



The  $H$ -adjoint of a real matrix  $A$  is the unique  $n \times n$  matrix, written  $A^\#$ , which satisfies

$$[Ax, y] = [x, A^\#y]$$

for all  $x, y \in \mathbb{C}^n$ . In particular, if  $A = A^\#$ , or equivalently, if  $A = HA^T H$ ,  $A$  is referred to as  $H$ -symmetric or pseudo-symmetric. Thus, the matrix  $J_n$  in (1.1) is pseudo-symmetric. If  $A^\#A = I_n$ ,  $A$  is called  $H$ -orthogonal or pseudo-orthogonal. Let  $\beta = (\beta_1, \dots, \beta_{n-1})$ , and let  $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1})$  be the so called sign vector. We denote the set of matrices of the form (1.1) by  $\mathcal{J}(n, \epsilon, \beta)$ . If  $\epsilon$  is a vector with all entries equal to one, then  $J_n$  reduces to a Jacobi matrix.

Pseudo-symmetric matrices usually appear in non-Hermitian quantum mechanics [12], where  $H$  is the sign operator (that is,  $H^2$  is the identity). A sign change in one of the components in  $\epsilon$  may lead to strong perturbations in the spectral properties of the matrices in  $\mathcal{J}(n, \epsilon, \beta)$ . The study of pseudo-Jacobi matrices extends the well-known theory of Jacobi matrices, which arise in a variety of applications in different fields such as classical moment problems [1], vibrating systems [15], etc. The discretization and truncation of the Schrödinger equation in non-Hermitian quantum mechanics leads to pseudo-Jacobi matrices [4]. Research on inverse eigenvalue problems for Jacobi matrices originated several fruitful results; see [2, 7, 9, 10, 11, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27, 28] and the references therein. In contrast, the theory concerning the pseudo-Jacobi case constitutes a small part of the literature. The problems in this area deserve attention in order to extend the classical theory of the Jacobi case. At present, some developments focusing on pseudo-Jacobi inverse eigenvalue problem, abbreviated by the acronym PJIEP, have been obtained; see [3, 4, 5, 6, 18, 21, 29, 30]. This paper is in the continuation of this research field. Our work proceeds along the conceptual lines of the standard Jacobi case, but some remarkable differences occur. First, as the matrix  $H$  that fixes the inner product is indefinite, there may appear Lanczos vectors with zero norm. Second, the mathematical manipulations are more involved.

In what follows, the principal submatrix of  $J_n$  in the lines  $(\omega, \dots, \nu)$ ,  $1 \leq \omega < \nu \leq n$ , will be denoted by

$$(1.2) \quad J_{\omega, \nu} = \begin{bmatrix} \alpha_\omega & \epsilon_\omega \beta_\omega & & & & \\ \beta_\omega & \alpha_{\omega+1} & \epsilon_{\omega+1} \beta_{\omega+1} & & & \\ & \beta_{\omega+1} & \ddots & \ddots & & \\ & & \ddots & \ddots & \epsilon_{\nu-1} \beta_{\nu-1} & \\ & & & \beta_{\nu-1} & \alpha_\nu & \end{bmatrix},$$

and  $J_{1, \nu}$  simply by  $J_\nu$ . We will consider the following inverse problem:

**PJIEP.** Let a sign vector  $\epsilon$  and the sets  $\lambda = \{\lambda_i\}_{i=1}^n \subset \mathbb{C}$ ,  $\mu_1 = \{\mu_i\}_{i=1}^r \subset \mathbb{R}$ , and  $\mu_2 = \{\mu_i\}_{i=r+1}^{n-1} \subset \mathbb{R}$ ,  $1 \leq r \leq n-2$  be given, where  $\lambda$  is closed under complex conjugation and the elements of both  $\mu_1$  and  $\mu_2$  are pairwise distinct. Construct a pseudo-Jacobi matrix  $J_n \in \mathcal{J}(n, \epsilon, \beta)$  such that  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  are, respectively, the spectra of the matrices  $J_n$ ,  $J_r$ , and  $J_{r+2, n}$ .

Before solving this computability problem, we determine a necessary and sufficient condition under which this problem has a solution. In [18], Mirzaei investigated the particular case of the PJIEP in which the elements of  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  are real pairwise distinct numbers and  $\mu_1 \cap \mu_2 = \emptyset$ . The special case in which  $H = I_r \oplus -I_1 \oplus I_{n-r-1}$  was solved by Xu, Bebbiano, and Chen [29].

This article is organized as follows. In Section 2, we present a modified unsymmetric Lanczos algorithm to construct a matrix  $\widehat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  whose eigenvalues are real and pairwise distinct. In Section 3, a necessary and sufficient condition under which the PJIEP has

a solution is stated in the cases when  $\mu_1 \cap \mu_2 = \emptyset$  and  $\mu_1 \cap \mu_2 \neq \emptyset$ . A numerical algorithm to solve the PJIEP is proposed in Section 4. In Section 5, numerical examples illustrate our approach to the PJIEP and test the efficiency and feasibility of the reconstruction algorithm. In Section 6, some conclusions are drawn. The theoretical results stated in Sections 2 and 3 are proved in the Appendices A and B.

**2. Modified unsymmetric Lanczos algorithms.** Throughout this paper, let  $\chi_{\omega,v}(\lambda) = \det(\lambda I_{v-\omega+1} - J_{\omega,v})$  and  $\sigma(J_{\omega,v})$  be, respectively, the characteristic polynomial and the spectrum of the pseudo-Jacobi matrix  $J_{\omega,v}$  in (1.2). For simplicity, we denote  $\chi_{1,n}(\lambda)$  as  $\chi_n(\lambda)$ . Let  $\widehat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  have real and distinct eigenvalues  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$  associated with the real eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , respectively, and let  $\Lambda = \text{diag}(\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n)$ . It can be easily shown that the eigenvectors  $\mathbf{v}_i, i = 1, 2, \dots, n$ , may be chosen so that they constitute an  $H$ -orthonormal basis of  $\mathbb{R}^n$ , i.e.,  $[\mathbf{v}_i, \mathbf{v}_j] = \delta_{ij} \delta_i$ , where  $\delta_{ij}$  denotes the Kronecker delta. Thus  $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] \in \mathbb{R}^{n \times n}$  is  $H$ -orthogonal, that is,  $V^\# V = I_n$  with  $V^\# = H V^T H$ .

Before presenting our modified Lanczos formalism, we state an useful extension of the Thompson-McEntegert-Paige theorem [19].

**THEOREM 2.1.** *The first and last entries of the  $H$ -orthonormal eigenvectors of a pseudo-Jacobi matrix with distinct real eigenvalues are both nonzero.*

The unsymmetric Lanczos algorithm in [8] can be used to reconstruct a pseudo-Jacobi matrix  $\widehat{J}_n$  from its distinct real eigenvalues  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$ , the first, or the last, entries of the corresponding  $H$ -orthonormal eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , and from its *pseudo-norms*  $\delta_1, \delta_2, \dots, \delta_n$ , where  $\delta_j := [\mathbf{v}_j, \mathbf{v}_j]$ . It should be noticed that  $H$  is indefinite and so the induced inner product lacks positivity. Therefore, it must be analyzed whether the  $H$ -norms of the computed Lanczos vectors do not vanish. By using the unsymmetric Lanczos algorithm for the matrices  $\text{diag}(\mu_1)$  and  $\text{diag}(\mu_2)$  with the appropriate starting vectors of order  $r$  and  $n - r - 1$ , respectively, the pseudo-Jacobi matrices  $J_r$  and  $J_{r+2,n}$  are obtained. The diagonal entry  $\alpha_{r+1}$  of  $J_n$  results from the trace condition and the neighboring off-diagonals come from the Lanczos procedures.

Firstly, we present the *backward modified unsymmetric Lanczos algorithm* to recover the matrix  $\widehat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  initialized from the last entries of its  $H$ -orthonormal eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

**THEOREM 2.2.** *Let  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$  be the real pairwise distinct eigenvalues of  $\widehat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$ . Given the last entries  $\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n}$  of the corresponding  $H$ -orthonormal eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , then  $\widehat{J}_n$  can be constructed by the backward modified unsymmetric Lanczos algorithm in Algorithm 1.*

Next, we give a *forward modified unsymmetric Lanczos algorithm*, initialized with the entries in the first row of the  $H$ -orthogonal matrix  $V$ . The proof of this result is similar to that of Theorem 2.2, and so it is omitted.

**THEOREM 2.3.** *Let  $\widehat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  have distinct real eigenvalues  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$ , and let  $\mathbf{v}_{11}, \mathbf{v}_{12}, \dots, \mathbf{v}_{1,n}$  be the first entries of the corresponding  $H$ -orthonormal eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . Then,  $\widehat{J}_n$  can be constructed by the forward modified unsymmetric Lanczos algorithm in Algorithm 2*

**REMARK 2.4.** By executing Algorithm 2 from Theorem 2.3, we find the matrices  $Y = [Y_1, Y_2, \dots, Y_n] = V^T$  and  $Z = [Z_1, Z_2, \dots, Z_n] = V^\#$  from the initial vector  $Y_1 = (\mathbf{v}_{11}, \mathbf{v}_{12}, \dots, \mathbf{v}_{1,n})^T$ .

**3. The construction of the pseudo-Jacobi matrix  $J_n$  in the PJIEP.** Let be given a sign vector  $\epsilon$  and the sets  $\lambda, \mu_1$ , and  $\mu_2$  as in the PJIEP. We construct a solution  $J_n \in \mathcal{J}(n, \epsilon, \beta)$

**Algorithm 1** Backward modified unsymmetric Lanczos algorithm.

- 1: **Initialize** two  $n$ -dimensional column vectors  $\mathbf{Y}_{n+1} = \mathbf{0}$  and  $\widehat{\mathbf{s}}_{n+1} = (\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n})^T$ .
- 2: **Set**  $\delta_{n+1} = 1, \widehat{\mathbf{r}}_{n+1} = H\widehat{\mathbf{s}}_{n+1}\delta_{n+1}, k = n + 1$ .
- 3: **while** ( $[\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_k]\delta_{k-1} > 0$ ) **do**
- 4:    $\beta_{k-1} = \sqrt{[\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_k]\delta_{k-1}}$
- 5:    $\gamma_{k-1} = \frac{\widehat{\mathbf{s}}_k^T \widehat{\mathbf{r}}_k}{\beta_{k-1}}$
- 6:    $\mathbf{Y}_{k-1} = \frac{\widehat{\mathbf{s}}_k}{\beta_{k-1}}$
- 7:    $\mathbf{Z}_{k-1} = H\mathbf{Y}_{k-1}\delta_{k-1}$
- 8:    $k = k - 1$
- 9:    $\alpha_k = \mathbf{Z}_k^T \Lambda \mathbf{Y}_k$
- 10:    $\widehat{\mathbf{s}}_k = (\Lambda - \alpha_k I_n)\mathbf{Y}_k - \gamma_k \mathbf{Y}_{k+1}$
- 11:    $\widehat{\mathbf{r}}_k = H\widehat{\mathbf{s}}_k\delta_k$
- 12: **end while**

