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A SIMPLIFICATION OF THE STATIONARY PHASE METHOD: APPLICATION TO THE ANGER AND WEBER FUNCTIONS*

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Abstract. The main difficulty in the practical use of the stationary phase method in asymptotic expansions of integrals is originated by a change of variables. The coefficients of the asymptotic expansion are the coefficients of the Taylor expansion of a certain function implicitly defined by that change of variables. In general, this function is not explicitly known, and then the computation of those coefficients is cumbersome. Using the factorization of the exponential factor used in previous works of [Tricomi, 1950], [Erdélyi and Wyman, 1963], and [Dingle, 1973], we obtain a variant of the method that avoids that change of variables and simplifies the computations. On the one hand, the calculation of the coefficients of the asymptotic expansion is remarkably simpler and explicit. On the other hand, the asymptotic sequence is as simple as in the standard stationary phase method: inverse powers of the asymptotic variable. New asymptotic expansions of the Anger and Weber functions $\mathbf{J}_{\lambda x}(x)$ and $\mathbf{E}_{\lambda x}(x)$ for large positive x and real parameter $\lambda \neq 0$ are given as an illustration.

Key words. asymptotic expansions, oscillatory integrals, method of the stationary phase, Anger and Weber functions

AMS subject classifications. 33B15, 41A60

1. Introduction.

We consider integrals of the form

$$F(x) := \int_a^b e^{i\,x f(t)} g(t) dt, \tag{1.1}$$

where (a, b) is a real interval (finite or infinite), x is a large positive parameter and the real functions $f(t)$ and $g(t)$ are smooth enough in (a, b). Long ago, Stokes and Kelvin [6, 12] made the observation that the major contribution to the value of the integral (1.1) comes from the neighborhoods of the end points of the interval (a, b) and from the neighborhoods of those points at which $f(t)$ is stationary, that is, $f'(t) = 0$. It is worth noting that, at the first order of the asymptotic approximation, the contribution of stationary points, if any, dominates the contribution of the end points.

If $f(t)$ has no stationary points in the interval (a, b), only the end points contribute to the asymptotic expansion of (1.1), which can be obtained by integrating by parts [14, Chap. 2, Sec. 3]:

$$F(x) = \int_a^b e^{i\,x f(t)} g(t) dt = \frac{e^{i\,x f(b)}}{i\,x f'(b)} g(b) - \frac{e^{i\,x f(a)}}{i\,x f'(a)} g(a) + \frac{1}{i\,x} \int_a^b e^{i\,x f(t)} g_1(t) dt, \tag{1.2}$$

where,

$$g_1(t) := -\frac{d}{dt}\left(\frac{g(t)}{f'(t)}\right).$$

The last integral in the right-hand side of (1.2) is of the same form as that in the left-hand side. Then, repeating this procedure K times we obtain

$$F(x) = \frac{e^{i\,x f(b)}}{i\,x f'(b)} \sum_{k=0}^{K} \frac{g_k(b)}{(i\,x)^k} - \frac{e^{i\,x f(a)}}{i\,x f'(a)} \sum_{k=0}^{K} \frac{g_k(a)}{(i\,x)^k} + \frac{1}{(i\,x)^{K+1}} \int_a^b e^{i\,x f(t)} g_{K+1}(t) dt, \tag{1.3}$$

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ETNA
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with $g_0(t) := g(t)$ and

$$g_{k+1}(t) := -\frac{d}{dt}\left(\frac{g_k(t)}{f'(t)}\right), \qquad k = 0, 1, 2, \ldots$$

Now, assume that $f(t)$ has one or more stationary points in $[a, b]$ of the first order. By subdividing the interval of integration if necessary, we may assume that $f(t)$ has only one stationary point in the integration interval and that it occurs at the lower limit $t = a$; that is, $f'(a) = 0$, $f''(a) \neq 0$. We may also assume that $f(t)$ is strictly increasing in (a, b). Then, the integral (1.1) can be transformed into a standard integral by using the change of variable [14, Chap. 2, Sec. 3]

$$f(t) - f(a) = u^2, \qquad \text{sign}(u) = \text{sign}(t - a). \tag{1.4}$$

The integral $F(x)$ may be written in the form

$$F(x) = e^{i\,x f(a)} \int_0^U e^{i\,x\,u^2} h(u)du, \qquad h(u) = g(t)\frac{dt}{du}, \tag{1.5}$$

where U is a positive parameter which follows from the change of variable. In order to derive the asymptotic expansion of $F(x)$ for large x, it is convenient to rewrite $F(x)$ in the form

$$F(x) = e^{i\,x f(a)} \left(\int_0^\infty e^{i\,x\,u^2} h(u)du - \int_U^\infty e^{i\,x\,u^2} h(u)du\right). \tag{1.6}$$

The approximation of the first integral in the right-hand side requires the assumption that $h(u)$ has a Taylor expansion at $u = 0$: $h(u) = \sum_{n=0}^\infty c_n\, u^n$. When we replace this expansion in (1.6) and interchange sum and integral (see [14, Chap. 2, p. 77-81] for more details) we obtain:

$$\int_0^\infty e^{i\,x\,s^2} h(s)ds \sim \sum_{n=0}^\infty c_n \int_0^\infty e^{i\,x\,u^2}\, u^n\, du = \sum_{n=0}^\infty \Gamma\left(\frac{n+1}{2}\right) \frac{c_n}{2(-i\,x)^{\frac{n+1}{2}}}. \tag{1.7}$$

The phase function in the second integral in the right-hand side of (1.6) has not any stationary point in $[U, \infty)$. Therefore, its asymptotic expansion follows straightforwardly from formula (1.3) replacing a by U and setting $b = \infty$.

