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EFFICIENT CUBATURE RULES*

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Abstract. 67 new cubature rules are found for three standard multi-dimensional integrals with spherically symmetric regions and weight functions using direct search with a numerical zero-finder. 63 of the new rules have fewer integration points than known rules of the same degree, and 20 are within three points of Möller's lower bound. Most have all positive coefficients, and most have some symmetry, including some supported by one or two concentric spheres. They include degree-7 formulas for the integration over the sphere and Gaussian-weighted integrals over the entire space, each in 6 and 7 dimensions, with 127 and 183 points, respectively.

Key words. multiple integrals, Gaussian weight, cubature formula, integration rule, numerical integration, regular simplex

AMS subject classifications. 65D30, 65D32, 41A55, 41A63

1. Introduction.

We are concerned with estimating multi-dimensional integrals of the form

$$\int_\Omega w(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}, \tag{1.1}$$

where $\mathbf{x} = [x_1\ x_2 \cdots x_n]^T$, for the integration regions Ω and weighting functions $w(\mathbf{x})$ given in Table 1.1. Applications of (1.1) include the evaluation of quantum-mechanical matrix elements with Gaussian wave functions in atomic physics [33], nuclear physics [18], and particle physics [19]. For applications in statistics, particularly Bayesian inference, see [11]. For applications in target tracking, see [2, 21].

We approximate these integrals using *cubature formulas* or *integration rules* of the form

$$\sum_{i=1}^{N} W_i f(\mathbf{x}_i), \tag{1.2}$$

where the *weights* W_i and *nodes* or *points* $\mathbf{x}_i$ are independent of the function f.

In the following, we use the notation G_n, $E_n^{r^2}$, E_n^r, and S_n for the integrals defined in Table 1.1. The first two integrals in the table are of course closely related. Given an approximation of G_n of the form (1.2), we can construct an equivalent approximation $E_n^{r^2} \approx \sum_{i=1}^N B_i f(\mathbf{b}_i)$, where $\mathbf{b}_i = \mathbf{x}_i/\sqrt{2}$ and $B_i = \pi^{n/2} W_i$. In this paper we address $E_n^{r^2}$ following the numerical analysis convention. However, in the supplemental material we quote the parameters for the corresponding G_n formulas for the convenience of researchers using another commonly used convention.

If an integration rule is exact for all polynomials up to and including degree d but not for some polynomial of degree $d+1$, then we say the rule has *algebraic degree of exactness* (or simply *degree*) d.

One can construct cubature formulas being exact for a space of polynomials by solving the large system of polynomial equations associated with it. In describing this method, Cools

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TABLE 1.1
Integrals studied.

Name	Region Ω	Weight Function $w(\mathbf{x})$
G_n	entire space $\mathbb{R}^n$	$(2\pi)^{-n/2}e^{-\mathbf{x}^T\mathbf{x}/2}$
$E_n^{r^2}$	entire space $\mathbb{R}^n$	$e^{-\mathbf{x}^T\mathbf{x}}$
E_n^r	entire space $\mathbb{R}^n$	$e^{-\sqrt{\mathbf{x}^T\mathbf{x}}}$
S_n	unit n-sphere $\mathbf{x}^T\mathbf{x} \le 1$	1

stated that "it is essential to restrict the search to cubature formulas with a certain structure" [8]. For example, in [1, "CUT4" formulas], points were assumed to take the form

$$\begin{array}{cccccccc} (0, & 0, & \cdots & 0, & 0) & W_0 & 1, \\ (\pm\eta, & 0, & \cdots & 0, & 0)_S & W_1 & 2n, \\ (\pm\nu, & \pm\nu, & \cdots & \pm\nu, & \pm\nu) & W_2 & 2^n, \end{array}$$

where the notation $(\cdots)_S$ indicates that all points obtained from these by permutation of coordinates are included and are assigned the same weight. The last column gives the number of points. This point set is *fully symmetric*, i.e., closed under all coordinate permutations and sign changes. However, relaxing this symmetry requirement may allow us to find formulas with fewer points [26, 43]. For example (as shown in Figure 6.1) in two dimensions, there is a formula of degree five with points at the vertices of a regular hexagon [40, formula V], which is closed under sign permutations but not coordinate permutations. There is also a formula of degree 4 with points at the vertices of a regular pentagon which is closed under sign changes in x_1 but not x_2, i.e., with bilateral symmetry.

The objective of this work was to test whether the continuing improvements in computer processing have made it feasible to find interesting new cubature rules by the "brute force" approach—using a numerical zero-finder to solve the moment constraint equations directly. We find that, for rules with up to approximately 3000 free parameters[1], it is no longer necessary to assume at the outset that the points have a particular structure. Relieving those assumptions has made it possible to discover rules with fewer points than known rules of the same degree, including twenty rules that come within three points of the lower bound found by Möller [25, 31].

Section 2 describes our search method, including several procedures that can improve a rule with negative weights or with little or no symmetry. Section 3 describes how the description of a rule can be simplified by orienting it to take best advantage of any symmetries and by finding closed-form expressions for point coordinates and weights. Sections 4–9 present the new rules that have significant symmetry. Section 10 lists and discusses all the new rules including those with little or no symmetry. The supplemental material[2] includes tables in double and quad precision of all the new rules.

2. Searching. An approximation is exact for all polynomials with degree $\le d$ if it is exact for all monomials

$$f(\mathbf{x}) = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \qquad 0 \le \alpha_1 + \cdots + \alpha_n \le d,$$

where the α_i are all nonnegative integers. If any of the α_i are odd, then the monomial integral is zero for any of our problems. Let $\alpha = [\alpha_1\ \alpha_2 \cdots \alpha_n]^T$, where all α_i are even, and

[1] Each of the N points has n coordinates and a weight, so a rule has $(n+1)N$ free parameters.

[2] The supplemental material is available at
http://etna.ricam.oeaw.ac.at/volumes/2011-2020/vol51/addition/files.zip.

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$\beta_i = (\alpha_i + 1)/2$. Then the monomial integral is [13, 39, 40]

$$I(\alpha) \equiv \int_\Omega w(\mathbf{x}) x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\, dx_1\, dx_2\, \cdots\, dx_n$$
$$= \begin{cases} \Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n) & \text{for } E_n^{r^2}, \\ \dfrac{2(\alpha_1 + \cdots + \alpha_n + n - 1)!}{\Gamma(\beta_1 + \cdots + \beta_n)}\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n) & \text{for } E_n^r, \\ \dfrac{1}{\Gamma(\beta_1 + \cdots + \beta_n + 1)}\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n) & \text{for } S_n. \end{cases}$$

For example, a rule of degree 4 for $E_2^{r^2}$ must satisfy the following 15 constraints, where x_{ij} is the jth coordinate of the ith point and each sum is over $i = 1 \ldots N$:

$$\begin{array}{lllll} \sum W_i = \pi & \sum W_i x_{i1} = 0 & \sum W_i x_{i1}^2 = \frac{\pi}{2} & \sum W_i x_{i1}^3 = 0 & \sum W_i x_{i1}^4 = \frac{3\pi}{4} \\ \sum W_i x_{i2} = 0 & \sum W_i x_{i1} x_{i2} = 0 & \sum W_i x_{i1}^2 x_{i2} = 0 & \sum W_i x_{i1}^3 x_{i2} = 0 & \\ \sum W_i x_{i2}^2 = \frac{\pi}{2} & \sum W_i x_{i1} x_{i2}^2 = 0 & \sum W_i x_{i1}^2 x_{i2}^2 = \frac{\pi}{4} & & \\ \sum W_i x_{i2}^3 = 0 & \sum W_i x_{i1} x_{i2}^3 = 0 & & & \\ \sum W_i x_{i2}^4 = \frac{3\pi}{4} & & & & \end{array}$$

The number of constraints increases rapidly with n and d. The rules in Section 7.2 of dimension and degree 7 satisfy 3432 constraints.