**Algorithm 2** Forward modified unsymmetric Lanczos algorithm.

- 1: **Initialize** two  $n$ -dimensional column vectors  $\mathbf{Z}_0 = \mathbf{0}$  and  $\mathbf{r}_0 = H(\mathbf{v}_{11}, \mathbf{v}_{12}, \dots, \mathbf{v}_{1,n})^T \delta_1$ .
- 2: **Set**  $\delta_0 = 1, \mathbf{s}_0 = H\mathbf{r}_0\delta_0, k = 0$ .
- 3: **while** ( $[\mathbf{r}_k, \mathbf{r}_k]\delta_{k+1} > 0$ ) **do**
- 4:    $\beta_k = \sqrt{[\mathbf{r}_k, \mathbf{r}_k]\delta_{k+1}}$
- 5:    $\gamma_k = \frac{\mathbf{s}_k^T \mathbf{r}_k}{\beta_k}$
- 6:    $\mathbf{Z}_{k+1} = \frac{\mathbf{r}_k}{\beta_k}$
- 7:    $\mathbf{Y}_{k+1} = H\mathbf{Z}_{k+1}\delta_{k+1}$
- 8:    $k = k + 1$
- 9:    $\alpha_k = \mathbf{Y}_k^T \Lambda \mathbf{Z}_k$
- 10:    $\mathbf{r}_k = (\Lambda - \alpha_k I_n)\mathbf{Z}_k - \gamma_{k-1}\mathbf{Z}_{k-1}$
- 11:    $\mathbf{s}_k = H\mathbf{r}_k\delta_k$
- 12: **end while**

for the PJIEP of the form

$$(3.1) \quad J_n = \begin{bmatrix} J_r & \epsilon_r \beta_r \mathbf{e}_r & \mathbf{0} \\ \beta_r \mathbf{e}_r^T & \alpha_{r+1} & \epsilon_{r+1} \beta_{r+1} \boldsymbol{\omega}_1^T \\ \mathbf{0} & \beta_{r+1} \boldsymbol{\omega}_1 & J_{r+2,n} \end{bmatrix},$$

where  $\mathbf{e}_r = (0, \dots, 0, 1)^T \in \mathbb{R}^r$ ,  $\boldsymbol{\omega}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^{n-r-1}$ ,  $\sigma(J_n) = \boldsymbol{\lambda}$ ,  $\sigma(J_r) = \boldsymbol{\mu}_1$ , and  $\sigma(J_{r+2,n}) = \boldsymbol{\mu}_2$ . Hence, the pseudo-Jacobi matrices  $J_r$  and  $J_{r+2,n}$  are, respectively,  $H_1$ -symmetric and  $H_2$ -symmetric for

$$H_1 = \text{diag}(1, \epsilon_1, \epsilon_1 \epsilon_2, \dots, \epsilon_1 \cdots \epsilon_{r-1}) = \text{diag}(\delta_1, \delta_2, \dots, \delta_r) \quad \text{and}$$

$$H_2 = \text{diag}(1, \epsilon_{r+2}, \epsilon_{r+2} \epsilon_{r+3}, \dots, \epsilon_{r+2} \cdots \epsilon_{n-1}) = \text{diag}\left(\frac{\delta_{r+2}}{\delta_{r+2}}, \frac{\delta_{r+3}}{\delta_{r+2}}, \dots, \frac{\delta_n}{\delta_{r+2}}\right).$$

Let  $\mathbf{u}_i^{(1)} = (u_{1,i}^{(1)}, u_{2,i}^{(1)}, \dots, u_{r,i}^{(1)})^T$  be the  $H_1$ -orthonormal eigenvector of  $J_r$  associated with its eigenvalue  $\mu_i$ , for  $i = 1, 2, \dots, r$ , and let  $\mathbf{u}_i^{(2)} = (u_{1,i}^{(2)}, u_{2,i}^{(2)}, \dots, u_{n-r-1,i}^{(2)})^T$  be the  $H_2$ -orthonormal eigenvector of  $J_{r+2,n}$  corresponding to its eigenvalue  $\mu_{r+i}$ , for  $i = 1, 2, \dots, n - r - 1$ . The matrices

$$U_1 = [\mathbf{u}_1^{(1)}, \mathbf{u}_2^{(1)}, \dots, \mathbf{u}_r^{(1)}] \quad \text{and} \\ U_2 = [\mathbf{u}_1^{(2)}, \mathbf{u}_2^{(2)}, \dots, \mathbf{u}_{n-r-1}^{(2)}]$$

are, respectively,  $H_1$ -orthogonal and  $H_2$ -orthogonal and satisfy

$$(3.2) \quad U_1^\# J_r U_1 = \Lambda_1 = \text{diag}(\boldsymbol{\mu}_1) \quad \text{and} \quad U_2^\# J_{r+2,n} U_2 = \Lambda_2 = \text{diag}(\boldsymbol{\mu}_2).$$

In Theorems 3.1 and 3.2 below, we construct the pseudo-Jacobi matrix  $J_n$  of the form (3.1) in the following two cases:

$$(1) \quad \boldsymbol{\mu}_1 \cap \boldsymbol{\mu}_2 = \emptyset \quad \text{and} \quad (2) \quad \boldsymbol{\mu}_1 \cap \boldsymbol{\mu}_2 \neq \emptyset.$$

Although the notation is inevitably heavy, our approach is conceptually simple and consists of the following three main steps.

(i) We recover  $J_r$  using Algorithm 1 with

$$H = H_1, \quad \Lambda = \Lambda_1,$$

and the initializing vector

$$\widehat{\mathbf{s}}_{r+1} = (u_{r,1}^{(1)}, u_{r,2}^{(1)}, \dots, u_{r,r}^{(1)})^T.$$

Algorithm 1 is used, replacing in steps 1, 2 and 10 all the subscripts  $n$  by  $r$ .

(ii) We recover  $J_{r+2,n}$  using Algorithm 2 with

$$H = H_2, \quad \Lambda = \Lambda_2,$$

and the initializing vector

$$\mathbf{r}_{r+1} = H_2(u_{11}^{(2)}, u_{12}^{(2)}, \dots, u_{1,n-r-1}^{(2)})^T \cdot \delta_{r+2}/\delta_{r+2}.$$

In this forward modified unsymmetric Lanczos algorithm, we replace all the subscripts  $n$  and  $k$  by  $n - r - 1$  and  $r + k + 1$ , respectively. Thus, all the  $\delta_{r+k+1}$  have a multiplicative factor  $1/\delta_{r+2}$ . In step 1, we replace  $\mathbf{r}_0 = H(\mathbf{v}_{11}, \mathbf{v}_{12}, \dots, \mathbf{v}_{1,n})^T \delta_1$  by  $\mathbf{r}_{r+1}$ . In step 2, we also replace all the subscripts 0 by  $r + 1$ .

(iii) In the backward modified unsymmetric Lanczos process,  $Y = [Y_1, Y_2, \dots, Y_r] = U_1^T$ . Similarly,  $Z = [Z_1, Z_2, \dots, Z_{n-r-1}] = U_2^\#$  in the forward modified unsymmetric Lanczos process. In both modified unsymmetric Lanczos processes we have  $\gamma_i = \epsilon_i \beta_i$ .

**THEOREM 3.1.** *Let be given a sign vector  $\epsilon$  and the sets  $\boldsymbol{\lambda}$ ,  $\boldsymbol{\mu}_1$ , and  $\boldsymbol{\mu}_2$  as in the PJIEP. Set  $x_j = -\prod_{i=1}^n (\lambda_i - \mu_j) \prod_{i=1, i \neq j}^{n-1} (\mu_i - \mu_j)^{-1}$ ,  $j = 1, 2, \dots, n - 1$ . If  $\boldsymbol{\mu}_1 \cap \boldsymbol{\mu}_2 = \emptyset$ , then the PJIEP has a solution if and only if the following conditions are satisfied, and in this case the solution is unique:*

$$(1) \quad \begin{cases} \delta_{r+1} \delta_j x_j > 0 & \text{if } j = 1, 2, \dots, r, \text{ and} \\ \delta_{r+1} \delta_{j+1} x_j > 0 & \text{if } j = r + 1, r + 2, \dots, n - 1, \end{cases} \\ (2) \quad \epsilon_r \sum_{j=1}^r x_j > 0 \text{ and } \epsilon_{r+1} \sum_{j=r+1}^{n-1} x_j > 0,$$

- (3)  $[\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_k] \delta_{k-1} > 0$  for  $k = r, r-1, \dots, 2$ , where  
 $\widehat{\mathbf{s}}_k = (\Lambda_1 - \alpha_k I_r) \mathbf{Y}_k - \gamma_k \mathbf{Y}_{k+1}$ ,  $\mathbf{Y}_k = (u_{k,1}^{(1)}, u_{k,2}^{(1)}, \dots, u_{k,r}^{(1)})^\top$ , and  $\gamma_k = \epsilon_k \beta_k$ ,
- (4)  $[\mathbf{r}_{r+k+1}, \mathbf{r}_{r+k+1}] \frac{\delta_{r+k+2}}{\delta_{r+2}} > 0$  for  $k = 1, 2, \dots, n-r-2$ , where  
 $\mathbf{r}_{r+k+1} = (\Lambda_2 - \alpha_{r+k+1} I_{n-r-1}) \mathbf{Z}_k - \gamma_{r+k} \mathbf{Z}_{k-1}$ ,  
 $\mathbf{Z}_k = H_2(u_{k,1}^{(2)}, u_{k,2}^{(2)}, \dots, u_{k,n-r-1}^{(2)})^\top \cdot \frac{\delta_{r+k+1}}{\delta_{r+2}}$ , and  
 $\gamma_{r+k} = \epsilon_{r+k} \beta_{r+k}$ .

**THEOREM 3.2.** *Let be given a sign vector  $\epsilon$  and the sets  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  as in the PJIEP. Let  $\mu_1 \cap \mu_2 = \{\mu_i\}_{i=1}^k$  and  $\mu_{r+i} = \mu_i$  for any  $i = 1, 2, \dots, k$  with  $k \leq \min\{r, n-r-1\}$ . Assume  $\lambda_i = \mu_i$ ,  $i = 1, 2, \dots, k$ , and set*

$$x_j = - \prod_{i=k+1}^n (\lambda_i - \mu_j) \prod_{i=k+1, i \neq j}^{n-1} (\mu_i - \mu_j)^{-1}, \quad j = k+1, \dots, n-1.$$

Then, the PJIEP has a solution if and only if

- (1) there exist real numbers  $\theta_j \notin \{0, 1\}$  such that

$$\delta_{r+1} \delta_j \theta_j x_{r+j} > 0 \quad \text{and} \quad \delta_{r+1} \delta_{r+j+1} (1 - \theta_j) x_{r+j} > 0, \quad \text{for } j = 1, 2, \dots, k,$$

$$(2) \begin{cases} \delta_{r+1} \delta_j x_j > 0 & \text{for } j = k+1, \dots, r, \text{ and} \\ \delta_{r+1} \delta_{j+1} x_j > 0, & \text{for } j = r+k+1, \dots, n-1, \end{cases}$$

$$(3) \epsilon_r (\sum_{j=1}^k \theta_j x_{r+j} + \sum_{j=k+1}^r x_j) > 0 \text{ and} \\ \epsilon_{r+1} (\sum_{j=1}^k (1 - \theta_j) x_{r+j} + \sum_{j=r+k+1}^{n-1} x_j) > 0,$$

- (4) Conditions (3) and (4) in Theorem 3.1 hold.

Furthermore, there are infinite many solutions.

**4. Algorithm.** Based on Theorems 3.1 and 3.2, we present the following algorithm to construct a pseudo-Jacobi matrix  $J_n$  in  $\mathcal{J}(n, \epsilon, \beta)$  that solves the PJIEP. The  $\theta_i$ ,  $i = 1, 2, \dots, k$ , in Theorem 3.2 are randomly selected in  $\mathbb{R} - \{0, 1\}$  most of the times.

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**Algorithm 3** A solution of the PJIEP.

