

# Electronic Transactions on Numerical Analysis

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### Contents

- 1 Bernstein fractal approximation and fractal full Müntz Theorems.  
*Vijender Nallapu.*

**Abstract.**

Fractal interpolation functions defined by means of suitable Iterated Function Systems provide a new framework for the approximation of continuous functions defined on a compact real interval. Convergence is one of the desirable properties of a good approximant. The goal of the present paper is to develop fractal approximants, namely Bernstein  $\alpha$ -fractal functions, which converge to the given continuous function even if the magnitude of the scaling factors does not approach zero. We use Bernstein  $\alpha$ -fractal functions to construct the sequence of Bernstein Müntz fractal polynomials that converges to either  $f \in \mathcal{C}(I)$  or  $f \in L^p(I)$ ,  $1 \leq p < \infty$ . This gives a fractal analogue of the full Müntz theorems in the aforementioned function spaces. For a given sequence  $\{f_n(x)\}_{n=1}^{\infty}$  of continuous functions that converges uniformly to a function  $f \in \mathcal{C}(I)$ , we develop a double sequence  $\{\{f_{n,l}^{\alpha}(x)\}_{l=1}^{\infty}\}_{n=1}^{\infty}$  of Bernstein  $\alpha$ -fractal functions that converges uniformly to  $f$ . By establishing suitable conditions on the scaling factors, we solve a constrained approximation problem of Bernstein  $\alpha$ -fractal Müntz polynomials. We also study the convergence of Bernstein fractal Chebyshev series.

**Key Words.**

Bernstein polynomials, fractal approximation, convergence, full Müntz theorems, Chebyshev series, box dimension.

**AMS Subject Classifications.**

41A30, 28A80, 41A17, 41A50.

- 15 Block-proximal methods with spatially adapted acceleration.  
*Tuomo Valkonen.*

**Abstract.**

We study and develop (stochastic) primal-dual block-coordinate descent methods for convex problems based on the method due to Chambolle and Pock. Our methods have known convergence rates for the iterates and the ergodic gap of  $O(1/N^2)$  if each block is strongly convex,  $O(1/N)$  if no convexity is present, and more generally a mixed rate  $O(1/N^2) + O(1/N)$  for strongly convex blocks if only some blocks are strongly convex. Additional novelties of our methods include blockwise-adapted step lengths and acceleration as well as the ability to update both the primal and dual variables randomly in blocks under a very light compatibility condition. In other words, these variants of our methods are doubly-stochastic. We test the proposed methods on various image processing problems, where we employ pixelwise-adapted acceleration.

**Key Words.**

PDHGM, Chambolle–Pock method, stochastic, doubly-stochastic, blockwise, primal-dual

**AMS Subject Classifications.**

49M29, 65K10, 65K15, 90C30, 90C47

- 50** Numerical treatment of singularly perturbed fourth-order two-parameter problems.  
*Mirjana Brdar, Sebastian Franz, and Hans-Grg Roos.*

**Abstract.**

A singularly perturbed fourth-order problem with two small parameters is considered in one dimension and on a smooth domain in two dimensions. Using its discretisation by a mixed finite element method on a properly defined Shishkin mesh, we prove convergence in the associated energy norm.

**Key Words.**

singular perturbation, fourth-order problem, two small parameters, mixed method, layer-adapted meshes

**AMS Subject Classifications.**

65L11, 65L60, 65N30, 65N50

- 63** A randomized multivariate matrix pencil method for superresolution microscopy.  
*Martin Ehler, Stefan Kunis, Thomas Peter, and Christian Richter.*

**Abstract.**

The matrix pencil method is an eigenvalue-based approach for the parameter identification of sparse exponential sums. We derive a reconstruction algorithm for multivariate exponential sums that is based on simultaneous diagonalization. Randomization is used and quantified to reduce the simultaneous diagonalization to the eigen-decomposition of a single random matrix. To verify feasibility, the algorithm is applied to synthetic and experimental fluorescence microscopy data.

**Key Words.**

frequency analysis, spectral analysis, exponential sum, moment problem, superresolution

**AMS Subject Classifications.**

65T40, 42C15, 30E05, 65F30

- 75** On the composite discontinuous Galerkin method for simulations of electric properties of semiconductor devices.