From a theoretical point of view, the problem of the derivation of an asymptotic expansion of $F(x)$ is solved. But from a practical point of view the situation is different. When the phase function $f(t)$ in (1.1) has no stationary points in $[a, b]$, the asymptotic expansion of $F(x)$ is given in formula (1.3); the computation of the terms of (1.3) is straightforward and the problem is over. But when $f(t)$ has stationary points, the computation of the asymptotic expansion (1.7) is not straightforward. In general, the computation of the coefficients c_n is quite complicated, depending on the difficulty of the change of variable $t \to u$. This is so because the coefficients c_n are the Taylor coefficients at $u = 0$ of the function $h(u) = g(t)\frac{dt}{du}$, which is defined in an implicit form because, in general, the function $t(u)$ is not explicitly known. In fact, traditional text books of asymptotic expansions of integrals like for example [1], [4] or [14] do not give an explicit and general analytic formula for these coefficients.

In the classical Laplace and saddle point methods, the difficulty of the computation of the coefficients of the expansion is also a drawback for the same reason: a change of the integration variable is defined in an implicit way. In previous papers [7, 8], inspired by the ideas of [3, p. 113], [5], and [13], we circumvented this problem by designing modified Laplace and saddle point methods that avoid the change of variable. In this way, these modified methods give asymptotic expansions where the coefficients are computed explicitly at any order of the approximation without complicating the computation of the asymptotic sequence. In this paper we pursue the same objective for the stationary phase method for the integral (1.1) when $f(t)$ has stationary points in $[a,b]$. In the remainder of the paper we assume (without loss of generality as we have argued above) that $f(t)$ has only one stationary point at $t=a$.

In the following section we specify with precision the conditions for the functions $f(t)$ and $g(t)$ in the integrand of (1.1) and establish some preliminary results. In Section 3 we introduce the modified method and summarize the discussion in Theorem 3.3. In Section 4 we use Theorem 3.3 to derive new asymptotic expansions of the Anger and Weber functions $\mathbf{J}_\nu(x)$ and $\mathbf{E}_\nu(x)$ for large index ν and argument x.

2. Preliminaries. We let the function $g(t)$ possess, perhaps, an algebraic branch point at $t=a$, that is, $g(t)=(t-a)^{s-1}\bar{g}(t)$, where $s\in(0,1]$ and $\bar{g}(t)$ is analytic at $t=a$. We also assume that the phase function $f(t)$ has a Taylor expansion at $t=a$. These analyticity conditions for $f(t)$ and $\bar{g}(t)$ may be relaxed and require only that both, $f(t)$ and $\bar{g}(t)$, have an asymptotic expansion at $t=a$. But for the sake of clarity in the exposition we require their analyticity; on the other hand, it is the usual situation in most practical examples. We also require that both, $f(t)$ and $\bar{g}(t)$, are infinitely differentiable in $[a,b]$ (or $[a,b)$ if $b=\infty$).

In principle, the functions $f(t)$ and $\bar{g}(t)$ are defined only in $[a,b]$. As in the classical method of the stationary phase, the modified method that we present here requires the extension of the functions $f(t)$ and $\bar{g}(t)$ to infinite differentiable functions defined in $[a,\infty)$ with $f(t)\equiv 0$ and $\bar{g}(t)\equiv 0$ in a neighborhood of infinity. As it is argued in [14, Chap. 2, Sec. 3], the explicit extension is not required; a construction of this extension is detailed in [11, p. 418]. Also, as in the classical method, we extend the integration interval in (1.1) up to the infinity by writing $F(x)=F_1(x)-F_2(x)$, with

$$F_1(x):=\int_a^\infty e^{i\,xf(t)}(t-a)^{s-1}\bar{g}(t)dt,\qquad F_2(x):=\int_b^\infty e^{i\,xf(t)}g(t)dt. \tag{2.1}$$

Eventually, when $b=\infty$, $F_2(x)=0$ and the extensions of the functions $f(t)$ and $\bar{g}(t)$ is not necessary. In any case, as we have explained in the introduction, all the difficulty is encoded in the approximation of $F_1(x)$, as the asymptotic approximation of $F_2(x)$ follows easily from (1.3). Now, the key point that makes the difference with respect to the standard stationary phase method is that we do not require any change of variable for the first integral $F_1(x)$. Instead, we divide the phase function $f(t)$ into a “main part":

$$f_m(t):=f(a)+\frac{f^{(m)}(a)}{m!}(t-a)^m$$

and a “secondary part" $f_p(t):=f(t)-f_m(t)$, where the integer m is the order of the first non-vanishing derivative of $f(t)$ at $t=a$ and p is the order of the next non-vanishing derivative. The “main part" $f_m(t)$ and $f(t)$ have the same asymptotic behavior at $t=a$: apart from the constant term $f(a)$, both behave as $(t-a)^m$, and this determines the asymptotic behavior of $F_1(x)$ for large x. Then, roughly speaking, the idea is the following: we will leave only $f_m(t)$ in the exponent of the integral $F_1(x)$ and will attach the exponential of $ixf_p(t)$ to the function $\bar{g}(t)$:

$$F_1(x)=e^{i\,xf(a)}\int_a^\infty e^{i\,x\frac{f^{(m)}(a)}{m!}(t-a)^m}(t-a)^{s-1}h(t,x)dt,\quad h(t,x):=e^{i\,xf_p(t)}\bar{g}(t). \tag{2.2}$$

Now, as a difference with respect to the classical method, the function that multiplies the exponential, $h(t,x)$, depends also on the asymptotic variable x. This technical complication,

ETNA
Kent State University and
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conveniently managed, does not distort the asymptotic analysis of $F_1(x)$. The key point is the following: the "new" phase function in the integral (2.2) is just a power of $t-a$, and then a change of variable of the form (1.4) is not required. The remaining steps are similar to those of the classical method. We develop these ideas in detail in the following section.