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**Input:**  $\epsilon$ ,  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  as in the PJIEP

**Output:**  $J_n$

1: **if**  $\mu_1 \cap \mu_2 = \emptyset$ , **then**

2:   Form

$$x_j = - \frac{\prod_{i=1}^n (\lambda_i - \mu_j)}{\prod_{i=1, i \neq j}^{n-1} (\mu_i - \mu_j)}, \quad j = 1, 2, \dots, n-1.$$

3: **if** the conditions (1)–(2) in Theorem 3.1 hold, **then**

4:   Calculate

$$\beta_r := \left( \epsilon_r \sum_{j=1}^r x_j \right)^{\frac{1}{2}}, \quad \beta_{r+1} := \left( \epsilon_{r+1} \sum_{j=r+1}^{n-1} x_j \right)^{\frac{1}{2}}.$$

5:   Compute

$$\begin{cases} u_{r,j}^{(1)} := \frac{\sqrt{\delta_{r+1} \delta_j x_j}}{\beta_r}, & j = 1, 2, \dots, r, \\ u_{1,j-r}^{(2)} := \frac{\sqrt{\delta_{r+1} \delta_{j+1} x_j}}{\beta_{r+1}}, & j = r+1, r+2, \dots, n-1. \end{cases}$$

6: **else**

- 7: **stop**  
 8: **end if**  
 9: **else**  
 10: Rearrange the elements in  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  such that  $\mu_1 \cap \mu_2 = \{\mu_i\}_{i=1}^k$ ,  $\mu_{r+i} = \mu_i$ , and  $\lambda_i = \mu_i$  for  $i = 1, 2, \dots, k$ .  
 11: Form
 
$$x_j = -\frac{\prod_{i=k+1}^n (\lambda_i - \mu_j)}{\prod_{i=k+1, i \neq j}^{n-1} (\mu_i - \mu_j)}, \quad j = k+1, \dots, n-1,$$
 12: Select  $\theta_i$  in  $\mathbb{R} - \{0, 1\}$  for all  $i = 1, 2, \dots, k$ .  
 13: **if** conditions (1)–(3) in Theorem 3.2 hold, **then**  
 14: Calculate
 
$$\beta_r := \left( \epsilon_r \left( \sum_{j=1}^k \theta_j x_{r+j} + \sum_{j=k+1}^r x_j \right) \right)^{\frac{1}{2}},$$

$$\beta_{r+1} := \left( \epsilon_{r+1} \left( \sum_{j=1}^k (1 - \theta_j) x_{r+j} + \sum_{j=r+k+1}^{n-1} x_j \right) \right)^{\frac{1}{2}}.$$
 15: Compute
 
$$u_{r,j}^{(1)} := \begin{cases} \frac{1}{\beta_r} \sqrt{\delta_{r+1} \delta_j \theta_j x_{r+j}}, & j = 1, 2, \dots, k, \\ \frac{1}{\beta_r} \sqrt{\delta_{r+1} \delta_j x_j}, & j = k+1, \dots, r, \end{cases}$$
 and
 
$$u_{1,j}^{(2)} := \begin{cases} \frac{1}{\beta_{r+1}} \sqrt{\delta_{r+1} \delta_{r+j+1} (1 - \theta_j) x_{r+j}}, & j = 1, 2, \dots, k, \\ \frac{1}{\beta_{r+1}} \sqrt{\delta_{r+1} \delta_{r+j+1} x_{r+j}}, & j = k+1, \dots, n-r-1. \end{cases}$$
 16: **else**  
 17: Go to step 12.  
 18: **end if**  
 19: **end if**  
 20: **if** the conditions (3)–(4) in Theorem 3.1 hold, **then**  
 21: Construct the pseudo-Jacobi matrix  $J_r$  from  $H_1$ ,  $\mu_1$ , and  $g_1 = (u_{r,1}^{(1)}, u_{r,2}^{(1)}, \dots, u_{r,r}^{(1)})^T$  by Algorithm 1.  
 22: Compute the pseudo-Jacobi matrix  $J_{r+2,n}$  from  $H_2$ ,  $\mu_2$ , and  $g_2 = (u_{11}^{(2)}, u_{12}^{(2)}, \dots, u_{1,n-r-1}^{(2)})^T$  by Algorithm 2.  
 23: **else**  
 24: **stop**  
 25: **end if**  
 26: Compute  $\alpha_{r+1} = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \mu_i$ .  
 27: **return**  $J_n$

Next, we discuss the computational complexity of the above algorithm considering just the case when  $\mu_1 \cap \mu_2 = \emptyset$ .

Step 1 requires  $\mathcal{O}(r(n-r-1))$  operations and step 2  $\mathcal{O}((n-1)(4n-5))$ . Step 3 requires  $\mathcal{O}(2n)$  operations, step 4 does not require operations, and the cost of step 5 is  $\mathcal{O}(n-1)$ . As the computational complexities of Algorithms 1 and 2 are both at most  $\mathcal{O}(15n^2)$ , the cost of steps 20–22 is at most  $\mathcal{O}(15(r^2 + (n-r-1)^2))$ . Finally, step 26 requires  $\mathcal{O}(2n-2)$  operations. Therefore, the total complexity of the algorithm when  $\mu_1 \cap \mu_2 = \emptyset$  is approximately  $\mathcal{O}(19n^2 + 29r^2 - 29nr - 34n + 29r + 17)$ .

**5. Numerical experiments.** In this section, we present some numerical examples illustrating that Algorithm 3 is theoretically effective to solve the PJIEP. All the tests are performed by using MATLAB R2016a. Because all the pseudo-Jacobi matrices  $J_n$  in  $\mathcal{J}(n, \epsilon, \beta)$  rely on the sign vector  $\epsilon$ , the main diagonal entries  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , and the subdiagonal entries  $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$ , they can be generated by the following MATLAB code:

```
diag(alpha)+diag(beta,-1)+diag(beta.*epsilon,1).
```

Let  $i = \sqrt{-1}$  be the imaginary unit.

EXAMPLE 5.1. Consider an extended harmonic oscillator [12]

$$H_\beta = \frac{\beta}{2}(p^2 + x^2) + i\sqrt{2}p, \quad \beta > 0,$$

which acts on  $L^2$  (the Hilbert space of square integrable differentiable functions of the real variable  $x$ ). The operator  $p: L^2 \rightarrow L^2$  is the differential operator  $f(x) \rightarrow -i\frac{df}{dx}$  and  $x: L^2 \rightarrow L^2$  is the multiplicative operator  $f(x) \rightarrow xf(x)$ . With respect to the orthonormal basis constituted by the eigenvectors of the harmonic oscillator,  $\phi_n(x) = K_n(x - \frac{d}{dx})^n \exp(-\frac{x^2}{2})$  ( $K_n$  is the normalization constant), the non-Hermitian operator  $H_\beta$  is represented by the non-selfadjoint infinite tridiagonal matrix

$$M_\beta = \begin{bmatrix} \frac{1}{2}\beta & -\sqrt{1} & & & & \\ \sqrt{1} & \frac{3}{2}\beta & -\sqrt{2} & & & \\ & \sqrt{2} & \frac{5}{2}\beta & -\sqrt{3} & & \\ & & \sqrt{3} & \frac{7}{2}\beta & \ddots & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

with real spectrum. The matrix  $M_\beta$  is pseudo-Hermitian for  $H = \text{diag}(1, -1, 1, -1, \dots)$ . Then, we consider the finite  $r \times r$  tridiagonal matrix

$$M_{\beta,r} = \begin{bmatrix} \frac{1}{2}\beta & -\sqrt{1} & & & & \\ \sqrt{1} & \frac{3}{2}\beta & -\sqrt{2} & & & \\ & \sqrt{2} & \frac{5}{2}\beta & -\sqrt{3} & & \\ & & \sqrt{3} & \ddots & \ddots & \\ & & & \ddots & \ddots & -\sqrt{r-1} \\ & & & & \sqrt{r-1} & \frac{2r-1}{2}\beta \end{bmatrix}.$$

Let a pseudo-Jacobi matrix  $J_n$  of the form (3.1) be given as follows:

$$\begin{aligned} J_r &= M_{\beta,r}, & J_{r+2,n} &= M_{\frac{\beta}{2},r}, \\ \beta_r &= \sqrt{r}, & \beta_{r+1} &= \sqrt{r+1}, \\ \alpha_{r+1} &= r, & \epsilon_r &= \epsilon_{r+1} = 1. \end{aligned}$$

Assume  $\lambda = \sigma(J_n)$ ,  $\mu_1 = \sigma(J_r)$ , and  $\mu_2 = \sigma(J_{r+2,n})$ . We choose the values of  $r$  and  $\beta$  such that the elements in  $\mu_1$  and  $\mu_2$  are all real and pairwise distinct and  $\mu_1 \cap \mu_2 = \emptyset$ .

By Algorithm 3 we can obtain a unique pseudo-Jacobi matrix  $\tilde{J}_n$ . Then, we compute the spectra  $\tilde{\lambda}$ ,  $\tilde{\mu}_1$ , and  $\tilde{\mu}_2$  of  $\tilde{J}_n$  and of its principal submatrices  $\tilde{J}_r$  and  $\tilde{J}_{r+2,n}$ . In Table 5.1, we present the comparison between  $\tilde{\lambda}$ ,  $\tilde{\mu}_1$ ,  $\tilde{\mu}_2$ ,  $\tilde{J}_n$ , and the initial counterparts. If the value of  $r$



is larger, then the errors  $\|\lambda - \tilde{\lambda}\|_2$ ,  $\|\mu_1 - \tilde{\mu}_1\|_2$ ,  $\|\mu_2 - \tilde{\mu}_2\|_2$ , and  $\|J_n - \tilde{J}_n\|_F$  will also be larger. Assume that there exist an  $H_1$ -orthogonal matrix  $\tilde{U}_1$  and an  $H_2$ -orthogonal matrix  $\tilde{U}_2$  such that  $\tilde{U}_1^\# \tilde{J}_r \tilde{U}_1 = \text{diag}(\tilde{\mu}_1)$  and  $\tilde{U}_2^\# \tilde{J}_{r+2,n} \tilde{U}_2 = \text{diag}(\tilde{\mu}_2)$ . From the Hoffman-Wielandt theorem for diagonalizable matrices [20], we have that

$$\begin{aligned} \|\mu_1 - \tilde{\mu}_1\|_2 &\leq \kappa(U_1)\kappa(\tilde{U}_1)\|J_r - \tilde{J}_r\|_F && \text{and} \\ \|\mu_2 - \tilde{\mu}_2\|_2 &\leq \kappa(U_2)\kappa(\tilde{U}_2)\|J_{r+2,n} - \tilde{J}_{r+2,n}\|_F \end{aligned}$$

by (3.2), where  $\kappa(X)$  denotes the condition number of the matrix  $X$ . Similarly, there exist pseudo-orthogonal matrices  $P$  and  $\tilde{P}$  such that  $P^\# J_n P = \text{diag}(\lambda)$  and  $\tilde{P}^\# \tilde{J}_n \tilde{P} = \text{diag}(\tilde{\lambda})$ . We also have that

$$\|\lambda - \tilde{\lambda}\|_2 \leq \kappa(P)\kappa(\tilde{P})\|J_n - \tilde{J}_n\|_F.$$

So the large variations in the errors for  $J_n$  may yield such small variations in the errors for the eigenvalues  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  in Table 5.1. Thus, the results agree with our theoretical results established in this paper and demonstrate the feasibility and effectiveness of Algorithm 3.

EXAMPLE 5.2. Let the vector  $\epsilon = (1, -1, -1, -1, 1, -1, -1, 1)$  and the matrix

$$J_9 = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 3 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -3 \end{bmatrix}$$

be given. Consider  $r = 4$  and let the spectra  $\lambda$ ,  $\mu_1$ ,  $\mu_2$  of  $J_9$  and of its principal submatrices  $J_4$  and  $J_{6,9}$  be as in Table 5.2. Then,  $\mu_1 \cap \mu_2 = \emptyset$ ,  $H_1 = \text{diag}(1, 1, -1, 1)$ , and  $H_2 = \text{diag}(1, -1, 1, 1)$ . By Algorithm 3, we get

$$\begin{aligned} x_1 &= -0.35726558990817, & x_2 &= -0.6666666666666666, & x_3 &= 1.9999999999999999, \\ x_4 &= -4.97606774342519, & x_5 &= 0.71665054819938, & x_6 &= -3.0000000000000001, \\ x_7 &= 9.0000000000000002, & x_8 &= 2.28334945180063. \end{aligned}$$

It is obvious that the conditions (1) and (2) in Theorem 3.1 hold. Thus,

$$\begin{aligned} g_1 &= (0.29885849072269, 0.40824829046386, 0.70710678118654, 1.11535507165041)^\top, \\ g_2 &= (0.28218405108868, 0.57735026918963, 1.000000000000000, 0.50369186477897)^\top. \end{aligned}$$

Continuing using Algorithm 3, the pseudo-Jacobi matrix  $\tilde{J}_9 \in \mathcal{J}(9, \epsilon, \tilde{\beta})$  in Table 5.3 can be obtained.

This matrix is the unique solution of the PJIEP because we have

$$\|J_9 - \tilde{J}_9\|_F = 1.80047652073822\text{e-}13.$$

Furthermore, we compute the spectra  $\tilde{\lambda}$ ,  $\tilde{\mu}_1$ ,  $\tilde{\mu}_2$  of  $\tilde{J}_9$  and of its principal submatrices  $\tilde{J}_4$  and  $\tilde{J}_{6,9}$ . In Figure 5.1 we compare the computed spectra with the original spectra  $\lambda$ ,  $\mu_1$ , and  $\mu_2$ .

TABLE 5.1  
 Numerical results of Example 5.1.