*Konrad Sakowski, Leszek Marcinkowski, Pawel Strak, Pawel Kempisty, and Stanislaw Krukowski.*

**Abstract.**

In this paper, a variant of the discretization of the van Roosbroeck equations in the equilibrium state with the composite discontinuous Galerkin method for the rectangular domain is discussed. It is based on the symmetric interior penalty Galerkin (SIPG) method. The proposed method accounts for lower regularity of the solution at the interfaces of the layers of the device. It is shown that the discrete problem is well defined and that the discrete solution is unique. Error estimates are derived. Finally, numerical simulations are presented.

**Key Words.**

composite discontinuous Galerkin method, drift-diffusion, van Roosbroeck equations

**AMS Subject Classifications.**

65N30, 65N15, 82D37

- 99** Simultaneous identification of volatility and interest rate functions—a two-parameter regularization approach.  
*Christopher Hofmann, Bernd Hofmann, and Alois Pichler.*

**Abstract.**

This paper investigates a specific ill-posed nonlinear inverse problem that arises in financial markets. Precisely, as a benchmark problem in the context of volatility surface calibration, we consider the simultaneous recovery of implied volatility and interest rate functions over a finite time interval from corresponding call- and put-price functions for idealized continuous families of European vanilla options over the same maturity interval. We prove identifiability of the pair of functions to be identified by showing injectivity of the forward operator in  $L^2$ -spaces. To overcome the ill-posedness we employ a two-parameter Tikhonov regularization with heuristic parameter choice rules and demonstrate chances and limitations by means of numerical case studies using synthetic data.

**Key Words.**

inverse option pricing, simultaneous identification, volatility, interest rate, regularization

**AMS Subject Classifications.**

65J20, 91G60, 47H30, 47A52

- 118** Revisiting aggregation-based multigrid for edge elements.  
*Artem Napov and Ronan Perrussel.*

**Abstract.**

We consider a modification of the Reitzinger-Schöberl algebraic multigrid method for the iterative solution of the curl-curl boundary value problem discretized with edge elements. The Reitzinger-Schöberl method is attractive for its low memory requirements and moderate cost per iteration, but the number of iterations typically tends to increase with the problem size. Here we propose several modifications to the method that aim at curing the size-dependent convergence behavior without significantly affecting the attractive features of the original method. The comparison with an auxiliary space preconditioner, a state-of-the-art solver for the considered problems, further indicates that both methods typically require a comparable amount of work to solve a given discretized problem but that the proposed approach requires less memory.

**Key Words.**

algebraic multigrid, edge elements, preconditioning, aggregation

**AMS Subject Classifications.**

65N12, 65N22, 65N55

- 135 A note on parallel preconditioning for all-at-once evolutionary PDEs.  
*Anthony Goddard and Andy Wathen.*

**Abstract.**

McDonald, Pestana, and Wathen [SIAM J. Sci. Comput., 40 (2018), pp. A1012–A1033] present a method for preconditioning time-dependent PDEs via an approximation by a nearby time-periodic problem, that is, they employ circulant-related matrices as preconditioners for the non-symmetric block Toeplitz matrices which arise from an all-at-once formulation. They suggest that such an approach might be efficiently implemented in parallel. In this short article, we present parallel numerical results for their preconditioner which exhibit strong scaling. We also extend their preconditioner via a Neumann series approach which also allows for efficient parallel execution. Results are shown for both parabolic and hyperbolic PDEs. Our simple implementation (in C++ and MPI) is available at the Git repository `PARALAAOMPI`.<sup>1</sup>

**Key Words.**

parallel-in-time, monolithic method, preconditioning

**AMS Subject Classifications.**

65M20, 65F08, 65Y05

- 151 Perturbation analysis for palindromic and anti-palindromic nonlinear eigenvalue problems.  
*Sk. Safique Ahmad.*

**Abstract.**

A structured backward error analysis for an approximate eigenpair of structured nonlinear matrix equations with *T-palindromic*, *H-palindromic*, *T-anti-palindromic*, and *H-anti-palindromic* structures is conducted. We construct a minimal structured perturbation in the Frobenius norm such that an approximate eigenpair becomes an exact eigenpair of an appropriately perturbed nonlinear matrix equation. The present work shows that our general framework extends existing results in the literature on the perturbation theory of matrix polynomials.