We initialized each search with normally distributed points, assigning initial weights of

$$W_i = e^{-\sqrt{\mathbf{x}_i^T \mathbf{x}_i}}$$

but then normalized them so that they sum up to V,

$$V \equiv \int_\Omega w(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n = \begin{cases} \pi^{n/2} & \text{for } E_n^{r^2}, \\ \dfrac{2(n-1)!\pi^{n/2}}{\Gamma(n/2)} & \text{for } E_n^r, \\ \dfrac{2\pi^{n/2}}{n\Gamma(n/2)} & \text{for } S_n, \end{cases}$$

thus the zeroth-degree constraint was satisfied exactly.

The points were then linearly scaled so that the second-degree constraints involving only one coordinate were also satisfied. For example for $E_n^{r^2}$, we want the points to satisfy

$$\sum_{i=1}^N W_i x_{i1}^2 = I([2\ 0 \ldots 0]^T) = \Gamma(3/2)\Gamma(1/2)^{n-1} = \frac{\pi^{n/2}}{2}\,.$$

We can achieve this by calculating a scaling factor

$$k = \sqrt{\frac{\pi^{n/2}/2}{\sum_{i=1}^N W_i x_{i1}^2}}$$

and making the replacement $x_{i1} \leftarrow k x_{i1}$. Other coordinates of the points were scaled similarly.

Stroud [35] showed that if there is an N point formula in n dimensions of degree d, then

$$N \geq \binom{n + \lfloor d/2 \rfloor}{\lfloor d/2 \rfloor}, \tag{2.1}$$

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where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Möller improved this bound for odd degrees [25, 31]. Let $d = 2s - 1$. Then, the Möller lower bound, N_{MLB}, is given by

$$N \geq N_{MLB} \equiv \begin{cases} \binom{n+s-1}{n} + \sum_{k=1}^{n-1} 2^{k-n} \binom{k+s-1}{k} & s \text{ even}, \\ \binom{n+s-1}{n} + \sum_{k=1}^{n-1} (1 - 2^{k-n}) \binom{k+s-2}{k} & s \text{ odd}. \end{cases} \tag{2.2}$$

However, a formula satisfying the bound exactly may not exist. We searched for the rule of a given degree with the fewest points using a binary search between Möller's lower bound and the number of points in a known formula of the given degree or of the next higher degree.

In most cases, the number of equations and unknowns were unequal (with almost all problems becoming over-determined before N reached Möller's lower bound), so many of the methods developed for solving nonlinear equations could not be applied. We used `fsolve` from the MATLAB Optimization Toolbox [6, 15, 29] or `UDL`[3] by Simonis [34].

Parameters of all searches were logged along with the results of all successful searches, hence the results of a lucky random starting point would not be lost. After a failure, the search was restarted with a new set of randomly distributed points. Some searches identified a valid rule with most points arranged symmetrically and with only a few distinct weight values and with the weights on the remaining points reduced to zero. Other searches finished with a valid rule but with no apparent pattern in point locations or weights.

A rule could sometimes be improved by projecting the innermost or outermost few points to (or toward) the same radius, giving them all the same weight and using that revised configuration to start a new search. In a few cases, this enabled us to eliminate negative weights or drop some points.

Some rules could be improved by restarting the search after dropping low-weight points, combining points with very near neighbors, adding moment constraints of the next higher degree of the form

$$\sum_{i=1}^{N} W_i x_{ij}^{d+1} = I([d+1\ 0 \dots 0]^T),$$

for $1 \leq j \leq n$, or by imposing symmetry as follows: We reoriented the rule so that the eigenvectors of the covariance of the unweighted points were aligned with the coordinate axes. We then tested whether the rule was close to bilaterally symmetric with respect to any of the axes. If so, we searched for a similar symmetric rule. The association of the original and reflected points was treated as a gated linear assignment problem and solved with the Jonker-Volgenant-Castanon (JVC) assignment algorithm [22, 28]. Each assigned point was moved to midway between its original location and that of its assigned reflection. If a point were assigned to its own reflection, then its adjusted position would automatically be on the symmetry plane. If a point was unassigned, then its reflection was added to the set (thereby increasing the number of points).

After finding a rule for one of the integrals, we also searched for similar rules for each of the other integrals starting with the same point layout and relative weights but normalizing the weights and scaling those points so that its zeroth- and second-degree constraints were satisfied exactly.

[3] We revised `UDL` by adding a stopping criterion: If, after any seven consecutive steps, the norm of the residual has decreased by less than seven percent, then the search is deemed a failure.

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3. Presentation.

3.1. Rotations. If a rule is symmetric, as evidenced by several points at the same radius and with equal weights, then it is desirable to determine its structure and, if possible, to express it in a simple form. Any orthogonal transformation of a set of points yields an equivalent set of points, and any orthogonal transformation can be expressed in terms of a skew-symmetric matrix via the Cayley transform [5]. Thus, for a rule in n dimensions there are $n(n-1)/2$ free parameters that can be used to orient it.

One approach is to concentrate on the sphere supporting the fewest points. Conceptually, we rotate to put one of those points on the first coordinate axis. Choosing points in that shell in order by increasing angular distance from that first point, we then rotate to put a second point in the plane defined by the first two axes, then a third point in the subspace defined by the first three axes, etc. We call this “aligning the axes” to the chosen points. It can be accomplished as follows:

Assume we have chosen n points. Reorder the rows of the point matrix so that those rows appear in order at the top forming an $n \times n$ submatrix which we call A. The remainder of the rows form a submatrix which we call B. Use the QR decomposition to factor the transpose of A so that

$$A^T = RU,$$

where R is orthogonal and U is upper triangular. Taking the transpose of both sides, we have

$$A = (RU)^T = U^T R^T.$$

Right multiplying by R, we have

$$AR = U^T R^T R = U^T.$$

Thus, right multiplying our original point matrix by R gives us

$$\begin{bmatrix} A \\ B \end{bmatrix} R = \begin{bmatrix} U^T \\ C \end{bmatrix},$$

where U^T is lower triangular. In its first row, only the first element is nonzero, so it represents a point along the first coordinate. The second row represents a point in the plane defined by the first two coordinates, etc. This satisfies the requirements set out above.

If a rule in n dimensions has $n+1$ points at the same radius (such as the 6 inner points in the 5-dimensional rules of Section 4.4), then they typically appear at the vertices of a regular n simplex. In that case, a simple description can be found by rotating one point to be equidistant from all coordinate axes with each of the other n points in the plane defined by that first point and one of the coordinate axes.

For example, the well-known second-degree rules have $n+1$ points at the vertices of an n simplex. They are often presented in a form like this for G_n [23, 42]:

$$\chi = \sqrt{n+1} \begin{bmatrix} \sqrt{\frac{1}{1\cdot 2}} & \sqrt{\frac{1}{2\cdot 3}} & \sqrt{\frac{1}{3\cdot 4}} & \cdots & \sqrt{\frac{1}{n(n+1)}} \\ -\sqrt{\frac{1}{1\cdot 2}} & \sqrt{\frac{1}{2\cdot 3}} & \sqrt{\frac{1}{3\cdot 4}} & \cdots & \sqrt{\frac{1}{n(n+1)}} \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3\cdot 4}} & \cdots & \sqrt{\frac{1}{n(n+1)}} \\ 0 & 0 & -\sqrt{\frac{3}{4}} & \cdots & \sqrt{\frac{1}{n(n+1)}} \\ \vdots & \vdots & \vdots & \ddots & \sqrt{\frac{1}{n(n+1)}} \\ 0 & 0 & 0 & \cdots & -\sqrt{\frac{n}{n+1}} \end{bmatrix},$$

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where each row represents a point and the weight for each point is $1/(n+1)$. Fan and You noticed that in three dimensions, the points can be expressed in the much simpler form [12]

$$\chi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

This can be generalized to other dimensions yielding the set of points

$$\chi = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

with the two solutions

$$a = \frac{-1 + (n-1)\sqrt{n+1}}{n}, \qquad b = \frac{-1 - \sqrt{n+1}}{n},$$

or

$$a = \frac{-1 - (n-1)\sqrt{n+1}}{n}, \qquad b = \frac{-1 + \sqrt{n+1}}{n}.$$

When a rule has $n+1$ points at the vertices of a regular simplex, it can be rotated into one of these orientations.