3. The modified stationary phase method.

3.1. The asymptotic analysis of $F_1(x)$. The derivation of the asymptotic expansion of $F_1(x)$ by using the classical stationary phase method requires the Taylor expansion of the function $h(u)$ in (1.5) at $u=0$. Analogously, we require here the Taylor expansion of $h(t,x)$ at $t=a$. We may derive this expansion from the Taylor expansions of its factors $e^{i\,x f_p(t)}$ and $\bar{g}(t)$. Therefore, we need to compute the coefficients $A_n(x)$ and B_n of the respective Taylor expansion of $e^{i\,x f_p(t)}$ and $\bar{g}(t)$ at $t=a$,

$$e^{i\,x f_p(t)} = \sum_{n=0}^{\infty} A_n(x)(t-a)^n, \qquad \bar{g}(t) = \sum_{n=0}^{\infty} B_n(t-a)^n.$$

Then, the coefficients $a_n(x)$ of the Taylor expansion of the function $h(t,x)$ at $t=a$,

$$h(t,x) = e^{i\,x f_p(t)}\bar{g}(t) = \sum_{n=0}^{N-1} a_n(x)(t-a)^n + h_N(t,x), \tag{3.1}$$

where $h_N(t,x)$ is the Taylor remainder, may be computed in the form

$$a_n(x) = \sum_{k=0}^{n} A_k(x)B_{n-k}.$$

REMARK 3.1. Consider the coefficients of the Taylor expansion of $f_p(t)$ at $t=a$, $f_p(t) = \sum_{n=p}^{\infty} \frac{f^{(n)}(a)}{n!}(t-a)^n$. Then, the coefficients $A_n(x)$ of the Taylor expansion of $e^{i\,x f_p(t)}$ at $t=a$ may be computed in terms of $f^{(n)}(a)$ by using the Faá di Bruno's formula [2]:

$$A_n(x) = \sum_{k=0}^{n} \frac{b_{n,k}}{n!}(ix)^k,$$

where $b_{n,k}$ are the partial ordinary Bell polynomials [10, p. 190]. They may be computed recursively in the following form [10, p. 190]: $b_{0,0}=1$; $b_{n,0}=0$, $n=1,2,3,...$; and

$$b_{n,k} = \sum_{j=p}^{n-k+1} \binom{n-1}{j-1} f^{(j)}(a)\; b_{n-j,k-1}, \qquad n=1,2,3,..., \quad k=1,2,3,...,n,$$

where empty sums must be understood as zero.

REMARK 3.2. It is straightforward to show that the coefficients $a_n(x)$ are polynomials in the variable x of degree $\lfloor n/p \rfloor$ (see [7] for a detailed proof in a similar situation). Therefore,

$$a_n(x) = \mathcal{O}\left(x^{\lfloor n/p \rfloor}\right) \quad \text{as} \quad x\to\infty.$$

When we replace the expansion (3.1) in (2.2) and interchange sum and integral we obtain:

$$F_1(x) = e^{i\,x f(a)}\left(\sum_{n=0}^{N-1} a_n(x)\Phi_n(x) + R_N(x)\right), \tag{3.2}$$

with

$$\Phi_n(x) := \int_a^\infty e^{i\,x\,\frac{f^{(m)}(a)}{m!}(t-a)^m}(t-a)^{n+s-1}dt = \int_0^\infty e^{i\,x\,\frac{f^{(m)}(a)}{m!}t^m}t^{n+s-1}dt \tag{3.3}$$

and

$$\begin{aligned} R_N(x) &:= \int_a^\infty e^{i\,x\,\frac{f^{(m)}(a)}{m!}(t-a)^m}(t-a)^{s-1}\,h_N(t,x)dt \\ &= \int_0^\infty e^{i\,x\,\frac{f^{(m)}(a)}{m!}t^m}t^{s-1}\,h_N(t+a,x)dt. \end{aligned} \tag{3.4}$$

We may compute the integral (3.3) by using the Cauchy's residue theorem: we replace the integration path $(0,\infty)$ in (3.3) by the straight line $\Gamma_+ := \{t = ue^{i\frac{\pi}{2m}}: 0 < u < \infty\}$ if $f^{(m)}(a) > 0$ or by the straight line $\Gamma_- := \{t = ue^{-i\frac{\pi}{2m}}: 0 < u < \infty\}$ if $f^{(m)}(a) < 0$. In any case we obtain:

$$\Phi_n(x) = \int_0^{\infty\, e^{\pm i\frac{\pi}{2m}}} e^{i\,x\,\frac{f^{(m)}(a)}{m!}t^m}t^{n+s-1}dt = \frac{1}{m}\left(\frac{i\,m!}{f^{(m)}(a)x}\right)^{(n+s)/m}\Gamma\left(\frac{n+s}{m}\right). \tag{3.5}$$

Obviously, we cannot compute exactly the integral (3.4), but we can determine its asymptotic behavior for large x. For this purpose we define inductively the sequence of functions $K_n(t)$ in the following way [14, Chap. 2, Sec. 3]:

$$K_{n+1}(t) := -\int_t^{t+\infty e^{i\,\alpha\,\frac{\pi}{2m}}} K_n(u)\,du, \qquad n = 0,1,2,\ldots,$$

with

$$K_0(t) := t^{s-1}e^{i\,x\,\frac{f^{(m)}(a)}{m!}t^m}.$$

In the above integral, $\alpha := sign\left(f^{(m)}(a)\right)$ and the path of integration is the ray $\arg(u - t) = (\alpha\pi)/(2m)$. Then, the function $K_{n+1}(t)$ is the $n+1$ iterated integral of the function $K_0(t)$, that may be written in the following way:

$$\begin{aligned} K_{n+1}(t) =& \frac{(-1)^{n+1}}{n!}\int_t^{t+\infty e^{i\,\alpha\,\frac{\pi}{2m}}}(u-t)^n u^{s-1}e^{i\,x\,\frac{f^{(m)}(a)}{m!}u^m}\,du \\ =& \frac{(-1)^{n+1}}{n!}t^{n+s}\int_1^{1+\infty e^{i\,\alpha\,\frac{\pi}{2m}}}(u-1)^n u^{s-1}e^{i\,x\,\frac{f^{(m)}(a)}{m!}(tu)^m}\,du. \end{aligned} \tag{3.6}$$

