathbf{curl}\,\boldsymbol{A}_{\rm C})\cdot\mathbf{curl}\,\boldsymbol{w}_{\rm C}\,d\boldsymbol{x} \\ + \nu_1\int_\Gamma\left[-\frac{1}{2}\psi - \mathcal{D}(\psi) + \mathcal{S}(\mathbf{curl}\,\boldsymbol{A}_{\rm C}\cdot\boldsymbol{n}_{\rm C})\right]\mathbf{curl}\,\boldsymbol{w}_{\rm C}\cdot\boldsymbol{n}_{\rm C}\,d\zeta \\ + \nu_1\boldsymbol{\alpha}^t\int_\Gamma \boldsymbol{Z}(\boldsymbol{w}_{\rm C}\times\boldsymbol{n}_{\rm C})\,d\zeta = \int_{\Omega_{\rm c}} \boldsymbol{J}\cdot\boldsymbol{w}_{\rm C}\,d\boldsymbol{x}, \end{aligned} \tag{3.10a}$$

$$\int_\Gamma\left[\frac{1}{2}\mathbf{curl}\,\boldsymbol{A}_{\rm C}\cdot\boldsymbol{n}_{\rm C} + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{A}_{\rm C}\cdot\boldsymbol{n}_{\rm C}) + \mathcal{H}(\psi)\right]\eta\,d\zeta = 0, \tag{3.10b}$$

$$\boldsymbol{\beta}^t\boldsymbol{N}\boldsymbol{\alpha} - \boldsymbol{\beta}^t\int_\Gamma \boldsymbol{Z}(\boldsymbol{A}_{\rm C}\times\boldsymbol{n}_{\rm C})\,d\zeta = \boldsymbol{\beta}^t\boldsymbol{N}\boldsymbol{\alpha}_0 - \boldsymbol{\beta}^t\int_\Gamma \boldsymbol{Z}(\boldsymbol{A}_{{\rm C},0}\times\boldsymbol{n}_{\rm C})\,d\zeta, \tag{3.10c}$$

for all $\boldsymbol{w}_{\rm C} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm c})$, $\eta \in \mathrm{H}_0^{1/2}(\Gamma)$, and $\boldsymbol{\beta} \in \mathbb{R}^L$, where $\boldsymbol{A}_{\rm C}$ satisfies the initial condition

$$\boldsymbol{A}_{\rm C}(0) = \boldsymbol{A}_{{\rm C},0}.$$

For theoretical analysis, it is convenient to eliminate $\boldsymbol{\alpha}$ and ψ from the previous formulation. To achieve this goal, we introduce the linear and bounded operator $\mathcal{J} : \mathbf{H}(\mathbf{curl};\Omega_{\rm c}) \longrightarrow \mathbb{R}^L$ defined by

$$\mathcal{J}(\boldsymbol{w}_{\rm C}) := \int_\Gamma \boldsymbol{Z}(\boldsymbol{w}_{\rm C}\times\boldsymbol{n}_{\rm C})\,d\zeta. \tag{3.11}$$

Then, we can rewrite (3.10c) as follows

$$\boldsymbol{\alpha} = \boldsymbol{N}^{-1}\mathcal{J}(\boldsymbol{A}_{\rm C}) + \boldsymbol{\alpha}_0 - \boldsymbol{N}^{-1}\mathcal{J}(\boldsymbol{A}_{{\rm C},0}).$$

On the other hand, to eliminate ψ, we can use Lemma 2.3 to define the linear and bounded operator $\mathcal{R} : \mathrm{H}_0^{-1/2}(\Gamma) \longrightarrow \mathrm{H}_0^{1/2}(\Gamma)$ characterized by

$$\int_\Gamma \mathcal{H}(\mathcal{R}(\xi))\eta\,d\zeta = \int_\Gamma \xi\eta\,d\zeta, \qquad \forall\eta \in \mathrm{H}_0^{1/2}(\Gamma),\ \forall\xi \in \mathrm{H}_0^{-1/2}(\Gamma). \tag{3.12}$$

Then, from (3.10b) it follows

$$\psi = -\mathcal{R}\left(\frac{1}{2}\mathbf{curl}\,\boldsymbol{A}_{\rm C}\cdot\boldsymbol{n}_{\rm C} + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{A}_{\rm C}\cdot\boldsymbol{n}_{\rm C})\right),$$

and, consequently, (3.10) admits the following equivalent reduced form:

Find $\boldsymbol{A}_{\rm C} \in L^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\rm c})) \cap \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\rm c})')$ such that

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\sigma}\boldsymbol{A}_{\rm C}(t),\boldsymbol{w}_{\rm C})_{0,\Omega_{\rm c}} + \langle\mathcal{E}\boldsymbol{A}_{\rm C}(t),\boldsymbol{w}_{\rm C}\rangle + \\ \langle\mathcal{B}\boldsymbol{A}_{\rm C}(t),\boldsymbol{w}_{\rm C}\rangle &= (\boldsymbol{J}(t),\boldsymbol{w}_{\rm C})_{0,\Omega_{\rm c}} + \boldsymbol{g}(\boldsymbol{w}_{\rm C}), \\ \boldsymbol{A}_{\rm C}(0) &= \boldsymbol{A}_{{\rm C},0} \quad \text{in} \quad \Omega_{\rm c}, \end{aligned} \tag{3.13}$$

for all $\boldsymbol{w}_{\mathrm{C}} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})$, where $\mathcal{E} : \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'$, $\mathcal{K} : \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathrm{H}_0^{-1/2}(\Gamma)$, $\mathcal{B} : \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'$ and $\boldsymbol{g} : \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbb{R}$ are given by

$$\langle \mathcal{E}\boldsymbol{F}, \boldsymbol{G}\rangle := \int_{\Omega_{\mathrm{c}}} \nu_{\mathrm{C}}(|\,\mathbf{curl}\,\boldsymbol{F}|)\,\mathbf{curl}\,\boldsymbol{F}\cdot\mathbf{curl}\,\boldsymbol{G}\,d\boldsymbol{x}, \tag{3.14}$$

$$\mathcal{K}(\boldsymbol{F}) := \frac{1}{2}\,\mathbf{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}_{\mathrm{C}} + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}_{\mathrm{C}}),$$

$$\langle \mathcal{B}\boldsymbol{F}, \boldsymbol{G}\rangle := \nu_1 \int_{\Gamma} \mathcal{S}(\mathbf{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}_{\mathrm{C}})\,\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}_{\mathrm{C}}\,d\zeta + \nu_1\int_{\Gamma}\mathcal{K}(\boldsymbol{G})\mathcal{R}(\mathcal{K}(\boldsymbol{F}))\,d\zeta + \nu_1(\mathcal{J}(\boldsymbol{G}))^t\boldsymbol{N}^{-1}\mathcal{J}(\boldsymbol{F}), \tag{3.15}$$

$$\boldsymbol{g}(\boldsymbol{F}) := \nu_1(\mathcal{J}(\boldsymbol{F}))^t\boldsymbol{N}^{-1}\mathcal{J}(\boldsymbol{A}_{\mathrm{C},0}) - \nu_1(\mathcal{J}(\boldsymbol{F}))^t\boldsymbol{\alpha}_0. \tag{3.16}$$