3.2. Closed-form expressions. If a rule has enough symmetry, then we attempt to express its points and weights in closed form. In some cases they are integers, simple fractions, or square roots of simple fractions, which can be identified by converting them to a simple continued fraction and looking for a repeating pattern [4]. To guard against accepting a solution that merely minimizes the residuals, or a mathematical coincidence (a closed form that only approximates the actual solution), our next step is to use Maxima [24, 41] to confirm that the resulting rule satisfies the moment constraint equations exactly or (if no closed-form solution was identified) with absolute error less than 10^{-55}.

4. Degree-4 rules.

4.1. Degree 4, dimension 3, 10-point rules. The points in these new formulas for $E_3^{r^2}$ and S_3 are closed with respect to sign changes along two of the three coordinates. The points form two pyramids with one offset and rotated from the other as illustrated in Figure 4.1. The configuration is given in Table 4.1. The Maxima program `m3_10_4.mac` verifying the correctness of these rules is provided in Figure 4.1. The supplemental material includes similar programs for the other new rules.

Becker found an 11-point cubature formula of degree 4 for S_3 [3], but we are not aware of any previous formulas of degree 4 for $E_3^{r^2}$.

4.2. Degree 4, dimension 3, 11-point rule. We were unable to find a 10-point rule for E_3^r, but we did find the 11-point rule given in Table 4.2. This rule has at least one remaining degree of freedom as the x_3.coordinate in the fourth line of the table need not be zero. An example with a nonzero value appears in the supplemental material.

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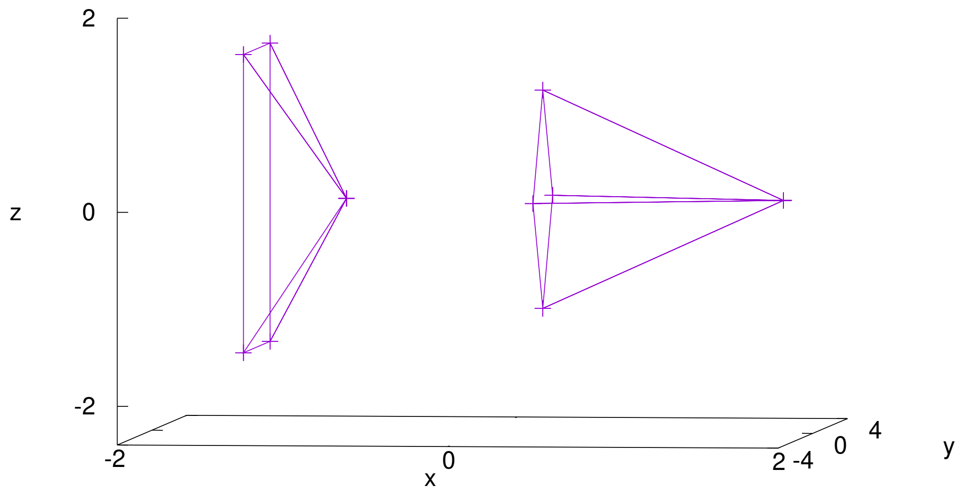

FIG. 4.1. *Configuration of the 10 points for the rule of degree 4 for* $E_3^{r^2}$.

4.3. Degree 4, dimension 4, 16-point rules. We found two sets of 16-point rules. Those in the first set have a central point, one shell of 10 points, and another shell of 5 points as shown in Table 4.3. The formula for S_4 has five points outside the region and one with a negative weight.

Each rule in the second set has a central point, a shell of 6 points, and another shell of 9 points as displayed in Table 4.4. The second formula for S_4 has zero weight on the central point making it a 15-point formula with all positive weights and nine points on the boundary.

Both of these formulas for S_4 are distinct from the 16-point formula found by Mysovskih [7, 32], which has positive weights and seven points on the boundary. We are not aware of previous degree-4 formulas for the other integrals.

4.4. Degree 4, dimension 5, 22-point rules. Each of these new formulas has a central point and two shells. We can describe the points using five generators as given in Table 4.5. The six points with weight W_0 are at the vertices of a regular 5-simplex. The 15 points with weight W_1 are the vertices of a rectified 5-simplex, i.e., each vertex being at the center of an edge of a regular 5-simplex. A MATLAB program to generate these rules (`c5_22_4.m`) is included in the supplemental materials.

4.5. Degree 4, dimension 6, 28-point rules. Each of these new formulas has a central point and 27 points all at the same radius. The rule is given in Table 4.6. Each point on the shell has 16 near neighbors (76 degrees away) and 10 more distant neighbors (120 degrees). This configuration is suggested in Figure 4.3, which displays the points in terms of their angular distance from a chosen point, though of course their distances from each other cannot be displayed realistically.

4.6. Degree 4, dimension 7, 38-point rules. We found 38-point rules for $E_7^{r^2}$ and S_7 each with two negative weights. A standard measure of the stability of an integration rule is the sum of the absolute value of the weights divided by the sum of the weights, which is a worst-case round-off error magnification factor [14]. These rules have stability factors of 7.18 for $E_7^{r^2}$ and 8.55 for S_7. They have some symmetry with the center point, a centered shell of 21 points, and two offset irregular 7-simplices. The configuration of the points with respect to one of the negative weight points is suggested in Figure 4.4. The rules are given in Table 4.7.

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TABLE 4.1
10-point rules of degree 4 for $E_3^{r^2}$ and S_3.

x_1	x_2	x_3	Weight	# Points
g	0	0	W_3	1
a	$(\pm c$	$0)_S$	W_2	4
$-b$	0	0	W_1	1
$-e$	$\pm f$	$\pm f$	W_4	4

	$E_3^{r^2}$	S_3
a	$(\sqrt{3}-1)/2$	$(2\sqrt{3}-1)/\sqrt{77}$
b	$(\sqrt{7}-1)/2$	$(2\sqrt{203}-\sqrt{77})/35$
c	$\sqrt{3-\sqrt{3}}$	$\sqrt{(48-8\sqrt{3})/77}$
e	$(\sqrt{3}+1)/2$	$(2\sqrt{3}+1)/\sqrt{77}$
f	$\sqrt{(3+\sqrt{3})/2}$	$\sqrt{(24+4\sqrt{3})/77}$
g	$(\sqrt{7}+1)/2$	$(2\sqrt{203}+\sqrt{77})/35$
W_1	$\pi^{3/2}(7+2\sqrt{7})/42$	$\pi(841+32\sqrt{11}\sqrt{29})/5220$
W_2	$\pi^{3/2}(2+\sqrt{3})/24$	$7\pi(13+4\sqrt{3})/720$
W_3	$\pi^{3/2}(7-2\sqrt{7})/42$	$\pi(841-32\sqrt{11}\sqrt{29})/5220$
W_4	$\pi^{3/2}(2-\sqrt{3})/24$	$7\pi(13-4\sqrt{3})/720$

TABLE 4.2
11-point rule of degree 4 for E_3^r.

x_1	x_2	x_3	Weight
±5.123512671436	4.925613098468	0.000000000000	0.379658096396
±4.102816292737	-1.218122471265	1.544992698170	1.815112382679
±3.636092685910	-1.218122471265	-4.843920857272	0.737101279022
0.000000000000	-1.836923221948	0.000000000000	8.813498359176
0.000000000000	-12.639707409137	-3.423767380484	0.036648025338
0.000000000000	1.948389609086	-1.422580596634	7.054048788228
0.000000000000	1.703608086180	3.398957047139	3.331366718822
0.000000000000	-8.635010968135	11.051160549267	0.033435820963

5. Degree-5 rules.

5.1. Degree 5, dimension 4, 23-point rules. This new family of degree-5 rules is provided in Table 5.1. Other than the central point, all points are at the same radius. However, two of those points have smaller weight than the others. The rule for E_5^r has fewer points as the known rules. A 22-point rule of degree 5 was known for S_n [39, S_n:5-1].