Integrating by parts N times in the rightmost integral in (3.4) and using the fact that $K'_{n+1}(t) = K_n(t)$, $K_n(\infty) = 0$ for all $n = 1,2,3,...,N$, and $h_N^{(n)}(a,x) = 0$ for all $n = 0,1,2,...,N-1$, the remainder $R_N(x)$ can be written in the form:

$$R_N(x) = (-1)^N\int_0^\infty K_N(t)\,h_N^{(N)}(t+a,x)dt. \tag{3.7}$$

Observe that, because $\bar{g}(t) = 0$ in a neighbourhood of infinity, then $h_N^{(N)}(t+a,x) = 0$ in a neighbourhood of infinity as well. From (3.1) and using Faá di Bruno's formula for the derivative of a composite function [2] we find that

$$h_N^{(N)}(t+a,x) = \left(e^{i\,x\,f_p(t+a)}\bar{g}(t+a)\right)^{(N)} = e^{i\,x\,f_p(t+a)}G(t+a,x), \tag{3.8}$$

ETNA
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with

$$G(t+a,x) := \sum_{k=0}^{N}\frac{(-N)_k(-1)^k}{k!}\left(\sum_{r=0}^{k}\frac{(i\,x)^r}{r!}\sum_{j=0}^{r}\frac{(-r)_j}{j!}\left(f_p^{r-j}(t+a)\right)^{(k)} f_p^j(t+a)\right)\bar{g}^{(N-k)}(t+a).$$

Rearranging sums in the above formula we can write $G(t+a,x)$ in the form of a polynomial in the variable x of degree at most N: $G(t+a,x)=\sum_{k=0}^{N}(i\,x)^k\,G_k(t+a)$, whose coefficients $G_k(t+a)$ are

$$G_k(t+a) := \sum_{r=k}^{N}\frac{(-N)_r(-1)^r}{r!\,k!}\sum_{j=0}^{k}\frac{(-k)_j}{j!}\left(f_p^{k-j}(t+a)\right)^{(r)} f_p^j(t+a)\,\bar{g}^{(N-r)}(t+a).$$

From the hypotheses for $f(t)$ and $\bar{g}(t)$ we know that the functions $G_k(t+a)$ are analytic functions of the variable t at $t=0$. On the other hand, it is straightforward to see that $G_k(t+a)\sim t^{\text{Max}\{pk-N,0\}}$ as $t\to 0$; in other words,

$$G_k(t+a)\sim\begin{cases} t^{pk-N} & \text{if} \quad k>\lfloor N/p\rfloor \\ 1 & \text{if} \quad k\le\lfloor N/p\rfloor \end{cases} \qquad \text{as} \quad t\to 0.$$

(Observe that, at $t=0$, $G(a,x)=e^{i\,x f(a)}a_N(x)$ and then $G_k(a,0)=0$ for $k>\lfloor N/p\rfloor$.) Then, from (3.6), (3.7), and (3.8) we find

(3.9)
$$R_N(x)=(-1)^N\int_1^{1+\infty e^{i\,\alpha\frac{\pi}{2m}}}(u-1)^{N-1}u^{s-1}du\int_0^\infty e^{i\,x\,F(t,u)}\,t^{N+s-1}\,G(t+a,x)dt,$$

with $F(t,u):=f(t+a)-f(a)-\dfrac{f^{(m)}(a)}{m!}t^m(1-u^m)$. The inner integral (in the variable t) may be written in the form

(3.10)
$$\begin{aligned} H(u,x) &:= \int_0^\infty e^{i\,x\,F(t,u)}\,t^{N+s-1}\,G(t+a,x)dt \\ &= \sum_{k=0}^{\lfloor N/p\rfloor}(i\,x)^k\int_0^\infty e^{i\,x\,F(t,u)}\,t^{N+s-1}\,G_k(t+a)dt \\ &\quad + \sum_{k=\lfloor N/p\rfloor+1}^{N}(i\,x)^k\int_0^\infty e^{i\,x\,F(t,u)}\,t^{N+s-1}\,G_k(t+a)dt. \end{aligned}$$

In every one of the above integrals, the phase function $F(t,u)$ has a unique stationary point at $t=0$ with $F^{(n)}(0,u)=0,\ n=0,1,2,...,m-1$ and $F^{(m)}(0,u)=f^{(m)}(a)u^m\neq 0$. After these preparations, we can apply the classical stationary phase first order approximation to every one of the integrals in (3.10), using formula [14, eq. (3.13)] with $\rho=m$, $n=0$ and $\lambda=N+s$ or $\lambda=pk+s$. We obtain

$$\begin{aligned} H(u,x) &= u^{-N-s}\sum_{k=0}^{\lfloor N/p\rfloor}(i\,x)^k\mathcal{O}\left(x^{-\frac{N+s}{m}}\right) \\ &\quad + \sum_{k=\lfloor N/p\rfloor+1}^{N}(i\,x)^k\,u^{-pk-s}\mathcal{O}\left(x^{-\frac{pk+s}{m}}\right) \qquad \text{as} \quad x\to\infty, \end{aligned}$$

ETNA
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where the $\mathcal{O}$ symbols stand uniformly for $|u| \in [1,\infty)$ along the ray $\arg(u-1) = (\alpha\pi)/(2m)$. The asymptotic behavior of the above sum for large x is dominated by the last term in the first sum, the term corresponding to $k = \lfloor N/p \rfloor$. Therefore

$$H(u,x) = u^{-N-s}\mathcal{O}\left(x^{\lfloor N/p\rfloor - \frac{N+s}{m}}\right). \tag{3.11}$$

When we substitute (3.11) into (3.9) we obtain

$$R_N(x) = \mathcal{O}\left(x^{\lfloor N/p\rfloor - \frac{N+s}{m}}\right) \quad \text{as} \quad x\to\infty,$$

and then,

$$\begin{aligned} F_1(x) = e^{i\,x f(a)} \Bigg[& \sum_{n=0}^{N-1} \frac{a_n(x)}{m}\Gamma\left(\frac{n+s}{m}\right)\left(\frac{i\,m!}{f^{(m)}(a)x}\right)^{(n+s)/m} \\ & + \mathcal{O}\left(x^{\left\lfloor \frac{N}{p}\right\rfloor - \frac{N+s}{m}}\right)\Bigg] \end{aligned} \tag{3.12}$$

as $x\to\infty$. The terms of the expansion (3.12) satisfy $a_n(x)\Phi_n(x) = \mathcal{O}(x^{\lfloor n/p\rfloor-(n+s)/m})$. Then, (3.12) is not a genuine asymptotic expansion, as the order of the terms of the expansion do not decrease linearly with n, but in the form of a saw-tooth: the order decreases $1/m$ units from every term to the next one, but it also increases 1 unit every p terms. Although this is not a standard Poincare-like expansion, it performs perfectly well in practical applications as it is always possible to select a prescribed order of approximation $(M+s)/m$ (error term of the order $\mathcal{O}(x^{-(M+s)/m})$) by choosing a number of terms $N = Mp/(p-m)$, with M an integer multiple of $p-m$.