	$\ \lambda - \tilde{\lambda}\ _2$	$\ \mu_1 - \tilde{\mu}_1\ _2$	$\ \mu_2 - \tilde{\mu}_2\ _2$	$\ J_n - \tilde{J}_n\ _F$	
$r = 5$	$\beta = 7.5$ $\beta = 7.05$ $\beta = 7.005$	1.21654087139106e-13 8.91280407333371e-14 5.74883418456454e-14	7.53906016829511e-14 3.31166706280011e-14 1.76968296418523e-14	1.07874211489352e-14 4.36203221376583e-14 5.23911753690996e-14	5.55406538452228e-07 6.75339350432302e-07 8.69512140667790e-09
$r = 10$	$\beta = 10.5$ $\beta = 10.05$ $\beta = 10.005$	1.03928498584809e-13 1.42836826118259e-13 1.01618101503372e-13	1.87991937470471e-14 6.37511677045892e-14 2.31608595598326e-14	2.45497399495158e-14 1.48885833566228e-14 2.04281036531029e-14	1.14483347887364e-06 8.23170442289946e-06 2.00292443464089e-06
$r = 15$	$\beta = 12.2$ $\beta = 12.27$ $\beta = 12.227$	1.63429971714009e-10 3.56072544922297e-10 7.66680284994086e-11	1.63136090375117e-10 3.10049934403217e-10 7.28806581445643e-11	6.45688930806847e-12 1.66967112863334e-10 2.29559276828696e-11	2.63164795080718e-06 8.95113200525828e-06 3.96950713189288e-06
$r = 20$	$\beta = 15.6$ $\beta = 15.62$ $\beta = 15.627$	1.58489386292105e-06 2.70117225939402e-07 8.29477849157585e-07	1.58492490913971e-06 2.70116033248486e-07 8.29470091227044e-07	3.17397792186766e-09 2.97767699245570e-11 3.67038177702291e-09	3.05364567938157e-05 2.82192471207050e-05 9.30526242363253e-06
$r = 25$	$\beta = 38.5$ $\beta = 38.54$ $\beta = 38.542$	2.67730161759526e-05 3.47348272517093e-05 4.68726739372201e-05	2.67730145907598e-05 3.47347596882915e-05 4.68725644670191e-05	1.43873524019575e-08 3.66138295955242e-08 2.17506827107243e-08	5.16235943827669e-05 6.20954702973390e-05 1.80684325832389e-04

TABLE 5.2  
 The spectra  $\lambda = \{\lambda_j\}_{j=1}^9$ ,  $\mu_1 = \{\mu_j\}_{j=1}^4$  and  $\mu_2 = \{\mu_j\}_{j=5}^8$ .

$j$	$\lambda_j$	$\mu_j$
1	2.97935111716971-0.16836804694051i	2.73205080756888
2	2.97935111716971+0.16836804694051i	-2.00000000000000
3	2.25663440991630	2.00000000000000
4	1.26336198716348-1.32257216682703i	-0.73205080756888
5	1.26336198716348+1.32257216682703i	2.37228132326901
6	-0.03727254885815	1.00000000000000
7	-1.96441245351364	-3.00000000000000
8	-4.46198378686847	-3.37228132326901
9	-3.27839182934239	

TABLE 5.3  
 Main diagonal  $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_9)$  and subdiagonal  $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_8)$  of  $\tilde{J}_9 \in \mathcal{J}(9, \epsilon, \tilde{\beta})$ .

$j$	$\tilde{\alpha}_j$	$\tilde{\beta}_j$
1	1.99999999999997	2.00000000000004
2	-0.99999999999996	1.00000000000001
3	2.99999999999997	1.99999999999997
4	-1.99999999999997	2.00000000000001
5	2.00000000000001	3.00000000000001
6	-3.99999999999999	1.99999999999998
7	0.99999999999994	1.41421356237315
8	3.00000000000007	1.41421356237305
9	-3.00000000000002	

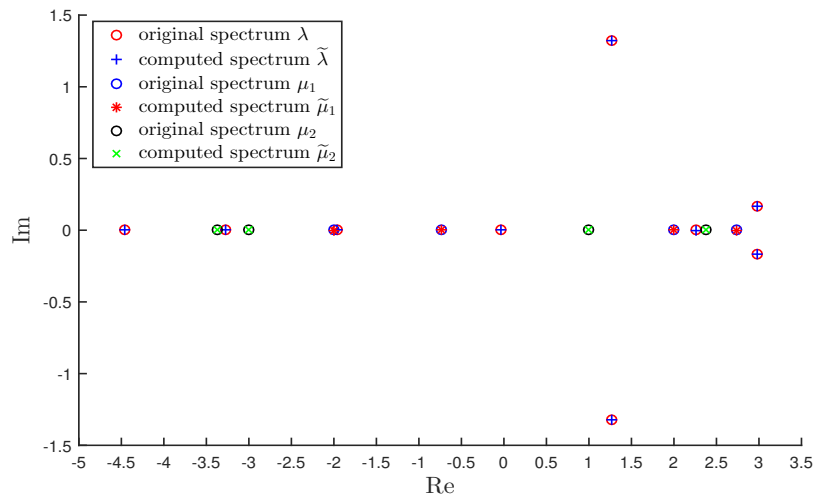


FIG. 5.1. Comparison between the original spectra  $\lambda$ ,  $\mu_1$ ,  $\mu_2$  and the computed spectra  $\tilde{\lambda}$ ,  $\tilde{\mu}_1$ ,  $\tilde{\mu}_2$ .

TABLE 5.4  
*Absolute errors between the original spectra  $\lambda$ ,  $\mu_1$ ,  $\mu_2$  and the computed spectra  $\tilde{\lambda}$ ,  $\tilde{\mu}_1$ ,  $\tilde{\mu}_2$ .*

$\ \lambda - \tilde{\lambda}\ _2$	$\ \mu_1 - \tilde{\mu}_1\ _2$	$\ \mu_2 - \tilde{\mu}_2\ _2$
6.27822826864024e-15	5.61843057806044e-15	7.93168690704416e-15

TABLE 5.5  
*The spectra  $\lambda = \{\lambda_j\}_{j=1}^9$ ,  $\mu_1 = \{\mu_j\}_{j=1}^4$  and  $\mu_2 = \{\mu_j\}_{j=5}^8$ .*

$j$	$\lambda_j$	$\mu_j$
1	2.00000000000000	2.00000000000000
2	-2.00000000000000	-2.00000000000000
3	6.58533601596052	2.73205080756888
4	3.40647425562398	-0.73205080756888
5	1.79809235457054	2.00000000000000
6	-1.96971063765212+2.86959352986032i	-2.00000000000000
7	-1.96971063765212-2.86959352986032i	3.37228132326901
8	-0.57578829818534	-2.37228132326901
9	-2.27469305266548	

In Table 5.4 we also present their respective absolute errors. The computed spectra are in total accordance with the original spectra within the machine precision.

EXAMPLE 5.3. Let  $\epsilon = (-1, -1, 1, 1, -1, -1, -1, 1)$ ,  $r = 4$ , and

$$J_9 = \begin{bmatrix} -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 4 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -2 \end{bmatrix}.$$

Then,  $H_1 = H_2 = \text{diag}(1, -1, 1, 1)$ , and the spectra  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  of  $J_9$  and of its principal submatrices  $J_4$  and  $J_{6,9}$  are given in Table 5.5.

Obviously,  $\mu_1 \cap \mu_2 = \{2.00000000000000, -2.00000000000000\}$  and  $k = 2$ . In Algorithm 3, we consider three of the solutions. Firstly, by selecting  $\theta_1 = 2$  and  $\theta_2 = -7$ , a pseudo-Jacobi matrix  $\tilde{J}_9^{(1)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(1)})$  is obtained in Table 5.6. Then, by choosing  $\theta_1 = 3$  and  $\theta_2 = -8$ , we get another pseudo-Jacobi matrix  $\tilde{J}_9^{(2)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(2)})$  whose entries are displayed in Table 5.7. Next, taking  $\theta_1 = 5$  and  $\theta_2 = -8$ , a pseudo-Jacobi matrix  $\tilde{J}_9^{(3)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(3)})$  is obtained and given in Table 5.8.

Finally, we compute the spectra  $\tilde{\lambda}^{(i)}$ ,  $\tilde{\mu}_1^{(i)}$ , and  $\tilde{\mu}_2^{(i)}$  of the pseudo-Jacobi matrices  $\tilde{J}_9^{(i)}$  and of their principal submatrices  $\tilde{J}_4^{(i)}$  and  $\tilde{J}_{6,9}^{(i)}$ , for  $i = 1, 2, 3$ . Comparing these spectra with the original spectra  $\lambda$ ,  $\mu_1$ , and  $\mu_2$ , Figures 5.2, 5.3, 5.4, and Table 5.9 illustrate that the computed spectra agree with the original spectra up to the machine precision. All the numerical results are in accordance with the theory developed in this paper.

**6. Conclusions.** In this paper, an inverse eigenvalue problem for Jacobi matrices that was investigated in [17] has been considered in the non-selfadjoint setting. This problem,

TABLE 5.6

Main diagonal  $\tilde{\alpha}^{(1)} = (\tilde{\alpha}_1^{(1)}, \tilde{\alpha}_2^{(1)}, \dots, \tilde{\alpha}_9^{(1)})$  and subdiagonal  $\tilde{\beta}^{(1)} = (\tilde{\beta}_1^{(1)}, \tilde{\beta}_2^{(1)}, \dots, \tilde{\beta}_8^{(1)})$  of  $\tilde{J}_9^{(1)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(1)})$ .

$j$	$\tilde{\alpha}_j^{(1)}$	$\tilde{\beta}_j^{(1)}$
1	-1.71754190328978	1.83482089091865
2	2.65888882292985	0.58272751993471
3	2.14866944697204	1.94778362432672
4	-1.09001636661211	10.09125033548037
5	1.99999999999999	9.63500562186306
6	-2.25134649910235	0.94668324899178
7	1.62607316500176	1.14864353913391
8	3.52046222224392	1.69309068435266
9	-1.89518888814334	

TABLE 5.7

Main diagonal  $\tilde{\alpha}^{(2)} = (\tilde{\alpha}_1^{(2)}, \tilde{\alpha}_2^{(2)}, \dots, \tilde{\alpha}_9^{(2)})$  and subdiagonal  $\tilde{\beta}^{(2)} = (\tilde{\beta}_1^{(2)}, \tilde{\beta}_2^{(2)}, \dots, \tilde{\beta}_8^{(2)})$  of  $\tilde{J}_9^{(2)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(2)})$ .

$j$	$\tilde{\alpha}_j^{(2)}$	$\tilde{\beta}_j^{(2)}$
1	-1.20263604281617	1.12069676849459
2	1.81727913229512	0.97735873260376
3	2.78441794338492	1.72672967266801
4	-1.39906103286386	10.31988372027510
5	1.99999999999999	9.87420882906570
6	-2.53333333333336	1.50122457136676
7	2.27605177993530	0.64677836372989
8	3.00807778349232	1.82098682825439
9	-1.75079623009426	

TABLE 5.8

Main diagonal  $\tilde{\alpha}^{(3)} = (\tilde{\alpha}_1^{(3)}, \tilde{\alpha}_2^{(3)}, \dots, \tilde{\alpha}_9^{(3)})$  and subdiagonal  $\tilde{\beta}^{(3)} = (\tilde{\beta}_1^{(3)}, \tilde{\beta}_2^{(3)}, \dots, \tilde{\beta}_8^{(3)})$  of  $\tilde{J}_9^{(3)} \in \mathcal{J}(9, \epsilon, \tilde{\beta}^{(3)})$ .

$j$	$\tilde{\alpha}_j^{(3)}$	$\tilde{\beta}_j^{(3)}$
1	-0.88332250873279	0.45831627309447
2	-1.02180790112723	3.59721086479957
3	5.83279767929948	1.17008113254617
4	-1.92766726943946	9.60034721594304
5	1.99999999999999	9.11957601353623
6	-3.31462925851710	2.60660360466998
7	3.19245936689820	0.38865718165124
8	2.80498308965325	1.87250089319368
9	-1.68281319803435	

TABLE 5.9

Absolute errors between the original spectra  $\lambda, \mu_1, \mu_2$  and the computed spectra  $\tilde{\lambda}^{(i)}, \tilde{\mu}_1^{(i)}, \tilde{\mu}_2^{(i)}$ .