We can notice that $\mathcal{E}$ is a nonlinear operator because ν_{C} is nonlinear, while the bilinear form $\mathcal{B}$ is bounded and nonnegative. On the other hand, to deduce the variational problem (3.13), we prove that if $(\boldsymbol{A}_{\mathrm{C}}, \psi, \boldsymbol{\alpha})$ is a solution of (3.10), then ψ and $\boldsymbol{\alpha}$ depend on $\boldsymbol{A}_{\mathrm{C}}$ through expressions

$$\psi := -\mathcal{R}\left(\mathcal{K}(\boldsymbol{A}_{\mathrm{C}})\right), \qquad \boldsymbol{\alpha} := \boldsymbol{N}^{-1}(\mathcal{J}(\boldsymbol{A}_{\mathrm{C}}) - \mathcal{J}(\boldsymbol{A}_{\mathrm{C},0})) + \boldsymbol{\alpha}_0. \tag{3.17}$$

Therefore, in order to show the existence and uniqueness of solutions to (3.10) it is enough to show the uniqueness of $\boldsymbol{A}_{\mathrm{C}}$.

4. Well-posedness of the nonlinear eddy current model. In this section, we will prove that the problem (3.13) has a unique solution. To this aim we need to add the following assumption about the reluctivity: $\nu_{\mathrm{C}} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a continuously differentiable function and there exist constants $m, M > 0$ satisfying

$$m \le \nu_{\mathrm{C}}'(s)s + \nu_{\mathrm{C}}(s) \le M, \qquad \forall s \ge 0. \tag{4.1}$$

Then, by using the Mean Value Theorem we can easily deduce that

$$m \le \nu_{\mathrm{C}}(s) \le M, \qquad \forall s \ge 0. \tag{4.2}$$

From now on, we will denote by $(\cdot,\cdot)_{\boldsymbol{\sigma}}$ the inner product in $\boldsymbol{L}^2(\Omega_{\mathrm{c}})$, which is defined by

$$(\boldsymbol{F}, \boldsymbol{G})_{\boldsymbol{\sigma}} := \int_{\Omega_{\mathrm{c}}} \boldsymbol{\sigma}\boldsymbol{F}\cdot\boldsymbol{G}\,d\boldsymbol{x}, \qquad \forall \boldsymbol{F}, \boldsymbol{G} \in \boldsymbol{L}^2(\Omega_{\mathrm{c}}),$$

and by $\|\cdot\|_{\boldsymbol{\sigma}}$ the corresponding induced norm. Since the norm $\|\cdot\|_{0,\Omega_{\mathrm{c}}}$ is equivalent to $\|\cdot\|_{\sigma}$, the norm $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega_{\mathrm{c}})}$ is equivalent to the norm $\|\cdot\|_{\sigma} + \|\,\mathbf{curl}(\cdot)\|_{0,\Omega_{\mathrm{c}}}$.

To show that problem (3.13) has a unique solution, we will use the substitution $\boldsymbol{U}_{\mathrm{C}}(t) := e^{-t}\boldsymbol{A}_{\mathrm{C}}(t)$, which allows to rewrite this problem in an equivalent form, for which it is possible to apply the theory of nonlinear parabolic equations. In fact, if $\boldsymbol{A}_{\mathrm{C}}(t) := e^t\boldsymbol{U}_{\mathrm{C}}(t)$, from (3.13), the following equivalent variational problem to (3.13), is obtained: Find $\boldsymbol{U}_{\mathrm{C}} \in L^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})) \cap \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})')$ such that

$$\frac{d}{dt}(\boldsymbol{U}_{\mathrm{C}}(t), \boldsymbol{w}_{\mathrm{C}})_{\boldsymbol{\sigma}} + \langle \mathcal{C}(t)\boldsymbol{U}_{\mathrm{C}}(t), \boldsymbol{w}_{\mathrm{C}}\rangle = e^{-t}\left[(\boldsymbol{J}(t), \boldsymbol{w}_{\mathrm{C}})_{0,\Omega_{\mathrm{c}}} + \boldsymbol{g}(\boldsymbol{w}_{\mathrm{C}})\right], \tag{4.3}$$

where, for each $t \in (0,T)$, $\mathcal{C}(t) : \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'$ is the nonlinear operator defined by

$$\langle \mathcal{C}(t)\boldsymbol{w}, \boldsymbol{v}\rangle := (\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{\sigma}} + \langle \mathcal{D}(t)\boldsymbol{w}, \boldsymbol{v}\rangle + \langle \mathcal{B}\boldsymbol{w}, \boldsymbol{v}\rangle,$$

with

$$\langle \mathcal{D}(t)\boldsymbol{w}, \boldsymbol{v}\rangle := \int_{\Omega_c} \nu_C(e^t|\operatorname{\mathbf{curl}} \boldsymbol{w}|) \operatorname{\mathbf{curl}} \boldsymbol{w} \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} \, d\boldsymbol{x}.$$

It can be verified that if $\boldsymbol{U}_C \in \mathbf{H}(\mathbf{curl}; \Omega_c)$ is a solution of (4.3), then $\boldsymbol{A}_C(t) := e^t \boldsymbol{U}_C(t)$ is the solution of (3.13). Consequently, to deduce the well-posedness it is enough to show that problem (4.3) has a unique solution, which is proved in the following result.

THEOREM 4.1. *Problem (4.3) has a unique solution.*

Proof. We are going to use [28, Theorem 30.A] to obtain the well-posedness of the problem. In fact, we need to prove that $\mathcal{C}(t)$ is hemicontinuous, monotone, coercive and bounded. We only prove here that $\mathcal{C}(t)$ is monotone, because the other three conditions can be easily verified. Taking into account that the operator $\mathcal{B}$ is nonnegative, it is enough to prove that $\mathcal{D}(t)$ is monotone. In fact, let $\boldsymbol{\alpha} := e^t \operatorname{\mathbf{curl}} \boldsymbol{v}$, $\boldsymbol{\beta} := e^t \operatorname{\mathbf{curl}} \boldsymbol{w}$ and

$$G(s) := \nu_C(|s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}|)(s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}) \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}).$$

Then, to deduce $\mathcal{D}(t)$ is monotone, it is enough to show that $G'(s) \geq 0$ for any s, which can be verified by checking that

$$G'(s) = \{[\nu_C'(\eta)\eta + \nu_C(\eta)]\cos^2\theta + \nu_C(\eta)\sin^2\theta\}|\boldsymbol{\alpha} - \boldsymbol{\beta}|^2,$$

where

$$\cos\theta := \frac{(s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}) \cdot (\alpha - \boldsymbol{\beta})}{|s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}||\boldsymbol{\alpha} - \boldsymbol{\beta}|}, \qquad \eta := |s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}|,$$

and by recalling that $\eta \longmapsto \nu_C(\eta)\eta$ is strictly increasing; see (4.1). ◻

REMARK 4.2. It is a simple matter to show that $(\boldsymbol{A}_C, \psi, \boldsymbol{\alpha})$ is the solution of problem (3.10), where $\boldsymbol{A}_C$ is the unique solution of problem (3.13), and ψ and $\boldsymbol{\alpha}$ are defined by (3.17). Moreover, by adapting the line of the proof of [11, Theorem 4.3], we can prove that there exist $\psi_I \in L^2(0, T; W^1(\Omega_I))$ and a function $C : [0, T] \longrightarrow \mathbb{R}$ such that $\psi = \psi_I|_\Gamma - C$ and $(\boldsymbol{A}_C, \psi_I, \boldsymbol{\alpha})$ is the unique solution of the strong problem (3.9).