3.2. Summary of the discussion. The asymptotic analysis of the integral $F_2(x)$ in (2.1) is much simpler than that of $F_1(x)$, since the phase function $f(t)$ has no stationary points in the integration interval $[b,\infty)$ and then we can just repeatedly integrate by parts to get an asymptotic expansion [14, Chap. 2, Sec. 3]. We give details in the following theorem, where we also summarize the discussion of Section 3.1 for the integral $F_1(x)$ and give the complete asymptotic expansion of the integral $F(x)$.

THEOREM 3.3. *Let the functions $f(t)$ and $\bar{g}(t) := (t-a)^{1-s}g(t)$ be infinitely differentiable in $[a,b]$, or in $[a,b)$ when $b=\infty$, for a certain $s\in(0,1]$. Let also the functions $f(t)$ and $\bar{g}(t)$ be analytic at $t=a$. Then, for N and $K = 1,2,3,\ldots$,*

$$\begin{aligned} \int_a^b e^{i\,x f(t)} g(t)dt \;=\; & e^{i\,x f(a)}\Bigg[\sum_{n=0}^{N-1}\frac{a_n(x)}{m}\Gamma\left(\frac{n+s}{m}\right)\left(\frac{i\,m!}{f^{(m)}(a)x}\right)^{(n+s)/m} \\ & + \mathcal{O}\left(x^{\left\lfloor \frac{N}{p}\right\rfloor - \frac{N+s}{m}}\right)\Bigg] \\ & + \frac{e^{i\,x f(b)}}{f'(b)}\left[\sum_{k=0}^{K-1} g_k(b)\left(-\frac{i}{x}\right)^{k+1} + \mathcal{O}\left(x^{-K-1}\right)\right], \end{aligned} \tag{3.13}$$

as $x\to\infty$, where

$$g_0(t) := g(t), \qquad g_{n+1}(t) := -\frac{d}{dt}\left(\frac{g_n(t)}{f'(t)}\right), \quad n = 0,1,2,\ldots, \tag{3.14}$$

the integer m is the degree of the first non-vanishing derivative of $f(t)$ at $t=a$ and p is the degree of the next non-vanishing derivative. For $n=0,1,2,\ldots$,

$$a_n(x) := \sum_{k=0}^{n} A_k(x) B_{n-k},$$

where $A_k(x)$ and B_k are the Taylor coefficients at $t=a$ of $e^{i\,x f_p(t)}$ and $\bar{g}(t)$ respectively:

$$e^{i\,x f_p(t)} = \sum_{n=0}^{\infty} A_n(x)(t-a)^n, \qquad \bar{g}(t) = \sum_{n=0}^{\infty} B_n(t-a)^n.$$

The coefficients $A_n(x)$ may be computed in the form:

$$A_n(x) = \sum_{k=0}^{n} \frac{b_{n,k}}{n!}(ix)^k,$$

where $b_{n,k}$ are the partial ordinary Bell polynomials. These polynomials may be computed recursively in the following form: $b_{0,0}=1$; $b_{n,0}=0$, $n=1,2,3,\ldots$; and

$$b_{n,k} = \sum_{j=p}^{n-k+1} \binom{n-1}{j-1} f^{(j)}(a)\ b_{n-j,k-1}, \qquad n=1,2,3,\ldots, \qquad k=1,2,3,\ldots,n,$$

where empty sums must be understood as zero. In order to get an approximation of the order $\mathcal{O}(x^{-(M+s)/m})$, we must take $N=Mp/(p-m)$ and $K=\lceil (M+s)/m-1\rceil$, where the symbol $\lceil q\rceil$ stands for the integer greater than or equal to q and $M=(p-m),2(p-m),3(p-m),\ldots$.

Proof. The integral in the left-hand side of (3.13) is the function $F(x)=F_1(x)-F_2(x)$, with $F_1(x)$ and $F_2(x)$ defined in (2.1). The first term in the right-hand side of (3.13) is just the asymptotic expansion of $F_1(x)$ derived in (3.2)–(3.5). The second term is the asymptotic expansion of $F_2(x)$ that may be derived straightforwardly: since the integrand in $F_2(x)$ is infinitely differentiable and $f(t)$ has no stationary points in $[b,\infty)$, by repeated integration by parts we get the second term in the right-hand side of (3.13) with $g_n(t)$ given in (3.14). The last sentence is trivial. ∎

4. An example. For $x>0$ and real parameter $\lambda\neq 0$, we consider the integral

$$F_\lambda(x) := \frac{1}{\pi}\int_0^{\pi} e^{i\,x(\lambda t-\sin(t))}dt. \tag{4.1}$$

We have that $\mathbf{J}_{\lambda x}(x)=\Re(F_\lambda(x))$ and $\mathbf{E}_{\lambda x}(x)=\Im(F_\lambda(x))$, where $\mathbf{J}$ and $\mathbf{E}$ are the Anger and Weber functions, respectively [9, Sec. 11.10]. The first order asymptotic approximation of $\mathbf{J}_{\lambda x}(x)$ and $\mathbf{E}_{\lambda x}(x)$ for large x may be found in [9, Sec. 11.11(iii)]; but only the first few coefficients of the expansion are given. We apply below Theorem 3.3 to derive the complete asymptotic expansions of these functions when $x\to\infty$ and λ is fixed. Conveniently reorganized in the form of Poincaré expansions, the approximations given below agree with those given in [9, Sec. 11.10] up to the order given there.