$i$	$\ \lambda - \tilde{\lambda}^{(i)}\ _2$	$\ \mu_1 - \tilde{\mu}_1^{(i)}\ _2$	$\ \mu_2 - \tilde{\mu}_2^{(i)}\ _2$
1	2.76195696656855e-14	1.39393795312866e-14	7.02517689335219e-15
2	7.85613248788838e-14	2.44249065417534e-15	2.97904098389673e-15
3	9.20595358801062e-13	2.13859186672314e-13	5.82417538579527e-15

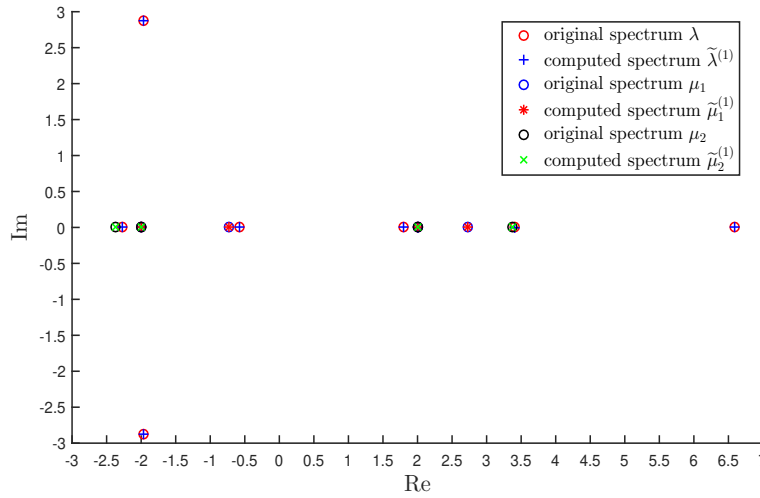


FIG. 5.2. Comparison between the original spectra  $\lambda$ ,  $\mu_1$ ,  $\mu_2$  and the computed spectra  $\tilde{\lambda}^{(1)}$ ,  $\tilde{\mu}_1^{(1)}$ ,  $\tilde{\mu}_2^{(1)}$ .

abbreviated as PJIEP, has been solved from the knowledge of a given sign vector  $\epsilon$  and from the prescribed spectra  $\lambda$ ,  $\mu_1$ , and  $\mu_2$  of  $J_n$  and of two complementary principal submatrices, where  $\lambda$  is closed under complex conjugation and all the elements in  $\mu_1$  and  $\mu_2$  are real pairwise distinct. Necessary and sufficient conditions for the existence of the solution have been found according to the two cases  $\mu_1 \cap \mu_2 = \emptyset$  and  $\mu_1 \cap \mu_2 \neq \emptyset$ . Then, the desired pseudo-Jacobi matrices have been constructed with the aid of a modified unsymmetric Lanczos algorithm. Furthermore, numerical experiments illustrate the efficiency and feasibility of the proposed construction algorithm (Algorithm 3). Our results extend the previous results obtained in [18] for the unique case when  $\mu_1 \cap \mu_2 = \emptyset$  as well as in [6] and [29] for the cases  $H = I_r \oplus -I_{n-r}$  and  $H = I_r \oplus -I_1 \oplus I_{n-r-1}$ , respectively. If the sets  $\mu_1$  and  $\mu_2$  in the PJIEP have complex elements and are closed under conjugation, then the present approach does not apply. This is an open problem that deserves future attention.

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**Appendix A. Proofs of Theorems 2.1 and 2.2.** The proofs of the theorems in Section 2 require the following lemmas.

LEMMA A.1. Let  $\hat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  have distinct real eigenvalues  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n$ . Then, the adjugate of  $\hat{\lambda}_j I_n - \hat{J}_n$  is

$$(A.1) \quad \text{adj}(\hat{\lambda}_j I_n - \hat{J}_n) = \chi'_n(\hat{\lambda}_j) \mathbf{v}_j \mathbf{v}_j^T H \delta_j,$$

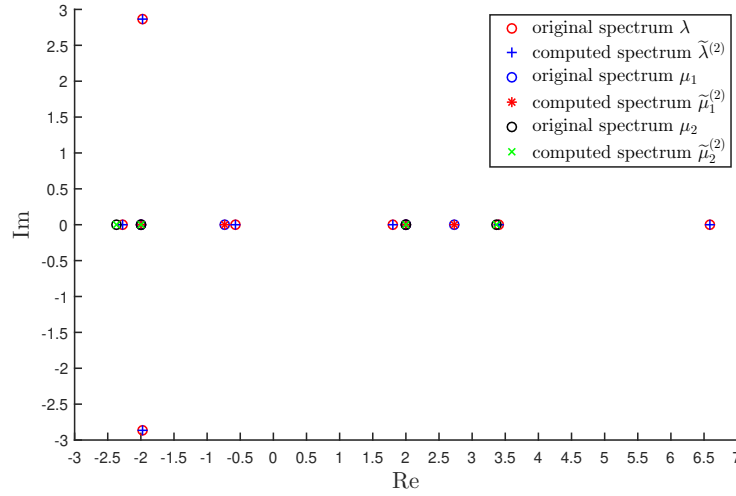


FIG. 5.3. Comparison between the original spectra  $\lambda, \mu_1, \mu_2$  and the computed spectra  $\tilde{\lambda}^{(2)}, \tilde{\mu}_1^{(2)}, \tilde{\mu}_2^{(2)}$ .

where  $\mathbf{v}_j$  is the  $j$ th column of an  $H$ -orthogonal diagonalizing matrix of  $\hat{J}_n$  and where  $\chi'_n(\hat{\lambda}_j) = \prod_{i=1, i \neq j}^n (\hat{\lambda}_j - \hat{\lambda}_i)$ .

*Proof.* Under the hypothesis on  $\hat{J}_n$ , there exists an  $H$ -orthogonal matrix  $V$  such that

$$\hat{J}_n V = V \Lambda, \quad \Lambda = \text{diag}(\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n).$$

Thus,

$$\begin{aligned} \text{adj}(\lambda I_n - \hat{J}_n) &= \det(\lambda I_n - \hat{J}_n) \cdot (\lambda I_n - \hat{J}_n)^{-1} = \det(\lambda I_n - \hat{J}_n) \cdot V(\lambda I_n - \Lambda)^{-1} V^\# \\ &= \sum_{i=1}^n \frac{\det(\lambda I_n - \hat{J}_n)}{\lambda - \hat{\lambda}_i} \mathbf{v}_i \mathbf{v}_i^\top H \delta_i. \end{aligned}$$

From the last equality, we easily obtain (A.1).  $\square$

REMARK A.2. Matrices in  $\mathcal{J}(n, \epsilon, \beta)$  may exist with multiple eigenvalues (see Examples 4.1 and 4.2 in [29]). It has been shown in Lemma A.1 that, if  $\hat{J}_n \in \mathcal{J}(n, \epsilon, \beta)$  is diagonalizable and has a multiple eigenvalue  $\hat{\lambda}_j$ , then the equality (A.1) also holds.

LEMMA A.3. Under the assumptions in Lemma A.1, both of the following statements hold:

(1) If  $\omega \leq v$ , then  $\chi_{1, \omega-1}(\hat{\lambda}_j) \beta_\omega \cdots \beta_{v-1} \chi_{v+1, n}(\hat{\lambda}_j) = \chi'_n(\hat{\lambda}_j) \mathbf{v}_{\omega j} \mathbf{v}_{v j} \delta_\omega \delta_j$ ;

(2) If  $\omega \geq v$ , then  $\chi_{1, v-1}(\hat{\lambda}_j) \beta_v \cdots \beta_{\omega-1} \chi_{\omega+1, n}(\hat{\lambda}_j) = \chi'_n(\hat{\lambda}_j) \mathbf{v}_{\omega j} \mathbf{v}_{v j} \delta_v \delta_j$ ,

where  $\mathbf{v}_{ij}$  is the  $i$ th component of  $\mathbf{v}_j$ .

*Proof.* If  $\omega \leq v$ , we consider the  $(\omega, v)$ th entry on both sides of the equality (A.1), and we get

$$\chi_{1, \omega-1}(\hat{\lambda}_j) \beta_\omega \cdots \beta_{v-1} \epsilon_\omega \cdots \epsilon_{v-1} \chi_{v+1, n}(\hat{\lambda}_j) = \chi'_n(\hat{\lambda}_j) \mathbf{v}_{\omega j} \mathbf{v}_{v j} \delta_v \delta_j.$$

Then, (1) holds because  $\epsilon_\omega \cdots \epsilon_{v-1} = \frac{\delta_v}{\delta_\omega}$ .

If  $\omega \geq v$ , we take the  $(\omega, v)$ th entry on both sides of the equality (A.1). Then,

$$\chi_{1, v-1}(\hat{\lambda}_j) \beta_v \cdots \beta_{\omega-1} \chi_{\omega+1, n}(\hat{\lambda}_j) = \chi'_n(\hat{\lambda}_j) \mathbf{v}_{\omega j} \mathbf{v}_{v j} \delta_v \delta_j,$$

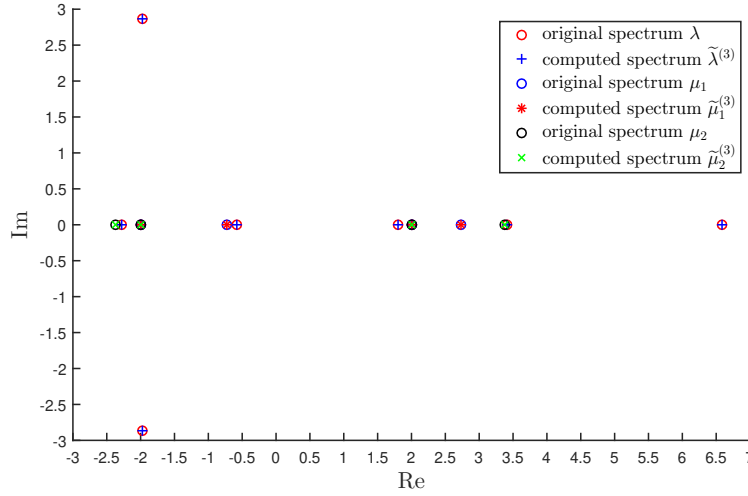


FIG. 5.4. Comparison between the original spectra  $\lambda, \mu_1, \mu_2$  and the computed spectra  $\tilde{\lambda}^{(3)}, \tilde{\mu}_1^{(3)}, \tilde{\mu}_2^{(3)}$ .

and (2) follows.  $\square$

When  $(\omega, \nu)$  is, respectively, taken to be  $(1, n), (n, 1), (1, 1)$ , and  $(n, n)$ , the following immediate consequence of Lemma A.3 holds.

LEMMA A.4. *Under the assumptions in Lemma A.1, we have*

- (1)  $\chi'_n(\hat{\lambda}_j) \mathbf{v}_{1j} \mathbf{v}_{nj} = \delta_1 \delta_j \beta_1 \beta_2 \cdots \beta_{n-1}$ ;
- (2)  $\chi'_n(\hat{\lambda}_j) \mathbf{v}_{1j}^2 = \delta_1 \delta_j \chi_{2,n}(\hat{\lambda}_j)$ ;
- (3)  $\chi'_n(\hat{\lambda}_j) \mathbf{v}_{nj}^2 = \delta_n \delta_j \chi_{1,n-1}(\hat{\lambda}_j)$ .