5. Fully-discrete scheme. In order to obtain a fully discrete finite element approximation scheme for problem (3.10), we need to consider a regular family of tetrahedral meshes $\{\mathcal{T}_h(\Omega_c)\}_h$ for the conducting domain Ω_c, where h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h(\Omega_c)$. Furthermore, let $\{\mathcal{T}_h(\Gamma)\}_h$ be the corresponding family of triangulations induced on Γ, $N \in \mathbb{N}$, $\Delta t := \frac{T}{N}$ and $t_n = n\Delta t$, for $n = 0, 1, \ldots, N$.

We will define the fully discrete scheme by using Nédélec finite elements. It is well known that the local representation on K of the lowest order Nédélec finite elements subspace is given by

$$\mathcal{N}(K) := \{\boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b} : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3, \boldsymbol{x} \in K\},$$

and its corresponding global Nédélec finite elements subspace $\mathbf{H}_h(\mathbf{curl}; \Omega_c)$ is the space of vector that are local in $\mathcal{N}(K)$ for all $K \in \mathcal{T}_h(\Omega_c)$. Furthermore, we define

$$\mathcal{L}_h(\Gamma) := \{\eta \in \mathrm{H}_0^{1/2}(\Gamma) : \eta|_F \in \mathbb{P}_1(F) \; \forall F \in \mathcal{T}_h(\Gamma)\},$$

which is a discrete subspace of $\mathrm{H}_0^{1/2}(\Gamma)$, where $\mathbb{P}_m(F)$ is the set of polynomial functions defined on F of degree not greater than m.

When Ω_c is not simply connected, problem (3.10) involves the matrices $\boldsymbol{N}$ and $\boldsymbol{Z}$ defined by (3.7) and (3.8), respectively. Consequently, in order to obtain the fully discrete approximation of (3.10), it is necessary to approximate numerically the basis $\{\widetilde{\nabla} z_k\}_{k=1}^L$ of the space of harmonic Neumann's vectorial fields defined in (2.3). In [21], the authors proposed a FEM-BEM coupling method to calculate the entries of the matrix $\boldsymbol{N}_h$ that approximates $\boldsymbol{N}$. Next, we briefly describe this method.

First, we consider a convex polyhedra Ω such that $\overline{\Omega_c} \cup \left(\cup_{k=1}^L \overline{\Sigma}_k^{\text{ext}}\right) \subset \Omega$ and let $Q := \Omega \setminus \overline{\Omega_c}$. For each $k = 1, \ldots, L$, we define $\boldsymbol{p}_k := \widetilde{\nabla} z_k|_Q$. Thus, given that z_k is solution to (2.4), we deduce that for each $k = 1, \ldots, L$, $\boldsymbol{p}_k$ belongs to the closed subspace $\mathbf{H}(\text{div}; Q)$ defined by

$$\mathcal{Y} := \{\boldsymbol{q} \in (L^2(Q))^3 : \text{div}\, \boldsymbol{q} = 0 \text{ in } Q \text{ and } \boldsymbol{q} \cdot \boldsymbol{n}_{\text{I}} = 0 \text{ on } \Gamma\}.$$

To approximate each $\boldsymbol{p}_k$ it is necessary to introduce the space of finite elements of Raviart-Thomas in Q to approximate $\mathbf{H}(\text{div}; Q)$. This subspace is defined by

$$\mathcal{RT}_h(Q) := \{\boldsymbol{q} \in \boldsymbol{H}(\text{div}; Q) : \boldsymbol{q}|_K \in \mathcal{RT}(K),\ \forall K \in \mathcal{J}_h(Q)\},$$

where

$$\mathcal{RT}(K) := \left\{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3, \boldsymbol{x} \in K\right\},$$

which precisely corresponds to the local representation on K of the subspace of finite elements of Raviart-Thomas of lowest order. The matrix $\boldsymbol{N}_h$ is defined by

$$\boldsymbol{N}_h := \left(\int_{\Sigma_j^{\text{ext}}} \boldsymbol{p}_{kh} \cdot \boldsymbol{n}_j \, d\zeta\right)_{1 \le k,j \le L}, \tag{5.1}$$

where $\boldsymbol{p}_{kh}$ is calculated by solving the following mixed problem (see [21]):

Find $\boldsymbol{p}_{kh} \in \mathcal{RT}_h^0(Q)$, $\phi_{kh} \in \Phi_h/\mathbb{R}$, and $\beta_{kh} \in M_h$ such that

$$\begin{aligned}
&\int_Q \boldsymbol{p}_{kh} \cdot \boldsymbol{q} \, d\boldsymbol{x} - \int_\Lambda \left[\frac{1}{2}\phi_{kh} + \mathcal{D}(\phi_{kh})\right] \boldsymbol{q} \cdot \boldsymbol{n} \, d\zeta \\
&+ \int_{\partial\Omega} \mathcal{S}\left(\boldsymbol{p}_{kh} \cdot \boldsymbol{n}\right) \boldsymbol{q} \cdot \boldsymbol{n} \, d\zeta + \int_Q \beta_{kh} \,\text{div}\, \boldsymbol{q} \, d\boldsymbol{x} = \int_{\Sigma_k^{\text{ext}}} \boldsymbol{q} \cdot \boldsymbol{n}_k \, d\zeta, \\
&\int_{\partial\Omega} \left[\frac{1}{2}\chi + \mathcal{D}(\chi)\right] \boldsymbol{p}_{kh} \cdot \boldsymbol{n} \, d\zeta + \int_{\partial\Omega} \mathcal{S}(\mathbf{curl}_\tau\, \phi_{kh})\, \mathbf{curl}_\tau\, \chi \, d\zeta = 0, \\
&\int_Q \text{div}\, \boldsymbol{p}_{kh} v \, d\boldsymbol{x} = 0,
\end{aligned}$$

for all function $\boldsymbol{q} \in \mathcal{RT}_h^0(Q)$, $\chi \in \Phi_h/\mathbb{R}$, and $v \in M_h$, where

$$\mathcal{RT}_h^0(Q) := \{\boldsymbol{q} \in \mathcal{RT}_h(Q) : \boldsymbol{q}|_\Gamma \cdot \boldsymbol{n}_{\text{I}} = 0\},$$

$$\Phi_h := \left\{\eta \in C^0(\partial\Omega) : \eta|_F \in \mathbb{P}_1(F)\ \forall F \in \mathcal{J}_h(\partial\Omega)\right\},$$

$$M_h := \left\{v \in L^2(Q) : v|_K \in \mathbb{P}_0(K)\ \forall K \in \mathcal{J}_h(Q)\right\}.$$

REMARK 5.1. The matrix $\boldsymbol{N}_h$ defined by (5.1) is symmetric and positive definite. Furthermore, there exists $h_0 > 0$ such that $\boldsymbol{N}_h$ is invertible for all $h \in (0, h_0)$ and the following approximation estimate holds

$$\|\boldsymbol{N} - \boldsymbol{N}_h\| + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\| \leq Ch^s \max_{1\leq k\leq L} \left(\|\boldsymbol{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right),$$

for any $s \in (1/2, s_Q)$, where $s_Q \in (1/2, 1)$ denotes the exponent of maximal regularity in Q of the solution of the Laplace operator with $L^2(Q)$ right-hand side and homogeneous Neumann boundary data; see Theorem 7.1 and Corollary 7.3 of [21] for the details.