5.2. Degree 5, dimension 6, 44-point rule. This new rule has points supported by two spheres as given in Table 5.2.

6. Degree-6 rules.

6.1. Degree 6, dimension 2, 10-point rule. A rule was found for $E_2^{r^2}$ with 10 points achieving Stroud's lower bound (2.1). This rule was known but unpublished [9, 20]. The points and weights are displayed in Table 6.1. The point layout is similar to that in the 10-point rule for S_2 by Wissmann and Becker [43, S_2:6-1]. The points are displayed in Figure 6.1,

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TABLE 4.3
16-point rules of degree 4 in 4 dimensions (group 1).

x_1	x_2	x_3	x_4	Weight	Radius	# Points
0	0	0	0	W_0	0	1
$(c$	c	$-b$	$-b)_S$	W_1	r_1	6
$(-e$	$-a$	$-a$	$-a)_S$	W_1	r_1	4
f	f	f	f	W_2	r_2	1
$(g$	$-e$	$-e$	$-e)_S$	W_2	r_2	4

	$E_4^{r^2}$	E_4^r	S_4
a	$(3\sqrt{3}-\sqrt{15})/12$	$(3\sqrt{42}-\sqrt{210})/12$	$(3\sqrt{3}-\sqrt{15})/24$
b	$(\sqrt{15}-\sqrt{3})/6$	$(\sqrt{210}-\sqrt{42})/6$	$(\sqrt{15}-\sqrt{3})/12$
c	$(\sqrt{15}+\sqrt{3})/6$	$(\sqrt{210}+\sqrt{42})/6$	$(\sqrt{15}+\sqrt{3})/12$
e	$(\sqrt{15}+\sqrt{3})/4$	$(\sqrt{210}+\sqrt{42})/4$	$(\sqrt{15}+\sqrt{3})/8$
f	$\sqrt{3}$	$\sqrt{42}$	$\sqrt{3}/2$
g	$(3\sqrt{15}-\sqrt{3})/4$	$(3\sqrt{210}-\sqrt{42})/4$	$(3\sqrt{15}-\sqrt{3})/8$
W_0	$\pi^2/12$	$29\pi^2/7$	$-\pi^2/9$
W_1	$9\pi^2/100$	$27\pi^2/35$	$3\pi^2/50$
W_2	$\pi^2/300$	$\pi^2/35$	$\pi^2/450$
r_1	$\sqrt{2}$	$\sqrt{28}$	$\sqrt{1/2}$
r_2	$\sqrt{12}$	$\sqrt{168}$	$\sqrt{3}$

TABLE 4.4
16- or 15-point rules of degree 4 in 4 dimensions (group 2).

x_1	x_2	x_3	x_4	Weight	Radius	# Points
0	0	0	0	W_0	0	1
0	0	0	$-c$	W_1	r_1	1
0	0	c	0	W_1	r_1	1
$\pm b$	0	$-a$	0	W_1	r_1	2
0	$\pm b$	0	a	W_1	r_1	2
$\pm b$	$\pm b$	a	$-a$	W_2	r_2	4
0	$\pm b$	$-c$	$-a$	W_2	r_2	2
$\pm b$	0	a	c	W_2	r_2	2
0	0	$-c$	c	W_2	r_2	1

	$E_4^{r^2}$	E_4^r	S_4
a	$\sqrt{1/2}$	$\sqrt{7}$	$\sqrt{1/8}$
b	$\sqrt{3/2}$	$\sqrt{21}$	$\sqrt{3/8}$
c	$\sqrt{2}$	$\sqrt{28}$	$\sqrt{1/2}$
W_0	$\pi^2/4$	$39\pi^2/7$	0
W_1	$\pi^2/12$	$5\pi^2/7$	$\pi^2/18$
W_2	$\pi^2/36$	$5\pi^2/21$	$\pi^2/54$
$c = r_1$	$\sqrt{2}$	$\sqrt{28}$	$\sqrt{1/2}$
r_2	2	$\sqrt{56}$	1

```
rule:"3_10_4";
n:3;    /* 3 dimensions */
N:10;   /* 10 points */
deg:4;  /* degree 4 */
/*            G_n                           E_n^{r^2}                S_n */
a_vals:[(sqrt(6)-sqrt(2))/2 ,(sqrt(6)-sqrt(2))/sqrt(8) ,0,(2*sqrt(3)-1)/sqrt(77)         ];
b_vals:[(sqrt(14)-sqrt(2))/2,(sqrt(14)-sqrt(2))/sqrt(8),0,(2*sqrt(203)-sqrt(77))/35      ];
c_vals:[sqrt(6 - 2*sqrt(3)) ,sqrt((6 - 2*sqrt(3))/2)   ,0,sqrt((48-8*sqrt(3))/77)        ];
e_vals:[(sqrt(6)+sqrt(2))/2 ,(1+sqrt(3))/2             ,0,(2*sqrt(3)+1)/sqrt(77)         ];
f_vals:[sqrt(sqrt(3)+3)     ,sqrt((sqrt(3)+3)/2)       ,0,sqrt((24+4*sqrt(3))/77)        ];
g_vals:[(1+sqrt(7))/sqrt(2) ,(1+sqrt(7))/2             ,0,(2*sqrt(203)+sqrt(77))/35      ];
w1vals:[(7+2*sqrt(7))/42,%pi^(3/2)*(2*sqrt(7)+7)/42,0,%pi*(841+32*sqrt(11)*sqrt(29))/5220];
w2vals:[(sqrt(3)+2)/24  ,%pi^(3/2)*(sqrt(3)+2)/24  ,0,%pi*(13+4*sqrt(3))*7/720          ];
w3vals:[(7-2*sqrt(7))/42,%pi^(3/2)*(7-2*sqrt(7))/42,0,%pi*(841-32*sqrt(11)*sqrt(29))/5220];
w4vals:[(2-sqrt(3))/24  ,%pi^(3/2)*(2-sqrt(3))/24  ,0,%pi*(13-4*sqrt(3))*7/720          ];
plabel:["      Gn","E_n^{r^2}","      E_n","      Sn"];
for problem in [1,2,4] do block(
a: a_vals[problem],
b: b_vals[problem],
c: c_vals[problem],
e: e_vals[problem],
f: f_vals[problem],
g: g_vals[problem],
w1:w1vals[problem],
w2:w2vals[problem],
w3:w3vals[problem],
w4:w4vals[problem],
x:matrix([ g, 0, 0, w3],
         [ a, c, 0, w2],
         [ a,-c, 0, w2],
         [ a, 0, c, w2],
         [ a, 0,-c, w2],
         [-b, 0, 0, w1],
         [-e, f, f, w4],
         [-e, f,-f, w4],
         [-e,-f, f, w4],
         [-e,-f,-f, w4]),
ex(a,p) := if p=0 then 1 else a^p,
alpha : makelist(0,i,n),
pass : true,
for i1 : 0 thru deg do
  (alpha[1] : i1,
  for i2 : 0 thru deg-i1 do
    (alpha[2] : i2,
    for i3 : 0 thru deg-i1-i2 do
      (alpha[3] : i3,
       if sum(alpha[i],i,1,n)<=deg then
       (beta : (alpha+makelist(1,i,n))/2,
         odd : false, for i : 1 thru n do odd : odd or oddp(alpha[i]),
  m :  if odd then 0 else
       if problem=1 then 2^sum(beta[ii],ii,1,n)*product(gamma(beta[ii]),ii,1,n)
         /(2*%pi)^(n/2) else
       if problem=2 then product(gamma(beta[ii]),ii,1,n) else
       if problem=3 then 2*gamma(sum(alpha[ii],ii,1,n)+n)
         *product(gamma(beta[ii]),ii,1,n)/gamma(sum(beta[ii],ii,1,n)) else
       if problem=4 then product(gamma(beta[ii]),ii,1,n)/sum(beta[ii],ii,1,n)
         /gamma(sum(beta[ii],ii,1, else
       if problem=5 then 2*product(gamma(beta[ii]),ii,1,n)
          /gamma(sum(beta[ii],ii,1,n)),
          value : radcan(sum(product(ex(x[i,j],alpha[j]),j,1,n)*x[i,n+1],i,1,N)),
          subtest : value=m, /* test one constraint */
          pass : pass and subtest)))),
print(plabel[problem],":",rule,"...", if pass then "pass" else "FAIL" ))$
```

FIG. 4.2. *The Maxima program (*`m3_10_4.mac`*) verifying the correctness of the rules of degree 4 in 3 dimensions.*

along with those for known formulas of degree 3, 4, 5, and 7, and the new formula of degree 8 discussed below. Note that in the figure, the rules of odd degree have central symmetry (for every point x there is also a point $-x$ with the same weight), while those of even degree are only bilaterally symmetric.