*Proof of Theorem 2.1.* The first item in the above lemma was obtained in [5, Lemma 2.2]. Because  $0 \neq \chi'_n(\hat{\lambda}_j) \in \mathbb{R}$ , from (1) it follows that  $0 \neq \mathbf{v}_{1j} \mathbf{v}_{nj} \in \mathbb{R}$ .  $\square$

*Proof of Theorem 2.2.* From Lemma A.4 (1) it follows that  $\beta_1 \beta_2 \cdots \beta_{n-1} \neq 0$  and  $0 \neq \mathbf{v}_{n,i} \in \mathbb{R}$ , for  $i = 1, 2, \dots, n$ . Since  $(\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n})$  is the last row of the  $H$ -orthogonal matrix  $V$ , it follows that

$$(A.2) \quad \delta_1 \mathbf{v}_{n,1}^2 + \delta_2 \mathbf{v}_{n,2}^2 + \cdots + \delta_n \mathbf{v}_{n,n}^2 = \delta_n.$$

Assume that  $[\hat{\mathbf{s}}_k, \hat{\mathbf{s}}_k] \delta_{k-1} > 0$ , for  $k = n, n-1, \dots, 2$ , and also  $[\hat{\mathbf{s}}_{n+1}, \hat{\mathbf{s}}_{n+1}] \delta_n > 0$ . We first show that  $\mathbf{Y}_n, \mathbf{Y}_{n-1}, \dots, \mathbf{Y}_1$ , computed by this algorithm, are the columns of the  $H$ -orthogonal matrix  $Y = [\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n]$ , and we demonstrate that the following pseudo-orthogonality relations hold:

$$(A.3) \quad [\mathbf{Y}_i, \mathbf{Y}_j] = \delta_{ij} \delta_i, \quad \text{for } j = n, n-1, \dots, i \text{ and } i = n, n-1, \dots, 1.$$

From (A.2) and steps 4 and 6, (A.3) follows for  $i = n$ . If (A.3) holds for  $i = n, n-1, \dots, l$ , we then prove that it also holds for  $i = l-1$ .

If  $j = l-1$ , steps 4 and 6 imply that (A.3) holds because

$$[\mathbf{Y}_{l-1}, \mathbf{Y}_{l-1}] = \frac{1}{\beta_{l-1}^2} [\hat{\mathbf{s}}_l, \hat{\mathbf{s}}_l] = \delta_{l-1}.$$



For  $j \geq l$ , from steps 6 and 10 we have

$$\begin{aligned} [\mathbf{Y}_{l-1}, \mathbf{Y}_j] &= \frac{1}{\beta_{l-1}} [\widehat{\mathbf{s}}_l, \mathbf{Y}_j] = \frac{1}{\beta_{l-1}} [(\Lambda - \alpha_l I_n) \mathbf{Y}_l - \gamma_l \mathbf{Y}_{l+1}, \mathbf{Y}_j] \\ &= \frac{1}{\beta_{l-1}} ([\Lambda \mathbf{Y}_l, \mathbf{Y}_j] - \alpha_l \delta_{lj} \delta_l - \gamma_l \delta_{l+1,j} \delta_{l+1}). \end{aligned}$$

Clearly steps 7 and 9 imply that the right-hand side of the above equality is zero for  $j = l$ . If  $j > l$ , we get

$$\begin{aligned} [\Lambda \mathbf{Y}_l, \mathbf{Y}_j] &= [\mathbf{Y}_l, \Lambda \mathbf{Y}_j] = [\mathbf{Y}_l, \widehat{\mathbf{s}}_j + \alpha_j \mathbf{Y}_j + \gamma_j \mathbf{Y}_{j+1}] \\ &= \beta_{j-1} \delta_{l,j-1} \delta_l + \alpha_j \delta_{l,j} \delta_l + \gamma_j \delta_{l,j+1} \delta_l = \beta_{j-1} \delta_{l,j-1} \delta_l \end{aligned}$$

from steps 6 and 10. Then

$$[\mathbf{Y}_{l-1}, \mathbf{Y}_j] = \frac{1}{\beta_{l-1}} (\beta_{j-1} \delta_{l,j-1} \delta_l - \gamma_l \delta_{l+1,j} \delta_{l+1}) = 0,$$

because  $\gamma_{k-1} = \frac{1}{\beta_{k-1}} [\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_k] \delta_k = \epsilon_{k-1} \beta_{k-1}$ , for  $k = n, n-1, \dots, 2$ , from steps 4, 5, and 11.

Next, we show that the matrix  $Z = [\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n]$  computed by this algorithm is an  $H$ -orthogonal matrix. As

$$[\mathbf{Z}_i, \mathbf{Z}_j] = [\mathbf{Y}_i, \mathbf{Y}_j] \delta_i \delta_j = \delta_{ij} \delta_j$$

for  $j = n, n-1, \dots, i$ , and  $i = n, n-1, \dots, 1$ , from step 7 and (A.3), we can also get  $Z = HYH$  as well as the biorthogonality condition  $Y^T Z = I_n$ .

Now, we show that  $\widehat{\mathbf{s}}_1 = \mathbf{0}$ . It is sufficient to prove that  $[\widehat{\mathbf{s}}_1, \mathbf{Y}_i] = 0$ , for  $i = 1, 2, \dots, n$ , because  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$  constitute an  $H$ -orthonormal basis of  $\mathbb{R}^n$ . If  $i = 1$ , we obtain

$$[\widehat{\mathbf{s}}_1, \mathbf{Y}_1] = [\Lambda \mathbf{Y}_1 - \alpha_1 \mathbf{Y}_1 - \gamma_1 \mathbf{Y}_2, \mathbf{Y}_1] = [\Lambda \mathbf{Y}_1, \mathbf{Y}_1] - \alpha_1 \delta_1 = 0$$

from steps 7, 9, and 10. If  $i = 2$ , we have

$$\begin{aligned} [\widehat{\mathbf{s}}_1, \mathbf{Y}_2] &= [\Lambda \mathbf{Y}_1 - \alpha_1 \mathbf{Y}_1 - \gamma_1 \mathbf{Y}_2, \mathbf{Y}_2] = [\mathbf{Y}_1, \Lambda \mathbf{Y}_2] - \gamma_1 \delta_2 \\ &= [\mathbf{Y}_1, \widehat{\mathbf{s}}_2 + \alpha_2 \mathbf{Y}_2 + \gamma_2 \mathbf{Y}_3] - \gamma_1 \delta_2 = \beta_1 \delta_1 - \gamma_1 \delta_2 = 0 \end{aligned}$$

from steps 6 and 10, and  $\gamma_1 = \epsilon_1 \beta_1$ . If  $i \geq 3$ , we obtain

$$\begin{aligned} [\widehat{\mathbf{s}}_1, \mathbf{Y}_i] &= [\Lambda \mathbf{Y}_1 - \alpha_1 \mathbf{Y}_1 - \gamma_1 \mathbf{Y}_2, \mathbf{Y}_i] = [\mathbf{Y}_1, \Lambda \mathbf{Y}_i] \\ &= [\mathbf{Y}_1, \widehat{\mathbf{s}}_i + \alpha_i \mathbf{Y}_i + \gamma_i \mathbf{Y}_{i+1}] = [\mathbf{Y}_1, \beta_{i-1} \mathbf{Y}_{i-1}] = 0 \end{aligned}$$

from steps 6 and 10. Then,  $\widehat{\mathbf{s}}_1 = \widehat{\mathbf{r}}_1 = \mathbf{0}$  holds from step 11. Thus, the algorithm will prematurely terminate in this case and it follows that

$$\Lambda Y = Y \widehat{J}^T \quad \text{and} \quad \Lambda Z = Z \widehat{J},$$

and so  $(Y^{-1})^T \widehat{J} Y^T = \Lambda$ .

Finally, we demonstrate that the constructed pseudo-Jacobi matrix  $\widehat{J}$  is unique. Let  $\widetilde{J}$  be a pseudo-Jacobi matrix characterized by the distinct real eigenvalues  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$ , the nonzero  $H$ -orthonormal vector  $(\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n})^T$ , and the pseudo-norms  $\delta_1, \delta_2, \dots, \delta_n$ . The column vectors  $\widetilde{\mathbf{Y}}_n, \widetilde{\mathbf{Y}}_{n-1}, \dots, \widetilde{\mathbf{Y}}_1$ , obtained by the algorithm, are pseudo-orthogonal with the

respective norms  $\delta_n, \delta_{n-1}, \dots, \delta_1$ . The computed column vectors  $\tilde{\mathbf{Z}}_n, \tilde{\mathbf{Z}}_{n-1}, \dots, \tilde{\mathbf{Z}}_1$  are also pseudo-orthogonal and satisfy the biorthogonality condition  $\tilde{\mathbf{Y}}_i^T \tilde{\mathbf{Z}}_j = \delta_{ij}$ , for  $i, j = 1, 2, \dots, n$ . Thus,

$$(A.4) \quad \Lambda \tilde{\mathbf{Y}} = \tilde{\mathbf{Y}} \tilde{\mathbf{J}}^T \quad \text{and} \quad \Lambda \tilde{\mathbf{Z}} = \tilde{\mathbf{Z}} \tilde{\mathbf{J}}$$

for  $\tilde{\mathbf{Y}} = [\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \dots, \tilde{\mathbf{Y}}_n]$  and  $\tilde{\mathbf{Z}} = [\tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2, \dots, \tilde{\mathbf{Z}}_n]$ . Because all the subdiagonal entries  $\tilde{\beta}_i$  and the superdiagonal entries  $\tilde{\gamma}_i$  of  $\tilde{\mathbf{J}}$  satisfy  $\tilde{\gamma}_i = \epsilon_i \tilde{\beta}_i$ , we only need to prove that all the main diagonal entries  $\tilde{\alpha}_i$  and subdiagonal entries  $\tilde{\beta}_i$  of  $\tilde{\mathbf{J}}$  are equal to the corresponding entries  $\alpha_i$  and  $\beta_i$  of the matrix  $\hat{\mathbf{J}}$  computed by the algorithm.

From the identities in (A.4), the columns  $\tilde{\mathbf{Y}}_k$  and  $\tilde{\mathbf{Z}}_k$  of the  $H$ -orthogonal matrices  $\mathbf{Y}$  and  $\mathbf{Z}$ , respectively, satisfy the following recurrence relations:

$$\begin{aligned} \tilde{\beta}_{k-1} \tilde{\mathbf{Y}}_{k-1} &= \tilde{\mathbf{s}}_k = (\Lambda - \tilde{\alpha}_k I_n) \tilde{\mathbf{Y}}_k - \tilde{\gamma}_k \tilde{\mathbf{Y}}_{k+1}, \\ \tilde{\gamma}_{k-1} \tilde{\mathbf{Z}}_{k-1} &= \tilde{\mathbf{r}}_k = (\Lambda - \tilde{\alpha}_k I_n) \tilde{\mathbf{Z}}_k - \tilde{\beta}_k \tilde{\mathbf{Z}}_{k+1} \end{aligned}$$

for  $k = n, n-1, \dots, 1$ , with  $\tilde{\mathbf{Y}}_{n+1} = \tilde{\mathbf{Z}}_{n+1} = \mathbf{0}$  and  $\tilde{\mathbf{s}}_1 = \tilde{\mathbf{r}}_1 = \mathbf{0}$ . As  $\tilde{\mathbf{Z}}_k = H \tilde{\mathbf{Y}}_k \delta_k$  and  $\tilde{\mathbf{r}}_k = H \tilde{\mathbf{s}}_k \delta_k$ , we only consider the first recurrence relation pre-multiplied by  $\tilde{\mathbf{Z}}_k^T$ . It follows that

$$\tilde{\alpha}_k = \tilde{\mathbf{Z}}_k^T \Lambda \tilde{\mathbf{Y}}_k, \quad k = n, n-1, \dots, 1.$$

Observing that

$$1 = \tilde{\mathbf{Y}}_{k-1}^T \tilde{\mathbf{Z}}_{k-1} = \frac{1}{\tilde{\beta}_{k-1} \tilde{\gamma}_{k-1}} \tilde{\mathbf{s}}_k^T \tilde{\mathbf{r}}_k,$$

we find that

$$\tilde{\mathbf{Y}}_{k-1} = \frac{1}{\tilde{\beta}_{k-1}} \tilde{\mathbf{s}}_k, \quad \tilde{\gamma}_{k-1} = \frac{1}{\tilde{\beta}_{k-1}} \tilde{\mathbf{s}}_k^T \tilde{\mathbf{r}}_k,$$

and so

$$[\tilde{\beta}_{k-1} \tilde{\mathbf{Y}}_{k-1}, \tilde{\beta}_{k-1} \tilde{\mathbf{Y}}_{k-1}] = [\tilde{\mathbf{s}}_k, \tilde{\mathbf{s}}_k].$$

Thus,

$$\tilde{\beta}_{k-1} = \sqrt{[\tilde{\mathbf{s}}_k, \tilde{\mathbf{s}}_k] \delta_{k-1}}, \quad k = n, n-1, \dots, 2.$$

By the sequence of computations in the backward modified unsymmetric Lanczos algorithm and considering  $\tilde{\mathbf{Y}}_n = (\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n})^T$ , we can easily show that the entries  $\tilde{\alpha}_n, \tilde{\beta}_{n-1}, \tilde{\alpha}_{n-1}, \tilde{\beta}_{n-2}, \dots, \tilde{\beta}_1, \tilde{\alpha}_1$  of  $\tilde{\mathbf{J}}$  coincide, respectively, with the entries  $\alpha_n, \beta_{n-1}, \alpha_{n-1}, \beta_{n-2}, \dots, \beta_1, \alpha_1$  of  $\hat{\mathbf{J}}$ .  $\square$

REMARK A.5. By executing the algorithm in Theorem 2.3, we can also get  $\mathbf{Y} = \mathbf{V}^T$  and  $\mathbf{Z} = \mathbf{V}^\#$  from the initial vector  $\mathbf{Y}_n = (\mathbf{v}_{n,1}, \mathbf{v}_{n,2}, \dots, \mathbf{v}_{n,n})^T$ .