On the other hand, it is also necessary to obtain a way to approximate the discrete form to the operator $\mathcal{J}$ defined in (3.11), for which it is necessary to approximate appropriately the matrix $\boldsymbol{Z}$. An appropriate approximation for the matrix $\boldsymbol{Z}$ was presented in [11], which consists of calculating the matrix $\boldsymbol{Z}_h$ defined by

$$\boldsymbol{Z}_h := [\widetilde{\nabla} z_{1h} \;\ldots\; \widetilde{\nabla} z_{Lh}]^T, \tag{5.2}$$

where $\widetilde{\nabla} z_{kh}$ (for $k = 1, \ldots, L$) is the zero extension to Q of ∇z_{kh} and z_{kh} is the solution of a weak discrete approximation problem for (2.4). Moreover, in [11, Lemma 5.1] it was proved that z_k and z_{kh} satisfy the following estimate

$$\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{kh}\|_{0,Q} \leq Ch^s. \tag{5.3}$$

Once the matrix $\boldsymbol{Z}_h$ defined in (5.2) has been calculated, let $\mathcal{T}_h : \mathbf{H}(\mathbf{curl}; \Omega_\mathrm{c}) \longrightarrow \mathbb{R}^L$ be the linear and bounded operator given by

$$\mathcal{T}_h(\boldsymbol{w}) := \int_\Gamma \boldsymbol{Z}_h(\boldsymbol{w} \times \boldsymbol{n}_\mathrm{C})\, d\zeta.$$

With the preceding discussion, we can introduce the fully discrete problem associated to (3.10):

For $n = 1, \ldots, N$, find $(\boldsymbol{A}_{\mathrm{C}h}^n, \psi_h^n, \boldsymbol{\alpha}_h^n) \in \mathbf{H}_h(\mathbf{curl}; \Omega_\mathrm{c}) \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$ such that

$$\begin{aligned}
&\int_{\Omega_\mathrm{c}} \boldsymbol{\sigma} \bar{\partial} \boldsymbol{A}_{\mathrm{C}h}^n \cdot \boldsymbol{w}_\mathrm{C}\, d\boldsymbol{x} + \int_{\Omega_\mathrm{c}} (\nu_\mathrm{C}(|\,\mathbf{curl}\, \boldsymbol{A}_{\mathrm{C}h}^n|)\, \mathbf{curl}\, \boldsymbol{A}_{\mathrm{C}h}^n) \cdot \mathbf{curl}\, \boldsymbol{w}_\mathrm{C}\, d\boldsymbol{x} \\
&\quad + \nu_1 \int_\Gamma \left[-\frac{1}{2}\psi_h^n - \mathcal{D}(\psi_h^n) + \mathcal{S}(\mathbf{curl}\, \boldsymbol{A}_{\mathrm{C}h}^n \cdot \boldsymbol{n}_\mathrm{C})\right] \mathbf{curl}\, \boldsymbol{w}_\mathrm{C} \cdot \boldsymbol{n}_\mathrm{C}\, d\zeta \\
&\qquad + \nu_1 (\boldsymbol{\alpha}_h^n)^t \mathcal{T}_h(w_\mathrm{C}) = \int_{\Omega_\mathrm{c}} \boldsymbol{J}(t_n) \cdot \boldsymbol{w}_\mathrm{C}\, d\boldsymbol{x}, \\
&\int_\Gamma \left[\frac{1}{2}\mathbf{curl}\, \boldsymbol{A}_{\mathrm{C}h}^n \cdot \boldsymbol{n}_\mathrm{C} + \mathcal{D}'(\mathbf{curl}\, \boldsymbol{A}_{\mathrm{C}h}^n \cdot \boldsymbol{n}_\mathrm{C}) + \mathcal{H}(\psi_h^n)\right] \eta\, d\zeta = 0, \\
&\boldsymbol{\beta}^t \boldsymbol{N}_h \boldsymbol{\alpha}_h^n - \boldsymbol{\beta}^t \mathcal{T}_h(\boldsymbol{A}_{\mathrm{C}h}^n) = \boldsymbol{\beta}^t \boldsymbol{N}_h \boldsymbol{\alpha}_0 - \boldsymbol{\beta}^t \mathcal{T}_h(\boldsymbol{A}_{\mathrm{C},0}),
\end{aligned} \tag{5.4}$$

for all $(\boldsymbol{w}_\mathrm{C}, \eta, \boldsymbol{\beta}) \in \mathbf{H}_h(\mathbf{curl}; \Omega_\mathrm{c}) \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$, with $\boldsymbol{A}_{\mathrm{C}h}^0 = \boldsymbol{A}_{\mathrm{C}h,0}$ in Ω_c, where $\boldsymbol{A}_{\mathrm{C}h,0} \in \mathbf{H}_h(\mathbf{curl}; \Omega_\mathrm{c})$ is an approximation of $\boldsymbol{A}_{\mathrm{C},0}$, and

$$\bar{\partial} \boldsymbol{A}_{\mathrm{C}h}^n := (\boldsymbol{A}_{\mathrm{C}h}^n - \boldsymbol{A}_{\mathrm{C}h}^{n-1})/\Delta t.$$

To prove the existence and uniqueness of the solutions to (5.4), we proceed first as in the continuous case and obtain the discrete form to the problem (3.13). Let $\mathcal{R}_h : \mathrm{H}_0^{-1/2}(\Gamma) \longrightarrow \mathcal{L}_h(\Gamma)$ be the operator defined by

$$\int_\Gamma \mathcal{H}(\mathcal{R}_h(\xi))\eta\, d\zeta = \int_\Gamma \xi\eta\, d\zeta, \qquad \forall \eta \in \mathcal{L}_h(\Gamma),\ \forall \xi \in \mathrm{H}_0^{-1/2}(\Gamma). \tag{5.5}$$

By using the Lax-Milgram Lemma, we have that $\mathcal{R}_h$ is a linear bounded operator which satisfies

$$\psi_h^n := -\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}_{\mathrm{C}h}^n)). \tag{5.6}$$