6.2. Degree 6, dimension 2, 11-point rule. This new rule for E_2^r has 11 points with bilateral symmetry. The points and weights are given in Table 6.2. This rule come close to pentagonal symmetry, but we were unable to adjust it for pentagonal symmetry.

7. Degree-7 rules. We found four rules of degree 7 with fewer points than previously reported.

7.1. Degree 7, dimension 6, 127-point rules. Each of these new rules has a central point and two shells. The inner shell has 54 points. Each of those has 10 nearest neighbors in that shell (60 degrees away) and 16, 16, 10, and 1 successively further away. The outer shell

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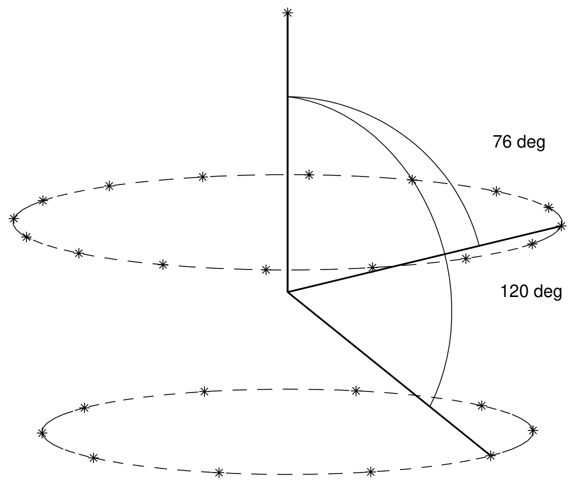

FIG. 4.3. *Configuration of the 27 non-central points for the rules of degree 4 in 6 dimensions.*

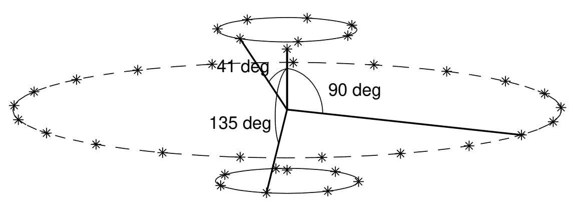

FIG. 4.4. *Configuration of the 37 non-central points for the rules of degree 4 in 7 dimensions.*

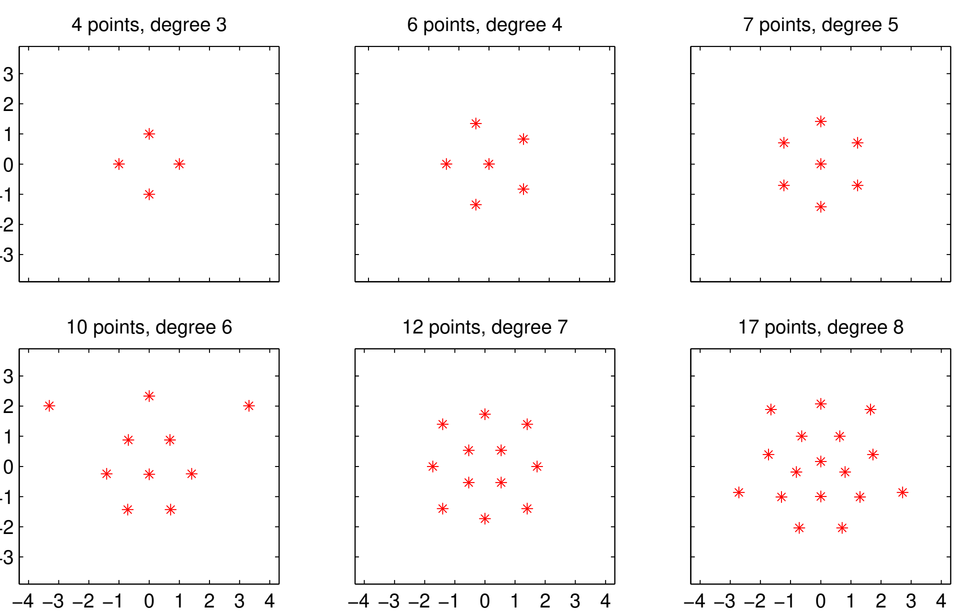

FIG. 6.1. *Points for $E_2^{r^2}$ rules. The rule of degree 8 is new.*

TABLE 4.5
22-point rules of degree 4 in 5 dimensions.

x_1	x_2	x_3	x_4	x_5	Weight	# Points
0	0	0	0	0	W_0	1
c	c	c	c	c	W_1	1
$(-h$	a	a	a	$a)_S$	W_1	5
$(-b$	$-b$	$-b$	g	$g)_S$	W_2	10
$(e$	$-f$	$-f$	$-f$	$-f)_S$	W_2	5

	$E_5^{r^2}$	E_5^r	S_5
a	$(2\sqrt{3}-\sqrt{2})/10$	$(4\sqrt{3}-2\sqrt{2})/5$	$(\sqrt{6}-1)/15$
b	$(2\sqrt{3}-\sqrt{2})/5$	$(8\sqrt{3}-4\sqrt{2})/5$	$(2\sqrt{6}-2)/15$
c	$\sqrt{1/2}$	$\sqrt{8}$	$1/3$
e	$(4\sqrt{3}-2\sqrt{2})/5$	$(16\sqrt{3}-8\sqrt{2})/5$	$(4\sqrt{6}-4)/15$
f	$(\sqrt{3}+2\sqrt{2})/5$	$(4\sqrt{3}+8\sqrt{2})/5$	$(\sqrt{6}+4)/15$
g	$(3\sqrt{3}+\sqrt{2})/5$	$(12\sqrt{3}+4\sqrt{2})/5$	$(3\sqrt{6}+2)/15$
h	$(8\sqrt{3}+\sqrt{2})/10$	$(16\sqrt{3}+2\sqrt{2})/5$	$(4\sqrt{6}+1)/15$
W_0	$\pi^{5/2}/4$	$28\pi^2$	$2\pi^2/105$
W_1	$\pi^{5/2}/18$	$8\pi^2/3$	$4\pi^2/105$
W_2	$\pi^{5/2}/36$	$4\pi^2/3$	$2\pi^2/105$
r_1	$\sqrt{5/2}$	$\sqrt{40}$	$\sqrt{5}/3$
r_2	2	8	$\sqrt{8}/3$

has 72 points. Each of those has 20 nearest neighbors and 30, 20, and 1 successively further away. The configuration of points is displayed in Table 7.1.

7.2. Degree 7, dimension 7, 183-point rules. In this case, we initialized a search with 226 points, and this new rule was found—the weights on the remaining 43 points having been driven to zero. It does not quite attain Möller's lower bound of $N = n/3(n^2 + 3n + 8) = 182$ for a degree-seven formula [25, 31].