**Key Words.**

nonlinear eigenvalue problem, even and odd matrix polynomials, palindromic matrix polynomial

**AMS Subject Classifications.**

65F15, 15A18, 65F35, 15A12

- 169 Topological derivative for the nonlinear magnetostatic problem.  
*Samuel Amstutz and Peter Gangl.*

**Abstract.**

The topological derivative represents the sensitivity of a domain-dependent functional with respect to a local perturbation of the domain and is a valuable tool in topology optimization. Motivated by an application from electrical engineering, we derive the topological derivative for an optimization problem which is constrained by the quasilinear equation of two-dimensional magnetostatics. Here, the main ingredient is to establish a sufficiently fast decay of the variation of the direct state

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<sup>1</sup><https://github.com/anthonyjamesgoddard/PARALAAOMPI>

at scale 1 as  $|x| \rightarrow \infty$ . In order to apply the method in a bi-directional topology optimization algorithm, we derive both the sensitivity for introducing air inside ferromagnetic material and the sensitivity for introducing material inside an air region. We explicitly compute the arising polarization matrices and introduce a way to efficiently evaluate the obtained formulas. Finally, we employ the derived formulas in a level-set based topology optimization algorithm and apply it to the design optimization of an electric motor.

**Key Words.**

topological derivative, nonlinear magnetostatics, topology optimization, electrical machine

**AMS Subject Classifications.**

35J62, 49Q10, 49Q12, 78M35, 78M50

**219** Efficient cubature rules.

*James R. Van Zandt.*

**Abstract.**

67 new cubature rules are found for three standard multi-dimensional integrals with spherically symmetric regions and weight functions using direct search with a numerical zero-finder. 63 of the new rules have fewer integration points than known rules of the same degree, and 20 are within three points of Möller's lower bound. Most have all positive coefficients, and most have some symmetry, including some supported by one or two concentric spheres. They include degree-7 formulas for the integration over the sphere and Gaussian-weighted integrals over the entire space, each in 6 and 7 dimensions, with 127 and 183 points, respectively.

**Key Words.**

multiple integrals, Gaussian weight, cubature formula, integration rule, numerical integration, regular simplex

**AMS Subject Classifications.**

65D30, 65D32, 41A55, 41A63

**240** Approximate residual-minimizing shift parameters for the low-rank ADI iteration.

*Patrick Kürschner.*

**Abstract.**

The low-rank alternating directions implicit (LR-ADI) iteration is a frequently employed method for efficiently computing low-rank approximate solutions of large-scale Lyapunov equations. In order to achieve a rapid error reduction, the iteration requires shift parameters whose selection and generation is often a difficult task, especially for nonsymmetric matrices in the Lyapunov equation. This article represents a follow up of Benner et al. [Electron. Trans. Numer. Anal., 43 (20142015), pp. 142–162] and investigates self-generating shift parameters based on a minimization principle for the Lyapunov residual norm. Since the involved objective functions are too expensive to evaluate and hence intractable, objective functions are introduced which are efficiently constructed from the available data generated by the LR-ADI iteration. Several numerical experiments indicate that these residual-minimizing shifts using approximated objective functions outperform existing precomputed and dynamic shift parameter selection techniques, although their generation is more involved.

**Key Words.**

Lyapunov equation, alternating directions implicit, low-rank approximation, shift parameters

**AMS Subject Classifications.**

15A06, 65F10, 65F30

- 262**  $\gamma\Phi$ -type inclusion set for eigenvalues of a tensor.  
*Xiao-Qiang Lei.*

**Abstract.**

In this paper, a new  $\gamma\Phi$ -type eigenvalue inclusion set for tensors is given, and some inclusion relations between this new inclusion set and other ones are presented. In addition, a new sufficient criterion for identifying nonsingular tensors is also provided by using the new  $\gamma\Phi$ -type eigenvalue inclusion set. Some numerical results are reported to show the superiority of the results.

**Key Words.**

tensor, eigenvalue, inclusion, nonsingular,  $\gamma\Phi$ -type.

**AMS Subject Classifications.**

15A69, 15A18, 65F15, 65H17, 15A15, 65F40

- 274** Bouligand-Levenberg-Marquardt iteration for a non-smooth ill-posed inverse problem.  
*Christian Clason and Vu Huu Nhu.*

**Abstract.**

In this paper, we consider a modified Levenberg-Marquardt method for solving an ill-posed inverse problem where the forward mapping is not Gateaux differentiable. By relaxing the standard assumptions for the classical smooth setting, we derive asymptotic stability estimates which are then used to prove convergence of the proposed method. This method can be applied to an inverse source problem for a non-smooth semilinear elliptic PDE where a Bouligand subdifferential can be used in place of the non-existing Frchet derivative, and we show that the corresponding *Bouligand-Levenberg-Marquardt iteration* is an iterative regularization scheme. Numerical examples illustrate the advantage over the corresponding Bouligand-Landweber iteration.