Furthermore, we can notice that (5.5) is a Galerkin discretization of the elliptic problem (3.12), hence from (3.12) and (5.5) we have the following Cea estimate: there exists $C > 0$ such that

$$\|\mathcal{R}(\xi) - \mathcal{R}_h(\xi)\|_{1/2,\Gamma} \leq C \inf_{\eta \in \mathcal{L}_h(\Gamma)} \|\mathcal{R}(\xi) - \eta\|_{1/2,\Gamma}, \qquad \forall \xi \in \mathrm{H}_0^{-1/2}(\Gamma). \tag{5.7}$$

Furthermore, from the third equation in (5.4) we have for all $\boldsymbol{w}_{\mathrm{C}} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})$

$$\begin{aligned}(\boldsymbol{\alpha}_h^n)^t \mathcal{T}_h(\boldsymbol{w}_{\mathrm{C}}) = (\mathcal{T}_h(\boldsymbol{w}_{\mathrm{C}}))^t \boldsymbol{N}_h^{-1} \mathcal{T}_h(\boldsymbol{A}_{\mathrm{C}h}^n) + (\mathcal{T}_h(\boldsymbol{w}_{\mathrm{C}}))^t \boldsymbol{\alpha}_0 \\ - (\mathcal{T}_h(\boldsymbol{w}_{\mathrm{C}}))^t \boldsymbol{N}_h^{-1} \mathcal{T}_h(\boldsymbol{A}_{\mathrm{C},0}),\end{aligned} \tag{5.8}$$

and thus, replacing (5.6) and (5.8) in the first equation of (5.4), we can see that (5.4) is equivalent to the following problem (which corresponds precisely to the discrete version of (3.13)):

For $n = 1, \ldots, N$, find $\boldsymbol{A}_{\mathrm{C}h}^n \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$ such that

$$(\bar{\partial}\boldsymbol{A}_{\mathrm{C}h}^n, \boldsymbol{w}_{\mathrm{C}})_{\boldsymbol{\sigma}} + \langle \mathcal{E}\boldsymbol{A}_{\mathrm{C}h}^n, \boldsymbol{w}_{\mathrm{C}}\rangle + \langle \mathcal{B}_h \boldsymbol{A}_{\mathrm{C}h}^n, \boldsymbol{w}_{\mathrm{C}}\rangle = (\boldsymbol{J}(t_n), \boldsymbol{w}_{\mathrm{C}})_{0,\Omega_{\mathrm{c}}} + \boldsymbol{g}_h(\boldsymbol{w}_{\mathrm{C}}), \tag{5.9}$$

for all $\boldsymbol{w}_{\mathrm{C}} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$, with $\boldsymbol{A}_{\mathrm{C}h}^0 = \boldsymbol{A}_{\mathrm{C}h,0}$ in Ω_{c}, where $\mathcal{E} : \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})'$ is the nonlinear operator defined in (3.14), and $\mathcal{B}_h : \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})'$ and $\boldsymbol{g}_h : \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbb{R}$ are respectively given by

$$\begin{aligned}\langle \mathcal{B}_h \boldsymbol{F}, \boldsymbol{G}\rangle &:= \nu_1 \int_\Gamma \mathcal{S}(\mathbf{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}_{\mathrm{C}})\,\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}_{\mathrm{C}}\,d\zeta \\ &\quad + \nu_1 \int_\Gamma \mathcal{K}(\boldsymbol{G})\mathcal{R}_h(\mathcal{K}(\boldsymbol{F}))\,d\zeta + \nu_1(\mathcal{T}_h(\boldsymbol{G}))^t \boldsymbol{N}_h^{-1}\mathcal{T}_h(\boldsymbol{F}), \\ \boldsymbol{g}_h(\boldsymbol{F}) &:= \nu_1(\mathcal{T}_h(\boldsymbol{F}))^t \boldsymbol{N}_h^{-1}\mathcal{T}_h(\boldsymbol{A}_{\mathrm{C},0}) - \nu_1(\mathcal{T}_h(\boldsymbol{F}))^t\boldsymbol{\alpha}_0.\end{aligned}$$

It is important to notice that the bilinear form $\mathcal{B}_h$ and the operator $\boldsymbol{g}_h$ are linear and bounded operators and they are the discrete versions of $\mathcal{B}$ and $\boldsymbol{g}$ respectively; see (3.15) and (3.16). Moreover, the bilinear form $\mathcal{B}_h$ is nonnegative, as it is the same case of its continuous version $\mathcal{B}$.

In the following result, we show some properties of the nonlinear operator $\mathcal{E}$, which are necessaries for the analysis of problem (5.9).

LEMMA 5.2. *If $\nu : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a continuously differentiable function satisfying* (4.1)–(4.2)*, then the nonlinear operator $\mathcal{E}$ defined by* (3.14) *satisfies*

$$\langle \mathcal{E}(\boldsymbol{u}) - \mathcal{E}(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \geq m\|\,\mathbf{curl}(\boldsymbol{u}-\boldsymbol{v})\|_{0,\Omega_{\mathrm{c}}}^2, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}), \tag{5.10}$$

and

$$\|\mathcal{E}(\boldsymbol{u}) - \mathcal{E}(\boldsymbol{v})\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'} \leq 3M\|\boldsymbol{u}-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})}, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}). \tag{5.11}$$

Proof. The proof is based on the ideas presented in [28, Lemma 25.26]; see also [25]. Using (4.1), (4.2), and the Mean Value Theorem we obtain

$$(\nu_{\mathrm{C}}(s)s - \nu_{\mathrm{C}}(t)t)\,(s-t) \geq m(s-t)^2, \qquad \forall s,t \geq 0, \tag{5.12}$$

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$$|\nu_{\mathrm{C}}(s)s - \nu_{\mathrm{C}}(t)t| \leq M|s-t|, \qquad \forall s,t \geq 0. \tag{5.13}$$

Then, using (5.12), we check

$$(\nu_{\mathrm{C}}(|\boldsymbol{p}|)\boldsymbol{p} - \nu_{\mathrm{C}}(|\boldsymbol{q}|)\boldsymbol{q}) \cdot (\boldsymbol{p}-\boldsymbol{q}) \geq m|\boldsymbol{p}-\boldsymbol{q}|^2, \qquad \forall \boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^3,$$

and consequently, by choosing $\boldsymbol{p} = \mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{v}$ in this inequality, we can conclude that (5.10) holds. Finally, from (3.14) and using (5.13), it is easy to verify (5.11). ☐

5.1. Well-posedness of the discrete problem. We recall that the discrete version of the problem (3.13) is:

Find $\boldsymbol{A}^n_{\mathrm{C}h} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$, $n = 1,\ldots,N$ such that

$$\left(\frac{\boldsymbol{A}^n_{\mathrm{C}h} - \boldsymbol{A}^{n-1}_{\mathrm{C}h}}{\Delta t}, \boldsymbol{w}_{\mathrm{C}}\right)_{\boldsymbol{\sigma}} + \langle \mathcal{E}\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}}\rangle + \langle \mathcal{B}_h\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}}\rangle = (\boldsymbol{J}(t_n), \boldsymbol{w}_{\mathrm{C}})_{0,\Omega_{\mathrm{c}}} + \boldsymbol{g}_h(\boldsymbol{w}_{\mathrm{C}}).$$