**Appendix B. Proofs of Theorems 3.1 and 3.2.** The proofs of the theorems in Section 3 use the following four lemmas. The first one gives a necessary and sufficient condition for  $J_r$  and  $J_{r+2,n}$  in (3.1) to share a common eigenvalue. As noticed by one of the referees, the

rational function  $F_1(\lambda)$  there appearing is the  $(r+1, r+1)$ -entry of the twisted factorization of the tridiagonal matrix  $\lambda I_n - J_n$  (see [22]).

LEMMA B.1. *Let  $\mu_1$  and  $\mu_2$  be the spectra of  $J_r$  and  $J_{r+2,n}$  in (3.1), respectively. Then, an element of  $\mu_1$  is an eigenvalue of  $J_n$  in (3.1) if and only if it is an element of  $\mu_2$  and vice-versa.*

*Proof.* From (3.2), it follows that  $U = U_1 \oplus I_1 \oplus U_2$  is a pseudo-orthogonal matrix with respect to  $\tilde{H} = H_1 \oplus I_1 \oplus H_2$ , and so  $U^\# = \tilde{H}U^T\tilde{H}$ . Thus, we get

$$U^\# J_n U = \begin{bmatrix} \Lambda_1 & \epsilon_r \beta_r U_1^\# \mathbf{e}_r & \mathbf{0} \\ \beta_r \mathbf{e}_r^T U_1 & \alpha_{r+1} & \epsilon_{r+1} \beta_{r+1} \omega_1^T U_2 \\ \mathbf{0} & \beta_{r+1} U_2^\# \omega_1 & \Lambda_2 \end{bmatrix}.$$

By using the Laplace expansion for the determinant, it follows that

(B.1)

$$\begin{aligned} \det(\lambda I_n - J_n) &= \det(\lambda I_n - U^\# J_n U) \\ &= \prod_{j=1}^{n-1} (\lambda - \mu_j) \left( \lambda - \alpha_{r+1} - \sum_{i=1}^r \frac{\delta_{r+1} (\beta_r u_{r,i}^{(1)})^2 \delta_i}{\lambda - \mu_i} - \sum_{i=r+1}^{n-1} \frac{\delta_{r+1} (\beta_{r+1} u_{1,i-r}^{(2)})^2 \delta_{i+1}}{\lambda - \mu_i} \right). \end{aligned}$$

Because  $\beta_r u_{r,i}^{(1)} \neq 0$ ,  $i = 1, 2, \dots, r$ , and  $\beta_{r+1} u_{1,i-r}^{(2)} \neq 0$ ,  $i = r+1, r+2, \dots, n-1$ , from Lemma A.4 (1) we find that

$$\begin{aligned} \det(\mu_j I_n - J_n) &= - \prod_{i=1, i \neq j}^{n-1} (\mu_j - \mu_i) \cdot \begin{cases} \delta_{r+1} (\beta_r u_{r,j}^{(1)})^2 \delta_j, & j = 1, 2, \dots, r, \\ \delta_{r+1} (\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1}, & j = r+1, r+2, \dots, n-1. \end{cases} \end{aligned}$$

Due to the fact that all the elements in  $\mu_1$  and  $\mu_2$  are pairwise distinct,  $\mu_j$  is an eigenvalue of  $J_n$  if and only if  $\prod_{i=1, i \neq j}^{n-1} (\mu_j - \mu_i) = 0$ , that is,  $\mu_j$  is a common eigenvalue of  $J_r$  and  $J_{r+2,n}$ .  $\square$

If  $J_r$  and  $J_{r+2,n}$  have no common eigenvalues, then the following holds.

LEMMA B.2. *Let  $\sigma(J_r) = \mu_1$  and  $\sigma(J_{r+2,n}) = \mu_2$ . If  $\sigma(J_r) \cap \sigma(J_{r+2,n}) = \emptyset$ , then the eigenvalues of  $J_n$  are the  $n$  zeros of the following rational function*

$$(B.2) \quad F_1(\lambda) = \lambda - \alpha_{r+1} - \sum_{i=1}^r \frac{\delta_{r+1} (\beta_r u_{r,i}^{(1)})^2 \delta_i}{\lambda - \mu_i} - \sum_{i=r+1}^{n-1} \frac{\delta_{r+1} (\beta_{r+1} u_{1,i-r}^{(2)})^2 \delta_{i+1}}{\lambda - \mu_i}.$$

*Proof.* By Lemma B.1,  $\mu_j \notin \sigma(J_n)$ , for any  $j = 1, 2, \dots, n-1$ , if  $\sigma(J_r) \cap \sigma(J_{r+2,n}) = \emptyset$ . Then,  $\det(\lambda I_n - J_n) = 0$  is equivalent to

$$F_1(\lambda) = \lambda - \alpha_{r+1} - \sum_{i=1}^r \frac{\delta_{r+1} (\beta_r u_{r,i}^{(1)})^2 \delta_i}{\lambda - \mu_i} - \sum_{i=r+1}^{n-1} \frac{\delta_{r+1} (\beta_{r+1} u_{1,i-r}^{(2)})^2 \delta_{i+1}}{\lambda - \mu_i} = 0$$

from equation (B.1), and the result holds.  $\square$

If  $J_r$  and  $J_{r+2,n}$  have common eigenvalues, then the following holds.

LEMMA B.3. *Let  $\sigma(J_r) = \mu_1$  and  $\sigma(J_{r+2,n}) = \mu_2$ . Assume  $\mu_1 \cap \mu_2 = \{\mu_i\}_{i=1}^k$  and  $\mu_{r+i} = \mu_i$ , for any  $i = 1, 2, \dots, k$ , with  $k \leq \min\{r, n-r-1\}$ . Then,  $\mu_1, \mu_2, \dots, \mu_k$  are*

eigenvalues of  $J_n$ , and the remaining eigenvalues of  $J_n$  are the  $n - k$  zeros of the rational function

$$\begin{aligned}
 (B.3) \quad F_2(\lambda) = & \lambda - \alpha_{r+1} - \sum_{i=1}^k \frac{\delta_{r+1}(\beta_r u_{r,i}^{(1)})^2 \delta_i + \delta_{r+1}(\beta_{r+1} u_{1,i}^{(2)})^2 \delta_{r+i+1}}{\lambda - \mu_i} \\
 & - \sum_{i=k+1}^r \frac{\delta_{r+1}(\beta_r u_{r,i}^{(1)})^2 \delta_i}{\lambda - \mu_i} - \sum_{i=r+k+1}^{n-1} \frac{\delta_{r+1}(\beta_{r+1} u_{1,i-r}^{(2)})^2 \delta_{i+1}}{\lambda - \mu_i}.
 \end{aligned}$$

*Proof.* Because  $\mu_i \in \sigma(J_r) \cap \sigma(J_{r+2,n})$ , for any  $i = 1, 2, \dots, k$ , then  $\mu_1, \mu_2, \dots, \mu_k$  are also eigenvalues of  $J_n$  by Lemma B.1. Hence the remaining eigenvalues of  $J_n$  are the zeros of the polynomial

$$G(\lambda) = \frac{\det(\lambda I_n - J_n)}{\prod_{i=1}^k (\lambda - \mu_i)} = \prod_{j=k+1}^{n-1} (\lambda - \mu_j) F_1(\lambda)$$

from equation (B.1). Since  $\mu_i \notin \sigma(J_r) \cap \sigma(J_{r+2,n})$  for any  $i \notin \{1, 2, \dots, k\} \cup \{r+1, r+2, \dots, r+k\}$ , from Lemma B.1  $\prod_{j=k+1}^{n-1} (\lambda - \mu_j) \neq 0$  for any  $\lambda \notin \{\mu_1, \mu_2, \dots, \mu_k\}$ . Hence,  $G(\lambda) = 0$  if and only if

$$\begin{aligned}
 F_2(\lambda) = F_1(\lambda) = & \lambda - \alpha_{r+1} - \sum_{i=1}^k \frac{\delta_{r+1}(\beta_r u_{r,i}^{(1)})^2 \delta_i + \delta_{r+1}(\beta_{r+1} u_{1,i}^{(2)})^2 \delta_{r+i+1}}{\lambda - \mu_i} \\
 & - \sum_{i=k+1}^r \frac{\delta_{r+1}(\beta_r u_{r,i}^{(1)})^2 \delta_i}{\lambda - \mu_i} - \sum_{i=r+k+1}^{n-1} \frac{\delta_{r+1}(\beta_{r+1} u_{1,i-r}^{(2)})^2 \delta_{i+1}}{\lambda - \mu_i} = 0.
 \end{aligned}$$

By construction,  $G(\lambda)$  has degree  $n - k$  and so  $G(\lambda)$  has  $n - k$  zeros. Thus,  $F_2(\lambda)$  also has  $n - k$  zeros.  $\square$

In order to prove the main theorems in Section 3, we recall the following crucial result presented in [5, 17].

LEMMA B.4. Let  $\{\xi_1, \xi_2, \dots, \xi_m\}$  be a set of complex numbers closed under conjugation, and let  $\{\eta_1, \eta_2, \dots, \eta_{m-1}\}$  be a set of distinct real numbers with  $\eta_j \notin \{\xi_1, \xi_2, \dots, \xi_m\}$ . Then, the following system of linear algebraic equations

$$\frac{x_1}{\xi_i - \eta_1} + \frac{x_2}{\xi_i - \eta_2} + \dots + \frac{x_{m-1}}{\xi_i - \eta_{m-1}} = \xi_i - a, \quad i = 1, 2, \dots, m,$$

has a unique solution  $\mathbf{x} = (x_1, x_2, \dots, x_{m-1})$  if and only if

$$x_j = -\frac{\prod_{i=1}^m (\xi_i - \eta_j)}{\prod_{i=1, i \neq j}^{m-1} (\eta_i - \eta_j)}, \quad j = 1, 2, \dots, m-1,$$

and  $a = \sum_{i=1}^m \xi_i - \sum_{i=1}^{m-1} \eta_i$ .

*Proof of Theorem 3.1. Necessity:* Assume that there exists a pseudo-Jacobi matrix  $J_n \in \mathcal{J}(n, \epsilon, \beta)$  as in (3.1) such that  $\sigma(J_n) = \lambda$ ,  $\sigma(J_r) = \mu_1$ , and  $\sigma(J_{r+2,n}) = \mu_2$ . Because  $\mu_1 \cap \mu_2 = \emptyset$ , then it follows from Lemma B.2 that the eigenvalues of  $J_n$  are the zeros of  $F_1(\lambda) = 0$  in (B.2). By Lemma B.4, we get

$$(B.4) \quad \begin{cases} \delta_{r+1}(\beta_r u_{r,j}^{(1)})^2 \delta_j = x_j, & j = 1, 2, \dots, r, \\ \delta_{r+1}(\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1} = x_j, & j = r+1, r+2, \dots, n-1, \end{cases}$$

where  $x_j = -\prod_{i=1}^n (\lambda_i - \mu_j) \prod_{i=1, i \neq j}^{n-1} (\mu_i - \mu_j)^{-1}$ . Since  $\beta_r, \beta_{r+1}, u_{r,j}^{(1)}, j = 1, 2, \dots, r$ , and  $u_{1,j-r}^{(2)}, j = r+1, r+2, \dots, n-1$ , are real, then condition (1) holds.