The rule has a central point, one shell of 56 points, and a second shell of 126 points. The inner shell is laid out the same way as for the 57-point formula of degree 5 by Stroud [39, $E_n^{r^2}$:5-1]. The points on the outer shell have vertex symmetry, but we have been unable to relate them to a known polytope.

The points are given in Table 7.2. We found closed-form expressions for the points on the outer shell and for the radius r_1 of the inner shell directly from their simple continued fractions. We were then able to find expressions for the ratios of the remaining coordinates to r_1. Maxima was then able to solve for the coordinates using the expressions for the points and three of the moment constraint equations.

8. Degree-8 rules.

8.1. Degree 8, dimension 2, 17-point rule. We found 17-point rules of degree 8 for all three integrals with all positive weights and bilateral symmetry. For details, see the supplemental material. A 16-point rule of degree 8 for S_2 was found by Wissmann and Becker [43]. We were unable, even using variations of that rule as starting guesses, to find a similar rule for E_2^r or $E_2^{r^2}$.

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TABLE 4.6
28-point rules of degree 4 in 6 dimensions.

x_1	x_2	x_3	x_4	x_5	x_6	Weight	Radius	# Points
0	0	0	0	0	0	W_0	0	1
$-c$	$\pm e$	0	0	0	0	W_1	r	2
$-c$	0	$\pm(b$	b	b	$-b)_S$	W_1	r	8
a	$-b$	$\pm(b$	b	b	$b\ )$	W_1	r	2
a	$-b$	$(b$	b	$-b$	$-b)_S$	W_1	r	6
f	0	0	0	0	0	W_1	r	1
a	b	$\pm(e$	0	0	$0)_S$	W_1	r	8

	$E_6^{r^2}$	E_6^r	S_6
a	$1/2$	$\sqrt{9/2}$	$\sqrt{1/20}$
b	$\sqrt{3/4}$	$\sqrt{27/2}$	$\sqrt{3/20}$
c	1	$\sqrt{18}$	$\sqrt{1/5}$
e	$\sqrt{3}$	$\sqrt{54}$	$\sqrt{3/5}$
$f = r$	2	$\sqrt{72}$	$\sqrt{4/5}$
W_0	$\pi^3/4$	$50\pi^3$	$\pi^3/96$
W_1	$\pi^3/36$	$70\pi^3/27$	$5\pi^3/864$

TABLE 4.7
38-point rules of degree 4 in 7 dimensions.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Weight	# Points
0	0	0	0	0	0	0	W_0	1
c	c	c	c	c	c	c	$-W_2$	1
$-b$	$-b$	$-b$	$-b$	$-b$	$-b$	$-b$	$-W_1$	1
$(\ f$	$-e$	$-e$	$-e$	$-e$	$-e$	$-e)_S$	W_3	7
$(\ h$	a	a	a	a	a	$a)_S$	W_4	7
$(-i$	$-i$	g	g	g	g	$g)_S$	W_5	21

	$E_7^{r^2}$	S_7
a	0.2286166663871	0.0974824740891
b	0.2590817563916	0.1104728321147
c	0.3117777721419	0.1329424887288
e	0.4422503418055	0.1885761793629
f	0.4505846393780	0.1921299357884
g	0.7531484451994	0.3211435760773
h	1.0981884332902	0.4682691213418
i	1.8927504201541	0.8070714909185
W_0	59.8014451908073	5.2337832579847
W_1	89.9014937680773	9.4465413692728
W_2	79.9432767398149	8.4001659957515
W_3	11.6616239025637	1.2253635397056
W_4	11.0688850060780	1.1630805645052
W_5	0.2803313076587	0.0294562546617

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TABLE 5.1
23-point rules of degree 5 in 4 dimensions.

x_1	x_2	x_3	x_4	Weight	Radius	# Points
0	0	0	0	W_0	0	1
$\pm h$	0	0	0	W_2	r	2
0	$\pm h$	0	0	W_1	r	2
$\pm c$	$\pm(b$	$-a)$	$\pm c$	W_1	r	8
$\pm c$	$\pm(b$	$e)$	0	W_1	r	4
0	$\pm(a$	$-g)$	0	W_1	r	2
0	$\pm(a$	$b)$	$\pm f$	W_1	r	4

	$E_4^{r^2}$	E_4^r	S_4
a	$\sqrt{1/3}$	$\sqrt{14/3}$	$\sqrt{1/12}$
b	$\sqrt{2/3}$	$\sqrt{28/3}$	$\sqrt{1/6}$
c	1	$\sqrt{14}$	$\sqrt{1/4}$
e	$\sqrt{4/3}$	$\sqrt{56/3}$	$\sqrt{1/3}$
f	$\sqrt{2}$	$\sqrt{28}$	$\sqrt{1/2}$
g	$\sqrt{8/3}$	$\sqrt{112/3}$	$\sqrt{2/3}$
$h = r$	$\sqrt{3}$	$\sqrt{42}$	$\sqrt{3/4}$
W_0	$\pi^2/3$	$44\pi^2/7$	$\pi^2/18$
W_1	$\pi^2/32$	$15\pi^2/56$	$\pi^2/48$
W_2	$\pi^2/48$	$5\pi^2/28$	$\pi^2/72$

TABLE 5.2
44-point rule of degree 5 for E_6^r.

x_1	x_2	x_3	x_4	x_5	x_6	Weight	Radius	# Points
(0	0	0	0	0	$\pm b)_S$	W_1	r_1	12
$(a$	a	a	a	a	$-a)_S$	W_2	r_2	6
$(a$	a	a	$-a$	$-a$	$-a)_S$	W_2	r_2	20
$(a$	$-a$	$-a$	$-a$	$-a$	$-a)_S$	W_2	r_2	6

	E_6^r
a	4.84099298434420
$b = r_1$	5.40578920173885
W_1	274.495347525855
W_2	13.3377822289287
r_2	11.8579626600364

TABLE 6.1
10-point rule of degree 6 for $E_2^{r^2}$.

x_1	x_2	Weight	Radius
±3.314013565941806	2.014171295633760	0.000757833922865	3.87809
±1.411670545911536	-0.242569904073576	0.236161927729435	1.43236
±0.713033732783175	-1.432390280414699	0.146082553662775	1.60005
±0.691608815107559	0.877693534044218	0.485399260031153	1.11744
0.000000000000000	-0.261367769356158	1.387418367858287	0.26137
0.000000000000000	2.335832264987514	0.017371135039050	2.33583

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TABLE 6.2
11-point rule of degree 6 for E_2^r.

x_1	x_2	Weight	Radius
0.000000000000000	0.000000000000000	3.927702275194840	0.00000
0.000000000000000	10.299713185154499	0.003846684331349	10.29971
0.000000000000000	-3.895765525253948	0.474246212300936	3.89577
±10.311630315898372	3.397224688449697	0.002841012046587	10.85683
± 6.251012172182811	-8.794364006109971	0.002944454683352	10.78962
± 3.752487980256190	-1.228482827331175	0.460111970539923	3.94846
± 2.312667676618243	3.141828043257887	0.472797630406369	3.90122

9. Degree-9 rules.

9.1. Degree 9, dimension 4, 124-point rule. We found a 124-point rule for $E_4^{r^2}$ with negative weights (stability factor 15.4) and central symmetry but no central point. We also found a 125-point rule for the same integrals with central symmetry and a central point. It also has negative weights but a somewhat better stability factor of 8.1. For details, see the supplemental material.

10. Summary.

10.1. Listings. The new cubature rules are listed in Tables 10.1, 10.2, and 10.3. In addition to those described above, we found many rules with only bilateral symmetry or no apparent symmetry, the details for which appear only in the supplemental material. Symmetry of “x_2, x_3" indicates a rule closed under sign changes in both of the indicated coordinates. Rules with the symmetry of a known polytope are indicated by that polytope. “Vertex” indicates symmetry with respect to the exchange of any two noncentral points but that the polytope has not been identified.