**Key Words.**

inverse problem, iterative regularization, Levenberg-Marquardt method, non-smooth equation

**AMS Subject Classifications.**

49K20, 49K40, 90C31

- 315** Analysis of the truncation error and barrier-function technique for a Bakhvalov-type mesh.  
*Thái Anh Nhan and Relja Vulanović.*

**Abstract.**

We use a barrier-function technique to prove the parameter-uniform convergence for singularly perturbed convection-diffusion problems discretized on a Bakhvalov-type

mesh. This is the first proof of this kind in the research literature, the barrier-function approach having only been applied so far to Shishkin-type meshes.

**Key Words.**

singular perturbation, convection-diffusion, boundary-value problem, Bakhvalov-type meshes, finite differences, uniform convergence

**AMS Subject Classifications.**

65L10, 65L12, 65L20, 65L70

- 331** Preconditioned gradient iterations for the eigenproblem of definite matrix pairs.

*Marija Miloloža Pandur.*

**Abstract.**

Preconditioned gradient iterations for large and sparse Hermitian generalized eigenvalue problems  $Ax = \lambda Bx$ , with positive definite  $B$ , are efficient methods for computing a few extremal eigenpairs. In this paper we give a unifying framework of preconditioned gradient iterations for definite generalized eigenvalue problems with indefinite  $B$ . More precisely, these iterations compute a few eigenvalues closest to the definiteness interval, which can be in the middle of the spectrum, and the corresponding eigenvectors of definite matrix pairs  $(A, B)$ , that is, pairs having a positive definite linear combination. Sharp convergence theorems for the simplest variants are given. This framework includes an indefinite locally optimal block preconditioned conjugate gradient (LOBPCG) algorithm derived by Kressner, Miloloža Pandur, and Shao [Numer. Algorithms, 66 (2014), pp. 681–703]. We also give a generic algorithm for constructing new “indefinite extensions” of standard (with positive definite  $B$ ) eigensolvers. Numerical experiments demonstrate the use of our algorithm for solving a product and a hyperbolic quadratic eigenvalue problem. With excellent preconditioners, the indefinite variant of LOBPCG is the most efficient method. Finally, we derive some ideas on how to use our indefinite eigensolver to compute a few eigenvalues around *any* spectral gap and the corresponding eigenvectors of definite matrix pairs.

**Key Words.**

eigenpair, definite matrix pair, definitizing shift, definiteness interval, spectral gap, preconditioned steepest descent/ascent iteration, indefinite LOBPCG

**AMS Subject Classifications.**

65F15, 65F08, 65F50

- 363** On the construction of real non-selfadjoint tridiagonal matrices with prescribed three spectra.

*Wei-Ru Xu, Natália Bebiano, and Guo-Liang Chen.*

**Abstract.**

Non-selfadjoint tridiagonal matrices play a role in the discretization and truncation of the Schrödinger equation in some extensions of quantum mechanics, a research field particularly active in the last two decades. In this article, we consider an inverse eigenvalue problem that consists of the reconstruction of such a real non-selfadjoint matrix from its prescribed eigenvalues and those of two complementary principal submatrices. Necessary and sufficient conditions under which the problem has a solution are presented, and uniqueness is discussed. The reconstruction is performed by using a modified unsymmetric Lanczos algorithm, designed to solve the proposed

inverse eigenvalue problem. Some illustrative numerical examples are given to test the efficiency and feasibility of our reconstruction algorithm.

**Key Words.**

inverse eigenvalue problem, non-selfadjoint tridiagonal matrix, modified unsymmetric Lanczos algorithm, spectral data

**AMS Subject Classifications.**

65F18, 65F15, 15A18, 15A29

- 387** Algebraic analysis of two-level multigrid methods for edge elements.  
*Artem Napov and Ronan Perrussel.*

**Abstract.**

We present an algebraic analysis of two-level multigrid methods for the solution of linear systems arising from the discretization of the curl-curl boundary value problem with edge elements. The analysis is restricted to the singular compatible linear systems as obtained by setting to zero the contribution of the lowest order (mass) term in the associated partial differential equation. We use the analysis to show that for some discrete curl-curl problems, the convergence rate of some Reitzinger-Schöberl two-level multigrid variants is bounded independently of the mesh size and the problem peculiarities. This covers some discretizations on Cartesian grids, including problems with isotropic coefficients, anisotropic coefficients and/or stretched grids, and jumps in the coefficients, but also the discretizations on uniform unstructured simplex grids.