According to Theorem 3.3, we have $f(t)=\lambda t-\sin(t)$, $f'(t)=\lambda-\cos(t)$ and $g(t)=1$, $s=1$. The location of the stationary points of $f(t)$ depends on λ and we find four different situations that we study separately in the following subsections.

4.1. Case 1: $-1<\lambda<1$ **and** $\lambda\neq 0$**.** The phase function f has a unique stationary point in $[0,\pi]$ at $t_0=\arccos(\lambda)$ with $0<\arccos(\lambda)<\pi$ and $f(t_0)=\lambda\, t_0-\sqrt{1-\lambda^2}$. We separate (4.1) in two integrals:

$$F_\lambda(x)=\frac{1}{\pi}\int_{-t_0}^{0}e^{i\,x(-\lambda t+\sin(t))}dt+\frac{1}{\pi}\int_{t_0}^{\pi}e^{i\,x(\lambda t-\sin(t))}dt.$$

According to the notation of Theorem 3.3, in the first integral in the right-hand side we have $a=-t_0$ and $b=0$. In this case the unique stationary point in the interval is found at $t=-t_0$. Hence $m=2$, $p=3$, and $s=1$. On the other hand, in the second integral we have $a=t_0$ and $b=\pi$. In this case the unique stationary point in the interval is found at $t=t_0$. Hence $m=2$, $p=3$, and $s=1$. Applying Theorem 3.3 to both integrals and after some manipulations we obtain, for $M=1,2,3,...$,

(4.2)
$$\begin{aligned}F_\lambda(x)=&\frac{e^{i\,x\,f(t_0)}}{\pi}\sum_{n=0}^{3M-1}\left(\frac{2\,i}{x\sqrt{1-\lambda^2}}\right)^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)P_n(x)+\\&-\frac{i}{\pi x}\sum_{k=0}^{\lceil(M+1)/2\rceil-2}\frac{(-1)^k}{x^{2k}}\left(\frac{g_{2k}(0)}{1-\lambda}+\frac{e^{i\,x\lambda\pi}g_{2k}(\pi)}{1+\lambda}\right)+\mathcal{O}\left(x^{-(M+1)/2}\right),\end{aligned}$$

where $g_0(t):=1$ and, for $n=0,1,2,...,$

$$g_{n+1}(t):=\frac{d}{d\,t}\left(\frac{g_n(t)}{\lambda-\cos(t)}\right),\qquad P_n(x):=\frac{1}{n!}\sum_{k=0}^{n}\frac{b_{n,k}^1+b_{n,k}^2}{2}(i\,x)^k,$$

$$b_{0,0}^\alpha:=1\ ,\ b_{n,0}^\alpha:=0\quad\text{and}\quad b_{n,k}^\alpha:=\sum_{j=3}^{n-k+1}\binom{n-1}{j-1}(-1)^{\alpha\,j}\,c_j\,b_{n-j,k-1}^\alpha\ ,\ \alpha=1,2,$$

with $c_j:=(-1)^{(j+1)/2}\lambda$ if j is odd and $c_j:=-(-1)^{j/2}\sqrt{1-\lambda^2}$ if j is even.

The real and imaginary parts of (4.2) constitute asymptotic expansions for the Anger and Weber functions respectively. Then, separating the polynomial $P_n(x)$ in real and imaginary parts,

$$\begin{aligned}P_n(x)=&\underbrace{\frac{1}{n!}\sum_{k=0}^{[n/2]}\frac{b_{n,2k}^1+b_{n,2k}^2}{2}(-1)^k\,x^{2k}}_{P_n^1(x)}\\&+\ i\ \underbrace{\left(\frac{1}{n!}\sum_{k=0}^{[(n-1)/2]}\frac{b_{n,2k+1}^1+b_{n,2k+1}^2}{2}(-1)^k\,x^{2k+1}\right)}_{P_n^2(x)},\end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

with $P_0^2(x) := 0$, we find

$$\mathbf{J}_{\lambda x}(x) = \frac{1}{\pi}\sum_{n=0}^{3M-1}\left(\frac{2}{x\sqrt{1-\lambda^2}}\right)^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)\left(\cos(\theta_1)\;P_n^1(x) - \sin(\theta_1)\;P_n^2(x)\right) + \frac{1}{\pi x}\sum_{k=0}^{\lceil (M+1)/2\rceil-2}\frac{(-1)^k}{x^{2k}}\frac{\sin(x\,\lambda\pi)}{1+\lambda}\,g_{2k}(\pi) + \mathcal{O}\left(x^{-(M+1)/2}\right)$$

and

$$\mathbf{E}_{\lambda x}(x) = \frac{1}{\pi}\sum_{n=0}^{3M-1}\left(\frac{2}{x\sqrt{1-\lambda^2}}\right)^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)\left(\cos(\theta_1)\;P_n^2(x) + \sin(\theta_1)\;P_n^1(x)\right) - \frac{1}{\pi x}\sum_{k=0}^{\lceil (M+1)/2\rceil-2}\frac{(-1)^k}{x^{2k}}\left(\frac{g_{2k}(0)}{1-\lambda} + \frac{g_{2k}(\pi)}{1+\lambda}\cos(\lambda\pi\,x)\right) + \mathcal{O}\left(x^{-(M+1)/2}\right).$$

In these formulas, $\theta_1 := x\left(\lambda\, t_0 - \sqrt{1-\lambda^2}\right) + \frac{n+1}{4}\pi$.