The “Quality” of a rule is given using the notation introduced in [27]. The first letter is P if all weights are positive or N if some weights are negative. For the integral S_n, there is a second letter which is I if all points are inside the region, B if some are on the boundary, or O if some points are outside the region.

Also shown is the Möller lower bound (MLB) for the number of points in a rule of the given degree from (2.2) and the smallest known rule of the given degree or the next higher degree. The new rules with points supported by one or two spherical shells are very efficient—within three points of the Möller lower bound. Those with little or no symmetry are much less efficient with over 40 percent more points than the Möller lower bound in the median, though still better than the previously known rules with the exceptions noted in the tables.

In most odd-degree formulas, points are supported by a few spherical shells with all weights positive. Most even-degree formulas lack symmetry, and they have more negative weights. We were unable to find rules for E_n^r and sometimes even S_n corresponding to some of the rules for $E_n^{r^2}$.

10.2. Examples. To illustrate the formulas, we numerically evaluate an integral used as an example by Stroud [36]:

$$J_4 = \int_{S_4} \cos(x_1 + \cdots + x_4)\, dx_1 \cdots dx_4 = 3.4823322817\,. \tag{10.1}$$

The values calculated using our seven formulas of dimension 4 plus the 16-point formula of Mysovskih [7, 32], the 31-point formula of degree 5 by Meng and Luo [30], and Stroud’s formulas of degrees 5 [37] and 7 [38] are given in Table 10.4 ordered by N.

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TABLE 7.1
127-point rules of degree 7 in 6 dimensions.

x_1	x_2	x_3	x_4	x_5	x_6	Weight	Radius	# Points
0	0	0	0	0	0	W_0	0	1
$\pm g$	0	0	0	0	0	W_1	r_1	2
$\pm c$	$(\pm f$	0	0	0	$0\,)_S$	W_1	r_1	20
$\pm(a$	b	b	b	b	$b\ )$	W_1	r_1	2
$\pm(a$	$(b$	b	b	$-b$	$-b)_S)$	W_1	r_1	20
$\pm(a$	$(b$	$-b$	$-b$	$-b$	$-b)_S)$	W_1	r_1	10
$\pm(h$	$(e$	e	e	e	$-e)_S)$	W_2	r_2	10
$\pm(h$	$(e$	e	$-e$	$-e$	$-e)_S)$	W_2	r_2	20
$\pm(h$	$-e$	$-e$	$-e$	$-e$	$-e\ )$	W_2	r_2	2
0	$(\pm i$	$\pm i$	0	0	$0\,)_S$	W_2	r_2	40

	$E_6^{r^2}$	S_6
$g = r_1$	$\sqrt{(4-\sqrt{6})\times 2}$	$\sqrt{2/3}$
c	$\sqrt{(4-\sqrt{6})/2}$	$\sqrt{1/6}$
f	$\sqrt{(4-\sqrt{6})\times 3/2}$	$\sqrt{1/2}$
a	$\sqrt{(4-\sqrt{6})/8}$	$\sqrt{1/24}$
b	$\sqrt{(4-\sqrt{6})\times 3/8}$	$\sqrt{1/8}$
e	$\sqrt{(6+\sqrt{6})/8}$	$\sqrt{1/8}$
h	$\sqrt{(6+\sqrt{6})\times 3/8}$	$\sqrt{3/8}$
i	$\sqrt{(6+\sqrt{6})/2}$	$\sqrt{1/2}$
r_2	$\sqrt{6+\sqrt{6}}$	1
W_0	$(16-\sqrt{6})\pi^3/100$	$\pi^3/240$
W_1	$(68+27\sqrt{6})\pi^3/9000$	$\pi^3/480$
W_2	$(54-19\sqrt{6})\pi^3/9000$	$\pi^3/1440$

10.3. Supplemental material. The supplemental material[4] includes plain-text listings of the new rules in double and quad precision with 15 and 32 decimal digits, respectively. Some known rules are included for comparison with sources indicated in the double precision listings. The quad precision listings are of two sorts both generated by programs in Maxima. Where closed form expressions were found for the parameters of a rule, those expressions were evaluated with 64 digit precision and printed with 32 digit precision. Otherwise, a simple root-finder using Newton's method with Moore-Penrose pseudoinverses was used to refine the double precision rule with an excessive $32d + 10$ digits of precision with the goal that the printed values of both node coordinates and weights would be correct to 32 digits. In either case, the constraint equations to the stated degree were evaluated and the maximum error was printed. The error is zero whenever the parameters were expressed in closed form and Maxima was able to simplify the resulting equations. Otherwise the error is the result of an extended precision calculation.

[4]http://etna.ricam.oeaw.ac.at/volumes/2011-2020/vol51/addition/files.zip

TABLE 7.2
183-point rules of degree 7 in 7 dimensions.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Weight	Radius	# Points
0	0	0	0	0	0	0	W_0	0	1
$\pm(-m$	0	0	0	0	0	$0)$	W_1	r_1	2
$\pm(-c$	k	0	0	0	0	$0)$	W_1	r_1	2
$\pm(-c$	$-f$	$(\pm i$	0	0	0	$0)_S)$	W_1	r_1	20
$\pm(-c$	a	e	e	e	e	$e)$	W_1	r_1	2
$\pm(-c$	a	$(e$	e	e	$-e$	$-e)_S)$	W_1	r_1	20
$\pm(-c$	a	$(e$	$-e$	$-e$	$-e$	$-e)_S)$	W_1	r_1	10
$\pm(j$	p	0	0	0	0	$0)$	W_2	r_2	2
$\pm(j$	b	g	g	g	g	$g)$	W_2	r_2	2
$\pm(j$	b	$(g$	g	g	$-g$	$-g)_S)$	W_2	r_2	20
$\pm(j$	b	$(g$	$-g$	$-g$	$-g$	$-g)_S)$	W_2	r_2	10
$\pm(j$	$-h$	$(\pm o$	0	0	0	$0)_S)$	W_2	r_2	20
$\pm(0$	l	$(g$	g	g	g	$-g)_S)$	W_2	r_2	10
$\pm(0$	l	$(g$	g	$-g$	$-g$	$-g)_S)$	W_2	r_2	20
$\pm(0$	l	$-g$	$-g$	$-g$	$-g$	$-g)$	W_2	r_2	2
0	0	$(\pm o$	$\pm o$	0	0	$0)_S$	W_2	r_2	40

	$E_7^{r^2}$	S_7
$m = r_1$	$\sqrt{(9-4\sqrt{3})\times 3/2}$	$\sqrt{(117-4\sqrt{78})\times 3/377}$
c	$\sqrt{(9-4\sqrt{3})/6}$	$\sqrt{(117-4\sqrt{78})/1131}$
k	$\sqrt{(9-4\sqrt{3})\times 4/3}$	$\sqrt{(117-4\sqrt{78})\times 8/1131}$
f	$\sqrt{(9-4\sqrt{3})/3}$	$\sqrt{(117-4\sqrt{78})\times 2/1131}$
i	$\sqrt{9-4\sqrt{3}}$	$\sqrt{(117-4\sqrt{78})\times 2/377}$
a	$\sqrt{(9-4\sqrt{3})/12}$	$\sqrt{(117-4\sqrt{78})/2262}$
e	$\sqrt{(9-4\sqrt{3})/4}$	$\sqrt{(117-4\sqrt{78})/754}$
j	$\sqrt{(\sqrt{3}+6)/3}$	$\sqrt{(\sqrt{78}+78)/273}$
p	$\sqrt{(\sqrt{3}+6)\times 2/3}$	$\sqrt{(\sqrt{78}+78)\times 2/273}$
b	$\sqrt{(\sqrt{3}+6)/24}$	$\sqrt{(\sqrt{78}+78)/2184}$
g	$\sqrt{(\sqrt{3}+6)/8}$	$\sqrt{(\sqrt{78}+78)/728}$
h	$\sqrt{(\sqrt{3}+6)/6}$	$\sqrt{(\sqrt{78}+78)/546}$
o	$\sqrt{(\sqrt{3}+6)/2}$	$\sqrt{(\sqrt{78}+78)/182}$
l	$\sqrt{(\sqrt{3}+6)\times 3/8}$	$\sqrt{(\sqrt{78}+78)\times 3/728}$
r_2	$\sqrt{\sqrt{3}+6}$	$\sqrt{(\sqrt{78}+78)/91}$
W_0	$(144-35\sqrt{3})\,\pi^{7/2}/1089$	$(6912-7\times 2^{11/2}\sqrt{39})\,\pi^3/2264031$
W_1	$(675+388\sqrt{3})\,\pi^{7/2}/95832$	$(104598+1085\times 2^{7/2}\sqrt{39})\,\pi^3/124521705$
W_2	$(90-37\sqrt{3})\,\pi^{7/2}/23958$	$(101088-235\times 2^{9/2}\sqrt{39})\,\pi^3/124521705$