**Key Words.**

convergence analysis, multigrid, algebraic multigrid, two-level multigrid, Reitzinger-Schöberl multigrid, preconditioning, aggregation, edge elements

**AMS Subject Classifications.**

65N55, 65N12, 65N22, 35Q60

- 412** Augmented GMRES-type versus CGNE methods for the solution of linear ill-posed problems.  
*Andreas Neubauer.*

**Abstract.**

In this paper we compare (augmented) GMRES-type methods and (augmented) CGNE methods. The numerical results show that the CGNE method is more robust and suitable for ill-posed problems with a much higher degree of ill-posedness. GMRES-type methods only yield useful results for very moderately ill-posed problems.

**Key Words.**

(augmented) CGNE and GMRES-type methods

**AMS Subject Classifications.**

65J20, 65R20, 65R30

- 432** Preconditioning the coarse problem of BDDC methods—Three-level, Algebraic Multigrid, and vertex-based preconditioners.  
*Axel Klawonn, Martin Lanser, Oliver Rheinbach, and Janine Weber.*

**Abstract.**

A comparison of three Balancing Domain Decomposition by Constraints (BDDC) methods with an approximate coarse space solver using the same software building blocks is attempted for the first time. The comparison is made for a BDDC method with an algebraic multigrid preconditioner for the coarse problem, a three-level BDDC method, and a BDDC method with a vertex-based coarse preconditioner. It is new that all methods are presented and discussed in a common framework. Condition number bounds are provided for all approaches. All methods are implemented in a common highly parallel scalable BDDC software package based on PETSc to allow for a simple and meaningful comparison. Numerical results showing the parallel scalability are presented for the equations of linear elasticity. For the first time, this includes parallel scalability tests for a vertex-based approximate BDDC method.

**Key Words.**

approximate BDDC, three-level BDDC, multilevel BDDC, vertex-based BDDC

**AMS Subject Classifications.**

68W10, 65N22, 65N55, 65F08, 65F10, 65Y05

**451** Biorthogonal rational Krylov subspace methods.

*Niel Van Buggenhout, Marc Van Barel, and Raf Vandebril.*

**Abstract.**

A general framework for oblique projections of non-Hermitian matrices onto rational Krylov subspaces is developed. To obtain this framework we revisit the classical rational Krylov subspace algorithm and prove that the projected matrix can be written efficiently as a structured pencil, where the structure can take several forms such as Hessenberg or inverse Hessenberg. One specific instance of the structures appearing in this framework for oblique projections is a tridiagonal pencil. This is a direct generalization of the classical biorthogonal Krylov subspace method, where the projection becomes a single non-Hermitian tridiagonal matrix and of the Hessenberg pencil representation for rational Krylov subspaces. Based on the compact storage of this tridiagonal pencil in the biorthogonal setting, we can develop short recurrences. Numerical experiments confirm the validity of the approach.

**Key Words.**

rational Krylov, biorthogonal, short recurrence, oblique projection, matrix pencil

**AMS Subject Classifications.**

15A22, 47A75, 65F99, 65Q30

**469** Flip-Flop Spectrum-Revealing QR Factorization and its applications to singular value decomposition.

*Yuehua Feng, Jianwei Xiao, and Ming Gu.*

**Abstract.**

We present the Flip-Flop Spectrum-Revealing QR (Flip-Flop SRQR) factorization, a significantly faster and more reliable variant of the QLP factorization of Stewart for low-rank matrix approximations. Flip-Flop SRQR uses SRQR factorization to initialize a partial column-pivoted QR factorization and then computes a partial LQ factorization. As observed by Stewart in his original QLP work, Flip-Flop SRQR tracks the exact singular values with “considerable fidelity”. We develop singular

value lower bounds and residual error upper bounds for the Flip-Flop SRQR factorization. In situations where singular values of the input matrix decay relatively quickly, the low-rank approximation computed by Flip-Flop SRQR is guaranteed to be as accurate as the truncated SVD. We also perform a complexity analysis to show that Flip-Flop SRQR is faster than the randomized subspace iteration for approximating the SVD, the standard method used in the Matlab tensor toolbox. We additionally compare Flip-Flop SRQR with alternatives on two applications, a tensor approximation and a nuclear norm minimization, to demonstrate its efficiency and effectiveness.