Therefore, for each iteration, we have to find $\boldsymbol{A}^n_{\mathrm{C}h} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$ such that

$$(\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}})_{\boldsymbol{\sigma}} + \Delta t\langle \mathcal{E}\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}}\rangle + \Delta t\langle \mathcal{B}_h\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}}\rangle = F_n(\boldsymbol{w}_{\mathrm{C}}), \tag{5.14}$$

for all $\boldsymbol{w}_{\mathrm{C}} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$, with $\boldsymbol{A}^0_{\mathrm{C}h} = \boldsymbol{A}_{\mathrm{C}h,0}$ in Ω_{c}, where

$$F_n(\boldsymbol{w}_{\mathrm{C}}) := \Delta t\left[(\boldsymbol{J}(t_n), \boldsymbol{w}_{\mathrm{C}})_{0,\Omega_{\mathrm{c}}} + \boldsymbol{g}_h(\boldsymbol{w}_{\mathrm{C}})\right] + (\boldsymbol{A}^{n-1}_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}})_{\boldsymbol{\sigma}}.$$

THEOREM 5.3. *The problem* (5.14) *is well posed, that is, it has a unique solution.*

Proof. First, we define the nonlinear operator $\mathcal{C}_h : \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}}) \longrightarrow \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})'$, given by

$$\langle \mathcal{C}_h\boldsymbol{F}, \boldsymbol{G}\rangle := (\boldsymbol{F}, \boldsymbol{G})_{\boldsymbol{\sigma}} + \Delta t\langle \mathcal{E}\boldsymbol{F}, \boldsymbol{G}\rangle + \Delta t\langle \mathcal{B}_h\boldsymbol{F}, \boldsymbol{G}\rangle,$$

then, problem (5.14) is equivalent to finding $\boldsymbol{A}^n_{\mathrm{C}h} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$ such that

$$\langle \mathcal{C}_h\boldsymbol{A}^n_{\mathrm{C}h}, \boldsymbol{w}_{\mathrm{C}}\rangle = F_n(\boldsymbol{w}_{\mathrm{C}}), \tag{5.15}$$

for all $\boldsymbol{w}_{\mathrm{C}} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$, with $\boldsymbol{A}^0_{\mathrm{C}h} = \boldsymbol{A}_{\mathrm{C}h,0}$ in Ω_{c}. To prove the well-posedness of (5.15) it is enough to show that the nonlinear operator $\mathcal{C}_h$ is strictly monotone and Lipschitz continuous in $\mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$ (see [28, Theorem 25.B]), which can be shown by adapting the line of the proof of Lemma 5.2. ☐

The proof of the following result is analogous to the continuous case.

THEOREM 5.4. *Let* $\boldsymbol{A}^n_{\mathrm{C}h} \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}})$ *be the solution of* (5.9). *If*

$$\psi^n_h := -\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}^n_{\mathrm{C}h})), \quad \boldsymbol{\alpha}^n_h := \boldsymbol{\alpha}_0 + \boldsymbol{N}_h^{-1}(\mathcal{J}_h(\boldsymbol{A}^n_{\mathrm{C}h}) - \mathcal{J}_h(\boldsymbol{A}_{\mathrm{C},0})),$$

then $(\boldsymbol{A}^n_{\mathrm{C}h}, \psi^n_h, \boldsymbol{\alpha}^n_h) \in \mathbf{H}_h(\mathbf{curl};\Omega_{\mathrm{c}}) \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$ *is the unique solution of* (5.4).

6. Error estimates. For any $s > 0$, consider the Sobolev space

$$\mathbf{H}^s(\mathbf{curl};\Omega_{\mathrm{c}}) := \{\boldsymbol{v} \in \boldsymbol{H}^s(\Omega_{\mathrm{c}}) : \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^s(\Omega_{\mathrm{c}})\},$$

endowed with the norm

$$\|\boldsymbol{v}\|^2_{\mathbf{H}^s(\mathbf{curl};\Omega_{\mathrm{c}})} := \|\boldsymbol{v}\|^2_{s,\Omega_{\mathrm{c}}} + \|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{s,\Omega_{\mathrm{c}}}.$$

By [4, Lemma 5.1], for each $s > 1/2$ the Nédélec interpolation operator

$$\mathcal{I}_h^N : \mathbf{H}^s(\mathbf{curl}; \Omega_c) \longrightarrow \mathbf{H}_h(\mathbf{curl}; \Omega_c)$$

is well defined. Moreover, for $1/2 < s \leq 1$, the following interpolation error estimate holds (see [4, Prop. 5.6])

$$\|\boldsymbol{v} - \mathcal{I}_h^N \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_c)} \leq C h^s \|\boldsymbol{v}\|_{\mathbf{H}^s(\mathbf{curl};\Omega_c)}, \qquad \forall \boldsymbol{v} \in \mathbf{H}^s(\mathbf{curl}; \Omega_c).$$

To simplify the notation, for any $\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}; \Omega_c)$, we introduce

$$f_h(\boldsymbol{w}) := \|(\mathcal{R} - \mathcal{R}_h)\mathcal{K}(\boldsymbol{w})\|_{1/2,\Gamma}.$$

LEMMA 6.1. *Let $(\boldsymbol{A}_{\mathrm{C}}, \psi, \boldsymbol{\alpha})$ and $(\boldsymbol{A}_{\mathrm{C}h}^n, \psi_h^n, \boldsymbol{\alpha}_h^n)$ be the solutions of problems (3.10) and (5.4), respectively, the last one with initial condition $\boldsymbol{A}_{\mathrm{C}h}^0 := \mathcal{I}_h^N(\boldsymbol{A}_{\mathrm{C},0})$. Suppose that*

$$\boldsymbol{A}_{\mathrm{C}} \in C^1([0,T]; \mathbf{H}(\mathbf{curl}; \Omega_c)) \cap \mathrm{C}^0([0,T]; \boldsymbol{H}^s(\mathbf{curl}; \Omega_c)),$$

with $s > 1/2$. Furthermore, let

$$\boldsymbol{\rho}^n := \boldsymbol{A}_{\mathrm{C}}(t_n) - \mathcal{I}_h^N \boldsymbol{A}_{\mathrm{C}}(t_n), \quad \boldsymbol{\delta}^n := \mathcal{I}_h^N \boldsymbol{A}_{\mathrm{C}}(t_n) - \boldsymbol{A}_{\mathrm{C}h}^n, \quad \boldsymbol{\tau}^n := \overline{\partial} \boldsymbol{A}_{\mathrm{C}}(t_n) - \partial_t \boldsymbol{A}_{\mathrm{C}}(t_n).$$

Then, there exists $C > 0$ independent of h and Δt such that

$$\begin{aligned} &\max_{1\leq n\leq N} \|\boldsymbol{\delta}^n\|_{\boldsymbol{\sigma}}^2 + \Delta t \sum_{k=1}^N \|\,\mathbf{curl}\,\delta^k\|_{0,\Omega_c}^2 \\ &\leq C \Bigg\{ (\|\boldsymbol{A}_{\mathrm{C},0}\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 + |\boldsymbol{\alpha}_0|^2) \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q}^2 + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2 \right) \\ &\quad + \Delta t \sum_{k=1}^N \Bigg[\|\overline{\partial} \boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 + f_h(\boldsymbol{A}_{\mathrm{C}}(t_k))^2 \\ &\qquad + \|\boldsymbol{A}_{\mathrm{C}}(t_k)\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q}^2 + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2 \right) \Bigg] \Bigg\}. \end{aligned}$$