4.2. Case 2: $\lambda = -1$**.** After the change of variable $t \to -t$ we get

$$F_{-1}(x) = \frac{1}{\pi}\int_{-\pi}^{0} e^{i\,x(t+\sin(t))}dt.$$

Following the notation of Theorem 3.3 we have $a = -\pi$ and $b = 0$. The unique stationary point of the phase function in the interval is found at $t = -\pi$. In this case $m = 3$, $p = 5$, and $s = 1$. From Theorem 3.3 we obtain, for $M = 2, 4, 6, \ldots$,

$$F_{-1}(x) = \frac{e^{-i\,x\,\pi}}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6i}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)P_n(x) - \frac{i}{2\pi x}\sum_{k=0}^{\lceil (M+1)/3\rceil-2}\frac{g_{2k}(0)}{(-x^2)^k} + \mathcal{O}\left(x^{-(M+1)/3}\right), \tag{4.3}$$

with $g_0(t) := 1$ and, for $n = 0, 1, 2, \ldots$,

$$g_{n+1}(t) := -\frac{d}{dt}\left(\frac{g_n(t)}{1+\cos(t)}\right), \qquad P_n(x) := \frac{1}{n!}\sum_{k=0}^{n} b_{n,k}(i\,x)^k,$$

$$b_{0,0} := 1\;,\quad b_{n,0} := 0 \quad \text{and} \quad b_{n,k} := -\sum_{j=5}^{n-k+1}\binom{n-1}{j-1}\sin\left(\frac{j}{2}\pi\right)\; b_{n-j,k-1}.$$

The real and imaginary parts of (4.3) constitute asymptotic expansions for the Anger and Weber functions respectively. Then, decomposing $P_n(x)$ into its real and imaginary parts,

$$P_n(x) = \underbrace{\frac{1}{n!}\sum_{k=0}^{[n/2]} b_{n,2k}(-1)^k\, x^{2k}}_{P_n^1(x)} + i\underbrace{\left(\frac{1}{n!}\sum_{k=0}^{[(n-1)/2]} b_{n,2k+1}(-1)^k\, x^{2k+1}\right)}_{P_n^2(x)}$$

with $P_0^2(x) := 0$, we find

$$\begin{aligned}\mathbf{J}_{-x}(x) =& \frac{1}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)\left(\cos(\theta_2)\,P_n^1(x)-\sin(\theta_2)\,P_n^2(x)\right)\\ &+\mathcal{O}\left(x^{-(M+1)/3}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{-x}(x) =& \frac{1}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)\left(\cos(\theta_2)\;P_n^2(x)+\sin(\theta_2)\;P_n^1(x)\right)\\ &-\sum_{k=0}^{\lceil (M+1)/3\rceil-2}\frac{(-1)^k\,g_{2k}(0)}{2\pi\,x^{2k+1}}+\mathcal{O}\left(x^{-(M+1)/3}\right).\end{aligned}$$

In these formulas, $\theta_2 := \pi\left(-x+\dfrac{n+1}{6}\right)$.

4.3. Case 3: $\lambda = 1$**.** The integral (4.1) reads

$$F_1(x) = \frac{1}{\pi}\int_0^{\pi} e^{i\,x(t-\sin(t))}dt.$$

In this particular case we have $a = 0$ and $b = \pi$ and the unique stationary point of the phase function in the interval is found at $t = 0$ with $m = 3$, $p = 5$, and $s = 1$. From Theorem 3.3 we obtain, for $M = 2, 4, 6, ...$,

$$\begin{aligned}F_1(x) =& \frac{1}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6i}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)P_n(x)\\ &+\frac{e^{i\,x\,\pi}}{2\,i\,\pi x}\sum_{k=0}^{\lceil (M+1)/3\rceil-2}\frac{g_{2k}(\pi)}{(-x^2)^k}+\mathcal{O}\left(x^{-(M+1)/3}\right),\end{aligned}\tag{4.4}$$

with $g_0(t) := 1$ and, for $n = 0, 1, 2,...,$

$$g_{n+1}(t) := \frac{d}{d\,t}\left(\frac{g_n(t)}{\cos(t)-1}\right), \qquad P_n(x) := \frac{1}{n!}\sum_{k=0}^{n} b_{n,k}(i\,x)^k,$$

$$b_{0,0} := 1\ ,\ \ b_{n,0} := 0 \quad \text{and}\ \ b_{n,k} := -\sum_{j=5}^{n-k+1}\binom{n-1}{j-1}\sin\left(\frac{j}{2}\pi\right)\ b_{n-j,k-1}.$$

The real and imaginary parts of (4.4) constitute asymptotic expansions for the Anger and Weber functions respectively. Therefore, separating $P_n(x)$ in real and imaginary parts,

$$P_n(x) = \underbrace{\frac{1}{n!}\sum_{k=0}^{[n/2]} b_{n,2k}(-1)^k\,x^{2k}}_{P_n^1(x)} + i\underbrace{\left(\frac{1}{n!}\sum_{k=0}^{[(n-1)/2]} b_{n,2k+1}(-1)^k\,x^{2k+1}\right)}_{P_n^2(x)},$$

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with $P_0^2(x) := 0$, we find

$$\begin{aligned}\mathbf{J}_x(x) =& \frac{1}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)\left(\cos\left(\frac{n+1}{6}\pi\right)P_n^1(x)\right.\\ &\left.-\sin\left(\frac{n+1}{6}\pi\right)P_n^2(x)\right)\\ &+\frac{\sin(\pi x)}{2\pi x}\sum_{k=0}^{\lceil (M+1)/3\rceil-2}\frac{g_{2k}(\pi)}{(-x^2)^k}+\mathcal{O}\left(x^{-(M+1)/3}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_x(x) =& \frac{1}{3\pi}\sum_{n=0}^{5M/2-1}\left(\frac{6}{x}\right)^{\frac{n+1}{3}}\Gamma\left(\frac{n+1}{3}\right)\left(\cos\left(\frac{n+1}{6}\pi\right)P_n^2(x)\right.\\ &\left.+\sin\left(\frac{n+1}{6}\pi\right)P_n^1(x)\right)\\ &-\frac{\cos(\pi x)}{2\pi x}\sum_{k=0}^{\lceil (M+1)/3\rceil-2}\frac{g_{2k}(\pi)}{(-x^2)^k}+\mathcal{O}\left(x^{-(M+1)/3}\right).\end{aligned}$$