By (3.2) we know that  $\sum_{j=1}^r (u_{r,j}^{(1)})^2 \delta_j = \delta_r$  and  $\sum_{j=r+1}^{n-1} (u_{1,j-r}^{(2)})^2 \delta_{j+1} = \delta_{r+2}$ . Thus, condition (2) follows from (B.4). Finally, conditions (3) and (4) are satisfied because

$$\begin{aligned} \beta_{k-1}^2 &= [\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_k] \delta_{k-1} > 0, & \text{for } k = r, r-1, \dots, 2, \text{ and} \\ \beta_{r+k+1}^2 &= [\mathbf{r}_{r+k+1}, \mathbf{r}_{r+k+1}] \delta_{r+k+2} / \delta_{r+2} > 0, & \text{for } k = 1, 2, \dots, n-r-2. \end{aligned}$$

*Sufficiency:* Assume that conditions (1)–(4) hold. Consider the sign vector  $\epsilon$  and the nonzero real numbers

$$(B.5) \quad x_j = -\frac{\prod_{i=1}^n (\lambda_i - \mu_j)}{\prod_{i=1, i \neq j}^{n-1} (\mu_i - \mu_j)}, \quad j = 1, 2, \dots, n-1.$$

Let us define

$$(B.6) \quad \beta_r := \left( \epsilon_r \sum_{j=1}^r x_j \right)^{\frac{1}{2}}, \quad \beta_{r+1} := \left( \epsilon_{r+1} \sum_{j=r+1}^{n-1} x_j \right)^{\frac{1}{2}},$$

and

$$(B.7) \quad \begin{cases} u_{r,j}^{(1)} := \frac{\sqrt{\delta_{r+1} \delta_j x_j}}{\beta_r}, & j = 1, 2, \dots, r, \\ u_{1,j-r}^{(2)} := \frac{\sqrt{\delta_{r+1} \delta_{j+1} x_j}}{\beta_{r+1}}, & j = r+1, r+2, \dots, n-1. \end{cases}$$

Then,  $g_1 = (u_{r,1}^{(1)}, u_{r,2}^{(1)}, \dots, u_{r,r}^{(1)})^T$  and  $g_2 = (u_{1,1}^{(2)}, u_{1,2}^{(2)}, \dots, u_{1,n-r-1}^{(2)})^T$  are, respectively, an  $H_1$ -orthonormal and an  $H_2$ -orthonormal vector. Furthermore, condition (3) ensures that a unique pseudo-Jacobi matrix  $J_r$  can be constructed from  $(H_1, \mu_1, g_1)$  by using the algorithm in Theorem 2.2. Similarly, condition (4) guarantees that the algorithm in Theorem 2.3 generates a unique pseudo-Jacobi matrix  $J_{r+2,n}$  given  $(H_2, \mu_2, g_2)$ . Since  $\alpha_{r+1} = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \mu_i$ , a unique pseudo-Jacobi matrix  $J_n$  is so constructed.

From equations (B.5) and (B.7), we find

$$x_j = \begin{cases} \delta_{r+1} (\beta_r u_{r,j}^{(1)})^2 \delta_j, & j = 1, 2, \dots, r, \\ \delta_{r+1} (\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1}, & j = r+1, r+2, \dots, n-1, \end{cases}$$

By Lemma B.4,  $F_1(\lambda_i) = 0$  holds for  $i = 1, 2, \dots, n$ , in Lemma B.2. Thus, it follows that  $\det(\lambda_i I_n - J_n) = 0, i = 1, 2, \dots, n$ . Therefore,  $\lambda = \sigma(J_n)$  and the constructed matrix  $J_n$  is the unique solution of the PJIEP.  $\square$

*Proof of Theorem 3.2. Necessity:* For the given sign vector  $\epsilon$  and the sets  $\lambda, \mu_1$ , and  $\mu_2$ , suppose that there exists a pseudo-Jacobi matrix  $J_n \in \mathcal{J}(n, \epsilon, \beta)$  as in (3.1). If  $\mu_1 \cap \mu_2 = \{\mu_i\}_{i=1}^k$  and  $\mu_{r+i} = \mu_i$ , for  $i = 1, 2, \dots, k$ , then  $\lambda_i = \mu_i, i = 1, 2, \dots, k$ , are also the eigenvalues of  $J_n$  by Lemma B.1. The remaining eigenvalues  $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$  of  $J_n$  are the zeros of  $F_2(\lambda) = 0$  in (B.3) by Lemma B.3. Thus, from Lemma B.4 we obtain

$$(B.8) \quad \begin{aligned} \delta_{r+1} (\beta_r u_{r,j}^{(1)})^2 \delta_j + \delta_{r+1} (\beta_{r+1} u_{1,j}^{(2)})^2 \delta_{r+j+1} &= x_{r+j}, & j = 1, 2, \dots, k, \\ \delta_{r+1} (\beta_r u_{r,j}^{(1)})^2 \delta_j &= x_j, & j = k+1, k+2, \dots, r, \\ \delta_{r+1} (\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1} &= x_j, & j = r+k+1, \dots, n-1, \end{aligned}$$

where  $x_j = -\prod_{i=k+1}^n (\lambda_i - \mu_j) \prod_{i=k+1, i \neq j}^{n-1} (\mu_i - \mu_j)^{-1}$ ,  $j = k + 1, \dots, n - 1$ . Then there exist real numbers  $\theta_j \notin \{0, 1\}$  such that

$$(B.9) \quad \delta_{r+1}(\beta_r u_{r,j}^{(1)})^2 \delta_j = \theta_j x_{r+j}, \quad \delta_{r+1}(\beta_{r+1} u_{1,j}^{(2)})^2 \delta_{r+j+1} = (1 - \theta_j) x_{r+j},$$

for  $j = 1, 2, \dots, k$ . Having in mind that  $\beta_r u_{r,j}^{(1)} \neq 0$ ,  $j = 1, 2, \dots, r$ , and  $\beta_{r+1} u_{1,j}^{(2)} \neq 0$ ,  $j = 1, 2, \dots, n - r - 1$ , from Lemma A.4, conditions (1) and (2) are satisfied.

Because  $\sum_{j=1}^r (u_{r,j}^{(1)})^2 \delta_j = \delta_r$  and  $\sum_{j=r+1}^{n-1} (u_{1,j-r}^{(2)})^2 \delta_{j+1} = \delta_{r+2}$  from (3.2), condition (3) holds by (B.8) and (B.9). Since  $\beta_{k-1}^2 = [\widehat{s}_k, \widehat{s}_k] \delta_{k-1} > 0$  for  $k = r, r - 1, \dots, 2$ , and  $\beta_{r+k+1}^2 = [r_{r+k+1}, r_{r+k+1}] \delta_{r+k+2} / \delta_{r+2} > 0$  for  $k = 1, 2, \dots, n - r - 2$ , conditions (3) and (4) in Theorem 3.1 follow.

*Sufficiency:* Because  $\mu_1 \cap \mu_2 = \{\mu_i\}_{i=1}^k$ , then  $\lambda_i = \mu_i$ ,  $i \in \{1, 2, \dots, k\}$  are eigenvalues of a pseudo-Jacobi matrix  $J_n$  which will be constructed in the sequel. If the conditions in this theorem are satisfied, let

$$(B.10) \quad x_j = -\frac{\prod_{i=k+1}^n (\lambda_i - \mu_j)}{\prod_{i=k+1, i \neq j}^{n-1} (\mu_i - \mu_j)}, \quad j = k + 1, \dots, n - 1,$$

where  $x_j$  are all nonzero real numbers. For the selected  $\theta_j \in \mathbb{R} - \{0, 1\}$ ,  $j = 1, 2, \dots, k$ , let us define

$$(B.11) \quad \beta_r := \left( \epsilon_r \left( \sum_{j=1}^k \theta_j x_{r+j} + \sum_{j=k+1}^r x_j \right) \right)^{\frac{1}{2}},$$

$$\beta_{r+1} := \left( \epsilon_{r+1} \left( \sum_{j=1}^k (1 - \theta_j) x_{r+j} + \sum_{j=r+k+1}^{n-1} x_j \right) \right)^{\frac{1}{2}},$$

$$(B.12) \quad u_{r,j}^{(1)} := \begin{cases} \frac{1}{\beta_r} \sqrt{\delta_{r+1} \delta_j \theta_j x_{r+j}}, & j = 1, 2, \dots, k, \\ \frac{1}{\beta_r} \sqrt{\delta_{r+1} \delta_j x_j}, & j = k + 1, \dots, r, \end{cases}$$

and

$$(B.13) \quad u_{1,j}^{(2)} := \begin{cases} \frac{1}{\beta_{r+1}} \sqrt{\delta_{r+1} \delta_{r+j+1} (1 - \theta_j) x_{r+j}}, & j = 1, 2, \dots, k, \\ \frac{1}{\beta_{r+1}} \sqrt{\delta_{r+1} \delta_{r+j+1} x_{r+j}}, & j = k + 1, \dots, n - r - 1. \end{cases}$$

Hence, we can construct a unique Jacobi matrix  $J_r$  from  $H_1$ ,  $\mu_1$  and also the  $H_1$ -orthonormal vector  $g_1 = (u_{r,1}^{(1)}, u_{r,2}^{(1)}, \dots, u_{r,r}^{(1)})^T$  by using the algorithm in Theorem 2.2 with the help of condition (3) in Theorem 3.1. In addition, condition (4) in Theorem 3.1 ensures that a unique pseudo-Jacobi matrix  $J_{r+2,n}$  can be constructed from  $H_2$ ,  $\mu_2$  and the  $H_2$ -orthonormal vector  $g_2 = (u_{11}^{(2)}, u_{12}^{(2)}, \dots, u_{1,n-r-1}^{(2)})^T$  by using the algorithm in Theorem 2.3. Then,  $\alpha_{r+1} = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \mu_i$  from (3.1), and a pseudo-Jacobi matrix  $J_n$  is so reconstructed.

As  $\beta_r$ ,  $\beta_{r+1}$ ,  $u_{r,j}^{(1)}$ , and  $u_{1,j}^{(2)}$  depend on  $\theta_j \in \mathbb{R} - \{0, 1\}$  and the  $\theta_j$  can be taken arbitrarily for  $j = 1, 2, \dots, k$ , there are infinite pseudo-Jacobi matrices  $J_r$  and  $J_{r+2,n}$  obtained from the algorithms in Theorems 2.2 and 2.3, and so infinite pseudo-Jacobi matrices  $J_n$  can be achieved.

Finally, we show that a reconstructed pseudo-Jacobi matrix  $J_n$  solves the PJIEP. By equations (B.10), (B.12), and (B.13), we have

$$x_j = \begin{cases} \delta_{r+1}(\beta_r u_{r,j-r}^{(1)})^2 \delta_{j-r} + \delta_{r+1}(\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1}, & j = r+1, r+2, \dots, r+k, \\ \delta_{r+1}(\beta_r u_{r,j}^{(1)})^2 \delta_j, & j = k+1, k+2, \dots, r, \\ \delta_{r+1}(\beta_{r+1} u_{1,j-r}^{(2)})^2 \delta_{j+1}, & j = r+k+1, \dots, n-1, \end{cases}$$

where  $\delta_{r+1}(\beta_r u_{r,j}^{(1)})^2 \delta_j = \theta_j x_{r+j}$  and  $\delta_{r+1}(\beta_{r+1} u_{1,j}^{(2)})^2 \delta_{r+j+1} = (1 - \theta_j) x_{r+j}$ ,  $\theta_j \in \mathbb{R} - \{0, 1\}$  for  $j = 1, 2, \dots, k$ . By Lemma B.4,  $F_2(\lambda_i) = \lambda_i - \alpha_{r+1} - \sum_{j=k+1}^{n-1} \frac{x_j}{\lambda_i - \mu_j} = 0$  holds for  $i = k+1, k+2, \dots, n$  in Lemma B.3. Then  $\det(\lambda_i I_n - J_n) = 0$ ,  $i = k+1, k+2, \dots, n$ , and  $\lambda_i$ ,  $i = k+1, k+2, \dots, n$  are the remaining eigenvalues of  $J_n$ . Thus,  $\lambda = \sigma(J_n)$ , and  $J_n$  is a solution of the PJIEP.  $\square$

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