Proof. It is straightforward to show that for all $\boldsymbol{v} \in \mathbf{H}_h(\mathbf{curl}; \Omega_c)$

$$\begin{aligned} &(\overline{\partial} \boldsymbol{\delta}^k, \boldsymbol{v})_{\boldsymbol{\sigma}} + \langle \mathcal{E}\mathcal{I}_h^N \boldsymbol{A}_{\mathrm{C}}(t_k) - \mathcal{E}\boldsymbol{A}_{\mathrm{C}h}^k, \boldsymbol{v}\rangle + \langle \mathcal{B}_h \boldsymbol{\delta}^k, \boldsymbol{v}\rangle \\ &\quad = -(\overline{\partial} \boldsymbol{\rho}^k, \boldsymbol{v})_{\boldsymbol{\sigma}} + (\boldsymbol{\tau}^k, \boldsymbol{v})_{\boldsymbol{\sigma}} - \left\langle \mathcal{E}\boldsymbol{A}_{\mathrm{C}}(t_k) - \mathcal{E}\mathcal{I}_h^N \boldsymbol{A}_{\mathrm{C}}(t_k), \boldsymbol{v}\right\rangle - \mathcal{B}_h(\boldsymbol{\rho}^k, \boldsymbol{v}) \\ &\qquad + \mathcal{B}_h(\boldsymbol{A}_{\mathrm{C}}(t_k), \boldsymbol{v}) - \mathcal{B}(\boldsymbol{A}_{\mathrm{C}}(t_k), \boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{g}_h(\boldsymbol{v}), \end{aligned} \tag{6.1}$$

as well as the following inequalities

$$(\overline{\partial} \boldsymbol{\delta}^k, \boldsymbol{\delta}^k)_{\boldsymbol{\sigma}} \geq \frac{1}{2\Delta t} \left(\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 - \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2 \right),$$

$$\begin{aligned} &\langle \mathcal{B}\boldsymbol{A}_{\mathrm{C}}(t_k), \boldsymbol{\delta}^k\rangle - \langle \mathcal{B}_h \boldsymbol{A}_{\mathrm{C}}(t_k), \boldsymbol{\delta}^k\rangle \\ &\quad \leq C \|\boldsymbol{A}_{\mathrm{C}}(t_k)\|_{\mathbf{H}(\mathbf{curl};\Omega_c)} \|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_c)} \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\| \right) \\ &\qquad + C\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}\, f_h(\boldsymbol{A}_{\mathrm{C}}(t_k)), \end{aligned}$$

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and

$$\boldsymbol{g}(\boldsymbol{\delta}^k) - \boldsymbol{g}_h(\boldsymbol{\delta}^k) \leq C\left(\|\boldsymbol{A}_{\mathrm{C},0}\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + |\boldsymbol{\alpha}_0|\right)\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} \left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|\right).$$

Taking $\boldsymbol{v} = \boldsymbol{\delta}^k$ in (6.1), using the fact that $\mathcal{B}_h$ is nonnegative and recalling the properties (5.10) and (5.11) of the nonlinear operator $\mathcal{E}$, we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\delta}^k\|^2_{\boldsymbol{\sigma}} - \|\boldsymbol{\delta}^{k-1}\|^2_{\boldsymbol{\sigma}}) + \|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega_\mathrm{c}} \\
&\quad\leq \|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}} + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}} + 3M\|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega_\mathrm{c}}\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} \\
&\qquad + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})}\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + C\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_\mathrm{c}}\, f_h(\boldsymbol{A}_\mathrm{C}(t_k)) \\
&\qquad + C\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})}\left(\|\boldsymbol{A}_\mathrm{C}(t_k)\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + \|\boldsymbol{A}_{\mathrm{C},0}\|_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + |\boldsymbol{\alpha}_0|\right) \\
&\qquad\quad \left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|\right).
\end{aligned}$$

Thus, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\|\boldsymbol{\delta}^k\|^2_{\boldsymbol{\sigma}} - \|\boldsymbol{\delta}^{k-1}\|^2_{\boldsymbol{\sigma}} + \Delta t\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega_\mathrm{c}} \\
&\quad\leq \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|^2_{\boldsymbol{\sigma}} + C\Delta t\Big\{\|\overline{\partial}\boldsymbol{\rho}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\tau}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + f_h(\boldsymbol{A}_\mathrm{C}(t_k))^2 \\
&\qquad + \left(\|\boldsymbol{A}_{\mathrm{C},0}\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + |\boldsymbol{\alpha}_0|^2 + \|\boldsymbol{A}_\mathrm{C}(t_k)\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})}\right) \\
&\qquad\quad \left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\right)\Big\}.
\end{aligned} \tag{6.2}$$

Then, by summing over k, recalling that $\boldsymbol{\delta}^0 = \mathbf{0}$ and using the discrete Gronwall Lemma (see [24, Lemma 1.4.2]), it follows that

$$\begin{aligned}
\|\boldsymbol{\delta}^n\|^2_{\boldsymbol{\sigma}} \leq\; & C\Big\{\Delta t\sum_{k=1}^n\Big[\|\overline{\partial}\boldsymbol{\rho}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\tau}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + f_h(\boldsymbol{A}_\mathrm{C}(t_k))^2 \\
&+ \left(\|\boldsymbol{A}_{\mathrm{C},0}\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + |\boldsymbol{\alpha}_0|^2 + \|\boldsymbol{A}_\mathrm{C}(t_k)\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})}\right) \\
&\quad \left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\right)\Big]\Big\}.
\end{aligned}$$

Therefore, using the preceding inequality in (6.2) and setting

$$\begin{aligned}
\theta^k :=\; & \|\overline{\partial}\boldsymbol{\rho}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\tau}^k\|^2_{\boldsymbol{\sigma}} + \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + f_h(\boldsymbol{A}_\mathrm{C}(t_k))^2 \\
&+ \|\boldsymbol{A}_\mathrm{C}(t_k)\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})}\left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\boldsymbol{\delta}^k\|^2_{\boldsymbol{\sigma}} - \|\boldsymbol{\delta}^{k-1}\|^2_{\boldsymbol{\sigma}}\right) + \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega_\mathrm{c}} \\
&\leq \frac{C}{4T}\Big\{\left(\|\boldsymbol{A}_{\mathrm{C},0}\|^2_{\mathbf{H}(\mathbf{curl};\Omega_\mathrm{c})} + |\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\leq i\leq L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,Q} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\right) \\
&\qquad + \Delta t\sum_{j=1}^k\theta^j\Big\}.
\end{aligned}$$