4.4. Case 4: $|\lambda| > 1$. In this case, the phase function $f(t) = \lambda t - \sin(t)$ has no stationary points in the interval of integration. From formula (1.3) with $a = 0$ and $b = \pi$ we obtain

$$F_\lambda(x) = \frac{i}{\pi}\sum_{m=0}^{M-1}\frac{(-1)^m}{x^{2m+1}}\left(\frac{g_{2m}(0)}{\lambda-1}-\frac{e^{ix\lambda\pi}g_{2m}(\pi)}{\lambda+1}\right)+\mathcal{O}\left(x^{-2M-1}\right), \tag{4.5}$$

with $g_0(t) := 1$ and, for $n = 0, 1, 2,..., g_{n+1}(t) = \dfrac{d}{dt}\left(\dfrac{g_n(t)}{\cos(t)-\lambda}\right)$. The real and imaginary parts of (4.5) constitute asymptotic expansions for the Anger and Weber functions respectively:

$$\mathbf{J}_{\lambda x}(x) = \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{(-1)^m}{x^{2m+1}}\frac{\sin(\lambda\pi x)}{\lambda+1}g_{2m}(\pi)+\mathcal{O}\left(x^{-2M-1}\right)$$

and

$$\mathbf{E}_{\lambda x}(x) = \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{(-1)^m}{x^{2m+1}}\left(\frac{g_{2m}(0)}{\lambda-1}-\frac{\cos(\lambda\pi x)g_{2m}(\pi)}{\lambda+1}\right)+\mathcal{O}\left(x^{-2M-1}\right).$$

4.5. Numerical experiments. Tables 4.1–4.4 show some numerical experiments that illustrate the accuracy of the above approximations for several values of λ and x and different orders M of the approximation. As the exact value of $F_\lambda(x)$, we have taken the numerical integration of (4.1) obtained with the program *Mathematica 11*.

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TABLE 4.1
Relative errors supplied by the approximation (4.2).

$\lambda = 1/2$	M				
x	1	2	3	4	5
15	$2.418 \cdot 10^{-1}$	$5.619 \cdot 10^{-3}$	$5.182 \cdot 10^{-3}$	$1.686 \cdot 10^{-3}$	$1.474 \cdot 10^{-3}$
50	$7.642 \cdot 10^{-2}$	$1.391 \cdot 10^{-3}$	$2.894 \cdot 10^{-4}$	$2.571 \cdot 10^{-5}$	$6.314 \cdot 10^{-6}$
125	$7.541 \cdot 10^{-2}$	$6.756 \cdot 10^{-4}$	$1.959 \cdot 10^{-5}$	$3.582 \cdot 10^{-6}$	$2.629 \cdot 10^{-7}$
500	$4.261 \cdot 10^{-2}$	$1.561 \cdot 10^{-4}$	$2.007 \cdot 10^{-6}$	$1.918 \cdot 10^{-7}$	$4.286 \cdot 10^{-9}$
1000	$3.077 \cdot 10^{-2}$	$7.829 \cdot 10^{-5}$	$4.386 \cdot 10^{-7}$	$4.841 \cdot 10^{-8}$	$5.018 \cdot 10^{-10}$

TABLE 4.2
Relative errors supplied by the approximation (4.3).

$\lambda = -1$	M			
x	2	4	6	8
15	$5.907 \cdot 10^{-2}$	$5.301 \cdot 10^{-4}$	$4.841 \cdot 10^{-5}$	$5.661 \cdot 10^{-6}$
50	$1.829 \cdot 10^{-2}$	$1.131 \cdot 10^{-4}$	$4.539 \cdot 10^{-6}$	$2.458 \cdot 10^{-7}$
125	$1.149 \cdot 10^{-2}$	$4.451 \cdot 10^{-5}$	$1.131 \cdot 10^{-6}$	$3.865 \cdot 10^{-8}$
500	$3.919 \cdot 10^{-3}$	$5.162 \cdot 10^{-6}$	$4.503 \cdot 10^{-8}$	$5.282 \cdot 10^{-10}$
1000	$2.467 \cdot 10^{-3}$	$2.045 \cdot 10^{-6}$	$1.125 \cdot 10^{-8}$	$8.297 \cdot 10^{-11}$

TABLE 4.3
Relative errors supplied by the approximation (4.4).

$\lambda = 1$	M			
x	2	4	6	8
15	$5.907 \cdot 10^{-2}$	$5.301 \cdot 10^{-4}$	$4.841 \cdot 10^{-5}$	$5.661 \cdot 10^{-6}$
50	$1.829 \cdot 10^{-2}$	$1.131 \cdot 10^{-4}$	$4.539 \cdot 10^{-6}$	$2.458 \cdot 10^{-7}$
125	$1.469 \cdot 10^{-2}$	$3.248 \cdot 10^{-5}$	$7.141 \cdot 10^{-7}$	$2.104 \cdot 10^{-8}$
500	$3.919 \cdot 10^{-3}$	$5.162 \cdot 10^{-6}$	$4.503 \cdot 10^{-8}$	$5.283 \cdot 10^{-10}$
1000	$2.467 \cdot 10^{-3}$	$2.045 \cdot 10^{-6}$	$1.125 \cdot 10^{-8}$	$8.324 \cdot 10^{-11}$

TABLE 4.4
Relative errors supplied by the approximation (4.5).

$\lambda = 3/2$	M			
x	1	2	3	4
20	$3.148 \cdot 10^{-2}$	$8.331 \cdot 10^{-3}$	$5.042 \cdot 10^{-3}$	$4.871 \cdot 10^{-3}$
50	$2.755 \cdot 10^{-3}$	$1.001 \cdot 10^{-4}$	$1.082 \cdot 10^{-5}$	$2.447 \cdot 10^{-6}$
125	$5.045 \cdot 10^{-4}$	$2.739 \cdot 10^{-6}$	$4.181 \cdot 10^{-8}$	$1.261 \cdot 10^{-9}$
500	$4.007 \cdot 10^{-5}$	$1.345 \cdot 10^{-8}$	$1.261 \cdot 10^{-11}$	$3.378 \cdot 10^{-14}$
1000	$1.001 \cdot 10^{-5}$	$8.401 \cdot 10^{-10}$	$2.081 \cdot 10^{-13}$	$6.386 \cdot 10^{-16}$

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