TABLE 10.1
*25 new cubature rules for $E_n^{r^2}$. A * indicates that a rule with fewer points was known.*

New Rule						N_{MLB}	Smallest Previous Rule		
n	N	d	Shells	Quality	Symmetry		N	d	Source
2	17	8		P	bilateral	15	18	9	[17]
2	24	10		N	bilateral	21	25	11	[16]
3	10	4		P	x_2, x_3	10	13	5	[40, VII]
3	22	6		P	bilateral	20	27	7	[39, $E_n^{r^2}$:7-1]
3	220	14		N	none	120	288	14	[39, $E_3^{r^2}$:14-1]
3	234	15		N	none	140	none		
4	16	4	1+10+5	P	4-simplex	15	22	5	[39, $E_n^{r^2}$:5-1]
4	16	4	1+6+9	P	x_1, x_3	15	22	5	[39, $E_n^{r^2}$:5-1]
4	23*	5	1+22	P	vertex	21	22	5	[39, $E_n^{r^2}$:5-1]
4	43	6		P	bilateral	35	49	7	[39, $E_n^{r^2}$:7-1]
4	105	8		N	none	70	193	9	[39, $E_n^{r^2}$:9-1]
4	124	9		N	central	91	193	9	[39, $E_n^{r^2}$:9-1]
4	125	9		N	central	91	193	9	[39, $E_n^{r^2}$:9-1]
4	213	10		N	none	126	417	11	[39, $E_n^{r^2}$:11-1]
5	22	4	1+6+15	P	5-simplex	21	32	5	[39, $E_n^{r^2}$:5-1]
5	80	6		P	none	56	83	7	[39, $E_n^{r^2}$:7-1]
5	224	8		N	none	126	395	9	[1, CUT8]
6	28	4	1+27	P	vertex	28	44	5	[39, $E_n^{r^2}$:5-1]
6	127	7	1+54+72	P	central	124	137	7	[39, $E_n^{r^2}$:7-1]
7	38	4	1+8+8+21	N	see text	36	57	5	[39, $E_n^{r^2}$:5-1]
7	183	7	1+56+126	P	central	182	227	7	[39, $E_n^{r^2}$:7-1]
8	339	6		N	none	165	705	7	[39, $E_n^{r^2}$:7-3]
9	76	4		P	none	55	111	5	[25, I]
10	96	4		P	none	66	133	5	[25, I]
11	119	4		N	none	78	157	5	[25, I]

Also included are several of the MATLAB/Octave and Maxima programs used to find these rules and to refine them to high precision.

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TABLE 10.2
21 new cubature rules for E_n^r.

New Rule						N_{MLB}	Smallest Previous Rule		
n	N	d	Shells	Quality	Symmetry		N	d	Source
2	11	6		P	bilateral	10	12	7	[40, VI]
2	17	8		P	bilateral	15	19	9	[17]
3	11	4		P	bilateral	10	13	5	[40, VII]
3	23	6		P	none	20	27	7	[39, E_n^r:7-1]
4	16	4	1+10+5	P	4-simplex	15	24	5	[39, E_n^r:5-2]
4	16	4	1+6+9	P	x_1, x_3	15	24	5	[39, E_n^r:5-2]
4	23	5	1+22	P	vertex	21	24	5	[39, E_n^r:5-2]
4	45	6		P	none	35	49	7	[39, E_n^r:7-1]
4	103	8		N	none	70	none		
4	154	9		N	none	91	none		
5	22	4	1+6+15	P	5-simplex	21	42	5	[39, E_n^r:5-2]
5	80	6		P	none	56	83	7	[39, E_n^r:7-1]
5	230	8		N	none	126	none		
6	28	4	1+27	P	vertex	28	57	5	[30]
6	44	5	12+32	P	central	43	57	5	[30]
7	46	4		P	none	36	99	5	[39, E_n^r:5-1]
7	223	6		P	none	120	227	7	[39, E_n^r:7-1]
8	59	4		P	none	45	129	5	[39, E_n^r:5-1]
9	78	4		P	none	55	163	5	[39, E_n^r:5-1]
10	107	4		P	none	66	201	5	[39, E_n^r:5-1]
11	133	4		P	none	78	243	5	[39, E_n^r:5-1]

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TABLE 10.3
*21 new cubature rules for S_n. A * indicates that a rule with fewer points was known.*

New Rule						N_{MLB}	Smallest Previous Rule		
n	N	d	Shells	Quality	Symmetry		N	d	Source
2	17*	8		PO	bilateral	15	16	8	[43]
2	23	10		PO	bilateral	21	25	11	[16]
3	10	4		PO	x_2, x_3	10	11	4	[3]
3	22	6		PO	bilateral	20	27	7	[39, S_n:7-1]
3	42	8		PO	none	35	45	8	[10]
4	16*	4	1+10+5	NO	4-simplex	15	16	4	[7, 32]
4	15	4	0+6+9	PB	x_1, x_3	15	16	4	[7, 32]
4	23*	5	1+22	PI	vertex	21	22	5	[39, S_n:5-1]
4	43	6		NO	bilateral	35	49	7	[39, S_n:7-1]
4	105	8		NO	none	70	193	9	[39, S_n:9-1]
4	147	9		NO	none	91	193	9	[39, S_n:9-1]
4	208	10		NO	bilateral	126	417	11	[39, S_n:11-1]
5	22	4	1+6+15	PI	5-simplex	21	32	5	[39, S_n:5-1]
5	80	6		NO	none	56	83	7	[39, S_n:7-1]
5	220	8		NO	none	126	421	9	[39, S_n:9-1]
6	28	4	1+27	PI	vertex	28	44	5	[39, S_n:5-1]
6	127	7	1+54+72	PB	central	124	137	7	[39, S_n:7-1]
7	38	4	1+8+8+21	NO	see text	36	57	5	[39, S_n:5-1]
7	183	7	1+56+126	PI	central	182	227	7	[39, S_n:7-1]
9	78	4		NO	none	55	163	5	[39, S_n:5-2]
10	96	4		NO	none	66	201	5	[39, S_n:5-2]
11	123	4		NO	none	78	243	5	[39, S_n:5-2]

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TABLE 10.4
Approximate values of J_4 in (10.1).

n	N	d	J_4 Estimates	Error	Source
4	15	4	3.4818127309	-0.0005195508	Table 4.4
4	16	4	3.4511488638	-0.0311834178	Table 4.3
4	16	4	3.4828928259	0.0005605442	[7, 32]
4	22	5	3.4403244866	-0.0420077951	[37],[39, S_n:5-1]
4	23	5	3.4838622252	0.0015299435	Table 5.1
4	31	5	3.4827186240	0.0003863423	[30]
4	43	6	3.4823547183	0.0000224367	Table 10.3
4	49	7	3.4823164472	-0.0000158345	[38],[39, S_n:7-1]
4	105	8	3.4823287423	-0.0000035394	Table 10.3
4	147	9	3.4823311982	-0.0000010835	Table 10.3
4	208	10	3.4823322804	-0.0000000012	Table 10.3

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