Then, by summing over n and recalling again that $\boldsymbol{\delta}^0 = \mathbf{0}$, we obtain for all $n = 1, \ldots, N$,

$$\begin{aligned}
&\|\boldsymbol{\delta}^n\|_{\boldsymbol{\sigma}}^2 + \Delta t \sum_{k=1}^{n} \|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_c}^2 \\
&\quad \le C \left\{ \left(\|\boldsymbol{A}_{\mathrm{C},0}\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 + |\boldsymbol{\alpha}_0|^2 \right) \left(\max_{1\le i\le L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q}^2 + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2 \right) \right. \\
&\qquad + \Delta t \sum_{k=1}^{n} \left[\|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^{\boldsymbol{k}}\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega_c)} + f_h(\boldsymbol{A}_{\mathrm{C}}(t_k))^2 \right. \\
&\qquad \left.\left. + \left(\|\boldsymbol{A}_{\mathrm{C}}(t_k)\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 \right) \left(\max_{1\le i\le L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,Q}^2 + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2 \right) \right] \right\}.
\end{aligned}$$

The conclusion of the lemma follows by taking maximum over $1 \le n \le N$ on both sides of the preceding inequality. ∎

LEMMA 6.2. *Let $(\boldsymbol{A}_{\mathrm{C}}, \psi, \boldsymbol{\alpha})$ be the solution of (3.10). If we assume that $\boldsymbol{A}_{\mathrm{C}} \in H^1(0,T;\boldsymbol{H}^s(\mathbf{curl};\Omega_c))$, $1/2 < s < s_Q$, then $\psi \in H^1(0,T;H^{s+1/2}(\Gamma))$ and the following estimate holds*

$$\inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\psi(t) - \eta\|_{1/2,\Gamma} \le Ch^s \|\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}}(t)\|_{s,\Omega_c}.$$

Proof. The result is obtained using Remark 4.2 and following the lines of [11, Lemma 5.1]. ∎

THEOREM 6.3. *Let $(\boldsymbol{A}_{\mathrm{C}}, \psi, \boldsymbol{\alpha})$ and $(\boldsymbol{A}_{\mathrm{C}h}^n, \psi_h^n, \boldsymbol{\alpha}_h^n)$, $n = 1, \ldots, N$, be the solutions of (3.10) and (5.4) respectively. Let us assume that $\boldsymbol{A}_{\mathrm{C}} \in H^1(0,T;\boldsymbol{H}^s(\mathbf{curl};\Omega_c)) \cap H^2(0,T;\boldsymbol{H}(\mathbf{curl};\Omega_c))$, with $s \in (1/2, s_Q)$. Then, there exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, the following estimate holds*

$$\begin{aligned}
&\max_{1\le n\le N} \|\boldsymbol{A}_{\mathrm{C}}(t_n) - \boldsymbol{A}_{\mathrm{C}h}^n\|_{\boldsymbol{\sigma}}^2 + \Delta t \sum_{n=1}^{N} \|\,\mathbf{curl}(\boldsymbol{A}_{\mathrm{C}} - \boldsymbol{A}_{\mathrm{C}h}^n)\|_{0,\Omega_c}^2 \\
&\quad \le Ch^{2s} \left\{ \max_{1\le n\le N} \|\boldsymbol{A}_{\mathrm{C}}(t_n)\|_{\boldsymbol{H}^s(\mathbf{curl};\Omega_c)}^2 + \left(\|\boldsymbol{A}_{\mathrm{C},0}\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 + |\boldsymbol{\alpha}_0|^2 \right.\right. \\
&\qquad \left. + \max_{1\le n\le N} \|\boldsymbol{A}_{\mathrm{C}}(t_n)\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2 \right) \left(\max_{1\le k\le L} \|\widetilde{\nabla} z_k\|_{s,Q}^2 + \|z_k\|_{s+1/2,\Gamma}^2 \right) \\
&\qquad \left. + \int_0^T \|\partial_t \boldsymbol{A}_{\mathrm{C}}(t)\|_{\boldsymbol{H}^s(\mathbf{curl};\Omega_c)}^2 dt \right\} + (\Delta t)^2 \int_0^T \|\partial_{tt} \boldsymbol{A}_{\mathrm{C}}(t)\|_{\mathbf{H}(\mathbf{curl};\Omega_c)}^2\, dt \\
&\quad \le C\left[h^{2s} + (\Delta t)^2 \right].
\end{aligned}$$

Proof. First we need to notice that from (5.7), using the fact that $\psi(t) = -\mathcal{R}(\mathcal{K}(\boldsymbol{A}_{\mathrm{C}}(t)))$ (see (3.17)) and Lemma 6.2, we can see

$$f_h(\boldsymbol{A}_{\mathrm{C}}(t_n)) \le Ch^s \|\,\mathbf{curl}\,\boldsymbol{A}_{\mathrm{C}}(t_n)\|_{s,\Omega_c}. \tag{6.3}$$

Consequently, the result follows from Lemma 6.1, Remark 5.1, and inequality (5.3), noticing that

$$\boldsymbol{A}_{\mathrm{C}}(t_n) - \boldsymbol{A}_{\mathrm{C}h}^n = \boldsymbol{\rho}^n + \boldsymbol{\delta}^n,$$

and proceeding as in [11, Theorem 6.1]. ∎

REMARK 6.4. It is a simple matter to deduce that Theorem 6.3 implies quasi-optimal convergence of the approximation error of the variable $\boldsymbol{A}_{\mathrm{C}}$ in the norm given by the discrete integral on the space $\boldsymbol{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}))$. More precisely,

$$\Delta t \sum_{k=1}^{N} \|\boldsymbol{A}_{\mathrm{C}}(t_n) - \boldsymbol{A}_{\mathrm{C}h}^n\|^2_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})} \leq C\left[h^{2s} + (\Delta t)^2\right].$$

For the other two variables ψ and $\boldsymbol{\alpha}$, quasi-optimal convergence of the approximation error is obtained too. In fact, by recalling

$$\psi(t_n) = -\mathcal{R}(\mathcal{K}(\boldsymbol{A}_{\mathrm{C}}(t_n)), \qquad \psi_h^n = -\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}_{\mathrm{C}h}^n)),$$

and using (6.3), the uniform bound of $\mathcal{R}_h$ with respect to h and Theorem 6.3, we obtain

$$\Delta t \sum_{n=1}^{N} \|\psi(t_n) - \psi_h^n\|^2_{1/2,\Gamma} \leq C\left[h^{2s} + (\Delta t)^2\right].$$

Finally, since

$$\boldsymbol{\alpha}(t_n) = \boldsymbol{\alpha}_0 + \boldsymbol{N}^{-1}(\mathcal{J}(\boldsymbol{A}_{\mathrm{C}}(t_n) - \boldsymbol{A}_{\mathrm{C},0})), \qquad \boldsymbol{\alpha}_h^n = \boldsymbol{\alpha}_0 + \boldsymbol{N}_h^{-1}(\mathcal{J}_h(\boldsymbol{A}_{\mathrm{C}h}^n - \boldsymbol{A}_{\mathrm{C},0})),$$

using Theorem 6.3 together with [11, Lemma 5.1], it follows that

$$\max_{1\leq n\leq N} |\alpha(t_n) - \alpha_h^n|^2 \leq C\left[h^{2s} + (\Delta t)^2\right].$$

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