**Key Words.**

QR factorization, randomized algorithm, low-rank approximation, approximate SVD, higher-order SVD, nuclear norm minimization

**AMS Subject Classifications.**

15A18, 15A23, 65F99

- 495** Preconditioned global Krylov subspace methods for solving saddle point problems with multiple right-hand sides.

*A. Badahmane, A. H. Bentbib, and H. Sadok.*

**Abstract.**

In the present paper, we propose a preconditioned global approach as a new strategy to solve linear systems with several right-hand sides coming from saddle point problems. The preconditioner is obtained by replacing a (2,2)-block in the original saddle-point matrix  $A$  by another well-chosen block. We apply the global GMRES method to solve this new problem with several right-hand sides and give some convergence results. Moreover, we analyze the eigenvalue distribution and the eigenvectors of the proposed preconditioner when the first block is positive definite. We also compare different preconditioned global Krylov subspace algorithms (CG, MINRES, FGMRES, GMRES) with preconditioned block (CG, GMRES) algorithms. Numerical results show that our preconditioned global GMRES method is competitive with other preconditioned global Krylov subspace and preconditioned block Krylov subspace methods for solving saddle point problems with several right-hand sides.

**Key Words.**

global Krylov subspace method, GMRES, MINRES, CG, preconditioner, saddle point problem

**AMS Subject Classifications.**

65F10, 65N22, 65F50

- 512** Adaptive Multilevel Krylov Methods.

*René Kehl, Reinhard Nabben, and Daniel B. Szyld.*

**Abstract.**

Inexact (variable) preconditioning of Multilevel Krylov methods (MK methods) for the solution of linear systems of equations is considered. MK methods approximate the solution of the local systems on a subspace using a few, but fixed, number of iteration steps of a preconditioned flexible Krylov method. In this paper, using the philosophy of inexact Krylov subspace methods, we use a theoretically-derived criterion to choose the number of iterations needed on each level to achieve a desired

tolerance. We use this criterion on one level and obtain an improved MK method. Inspired by these results, a second ad hoc method is also explored. Numerical experiments for the Poisson, Helmholtz, and the convection-diffusion equations illustrate the efficiency and robustness of this adaptive Multilevel Krylov method.

**Key Words.**

Multilevel Krylov methods, flexible GMRES, inexact Krylov subspace methods, inexact preconditioning

**AMS Subject Classifications.**

65F10, 65F50, 65N22, 65N55

- 529** Thick restarting the weighted harmonic Golub-Kahan-Lanczos algorithm for the linear response eigenvalue problem.  
*Hong-Xiu Zhong and Guo-Liang Chen.*

**Abstract.**

In this paper, we propose a weighted harmonic Golub-Kahan-Lanczos algorithm for the linear response eigenvalue problem (LREP). Convergence properties are established for the error bounds of the approximate eigenpairs. Moreover, we consider a practical thick-restart procedure to reduce the computational and memory costs and present a weighted harmonic Golub-Kahan-Lanczos algorithm with deflated restarting. Numerical tests show the efficiency of our new algorithms.

**Key Words.**

linear response eigenvalue problem, harmonic Rayleigh-Ritz projection, weighted Golub-Kahan-Lanczos algorithm, thick restart

**AMS Subject Classifications.**

65F15, 65F50, 15A18

- 547** Laminar-turbulent transition in channel flow: wall effects and critical Reynolds number.  
*Hidesada Kanda.*

**Abstract.**

This article describes a possible cause of natural laminar-turbulent transition in channel flow, and the minimum critical Reynolds number  $R_{c,\min}$  is determined. It is assumed that the mechanism of transition is the same for both circular pipe flow and channel flow since each flow has its own minimum critical Reynolds number. Our starting points are that under natural disturbance conditions, transition appears to take place only in the developing entrance region and that the critical Reynolds number  $R_c$  becomes  $R_{c,\min}$  when using a sharp-edged uniform channel. In our previous studies of circular pipe flow, we have developed a model for transition and obtained  $R_{c,\min} = 1910$  and  $1950$  in two mesh systems. In this study, for channel flow, the above transition model is verified by obtaining  $R_{c,\min} = 1190$  and  $1260$  in two mesh systems.

**Key Words.**

hydrodynamic stability, mesh refinement, thermodynamics

**AMS Subject Classifications.**

76E05, 65M50, 80A05