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# A NOTE ON OPTIMAL RATES FOR LAVRENTIEV REGULARIZATION WITH ADJOINT SOURCE CONDITIONS*

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**Abstract.** In a recent paper, Plato, Mathé, and Hofmann proved several convergence rate results for Lavrentiev regularization. Especially, they also proved new results for the case when the exact solution $u$ of an ill-posed linear problem $Au = f$ satisfies the adjoint source condition $u \in \mathcal{R}((A^*)^p)$, $0 < p \le \frac{1}{2}$. In this note we slightly improve the rate for $p = \frac{1}{2}$ and also prove the rate $O(\delta^{\frac{1}{3}})$ if $p > \frac{1}{2}$.

**Key words.** Lavrentiev regularization, convergence rates

**AMS subject classifications.** 47A52, 65J20

## 1. Introduction.

In the recent paper [5], Plato, Mathé, and Hofmann proved new convergence rate results for Lavrentiev regularization when the exact solution satisfies adjoint source conditions. Using their notations, we deal with the following problem: find $u \in \mathcal{H}$ in

$$Au = f\,, \qquad f \in \mathcal{R}(A)\,, \tag{1.1}$$

where $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear accretive operator in an infinite-dimensional and separable complex Hilbert space $\mathcal{H}$. Accretive means that

$$\mathrm{Re}\,\langle\, Au, u \,\rangle \ge 0 \qquad \text{for all } u \in \mathcal{H}\,. \tag{1.2}$$

We assume that the range $\mathcal{R}(A)$ is not closed, i.e., the problem of solving (1.1) is ill-posed and has to be regularized (see, e.g., [1]), especially since instead of the exact data $f$ one usually only has perturbed data $f^\delta \in \mathcal{H}$ with

$$\left\|f - f^\delta\right\| \le \delta\,,$$

where $\delta > 0$ denotes the noise level. As in [5] we consider Lavrientiev regularization, i.e., $u$ is approximated by

$$u_\gamma^\delta = (A + \gamma I)^{-1} f^\delta\,, \qquad \gamma > 0\,.$$

Using the estimate (see [5, (1.4)])

$$\left\|u - u_\gamma^\delta\right\| \le \left\|\gamma(A + \gamma I)^{-1} u\right\| + \frac{\delta}{\gamma}\,, \tag{1.3}$$

one can prove convergence rates if $u$ satisfies certain source conditions.

For selfadjoint operators it is well known that

$$\left\|u - u_{\gamma(\delta)}^\delta\right\| = O\left(\delta^{\frac{p}{p+1}}\right)$$

if $u \in \mathcal{R}(A^p)$, $0 < p \le 1$, and $\gamma(\delta) \sim \delta^{\frac{1}{p+1}}$. It is shown in [5, Proposition 4] that these rates are also true for general accretive operators. One can even prove converse and saturation results (see [4]).

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For adjoint source conditions

$$u = (A^*)^p v\,, \qquad v \in \mathcal{H}\,, \tag{1.4}$$

convergence rates are proven based on the following result by Kato [2]: let $0 < p < \frac{1}{2}$, then

$$\|(A^*)^p u\| \leq e_p \, \|A^p u\| \qquad \text{for all } u \in \mathcal{H}\,, \tag{1.5}$$

where

$$e_p := \tan \frac{\pi(1+2p)}{4}\,. \tag{1.6}$$

(Obviously, this estimate also holds for $p = 0$.) This implies that one can also get the rate $O(\delta^{\frac{p}{p+1}})$ for the source condition (1.4) if $p < \frac{1}{2}$ (see [5, Theorem 1]). For the case $p = \frac{1}{2}$, the rate

$$O\left( \left( \delta |\ln \delta|^2 \right)^{\frac{1}{3}} \right) \tag{1.7}$$

is shown if $\gamma(\delta)$ is chosen appropriately; see [5, Theorem 2]. We will improve this rate a little bit.

It is well known from [3] that for $p \geq 1$, the rate

$$O\left( \delta^{\frac{1}{3}} \right) \tag{1.8}$$

holds. In [5, Section 6], results on limit orders were obtained that suggest that this rate could also hold for $p > \frac{1}{2}$. We show in the next section that this is true. Moreover, we improve several constants appearing in certain estimates from [5].

**2. Improvements.** Using the formulas

$$\begin{aligned} \left\|(A+sI)^{-1}A\right\| &\leq 1\,, \qquad s > 0\,,\\ \left\|s(A+sI)^{-1}\right\| &\leq 1\,, \qquad s > 0\,,\\ A^p := \frac{\sin \pi p}{\pi} &\int_0^\infty s^{p-1}(A+sI)^{-1}A\, ds\,, \qquad 0 < p < 1\,, \end{aligned}$$

(see [5, Remark 3, (2.2), (2.3)]), it follows with $a > 0$ that

$$\begin{aligned} \|A^p x\| &\leq \frac{\sin \pi p}{\pi} \left( \int_0^a s^{p-1} \|x\| \; ds + \int_a^\infty s^{p-2} \|Ax\| \; ds \right)\\ &= \frac{\sin \pi p}{\pi} \left( \frac{1}{p} a^p \|x\| + \frac{1}{1-p} a^{p-1} \|Ax\| \right)\,. \end{aligned}$$

When $x \neq 0$, this bound is minimized for $a = \|Ax\| \, \|x\|^{-1}$ and yields

$$\|A^p x\| \leq c_p \, \|Ax\|^p \, \|x\|^{1-p}$$

with

$$c_p := \frac{\sin \pi p}{\pi p(1-p)} \leq c_{\frac{1}{2}} = \frac{4}{\pi}\,. \tag{2.1}$$

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For $x = 0$ the estimate trivially holds. This improves the estimate [5, (2.7)] since $c_p < 2$. Of course, then also the constant 2 in [5, Proposition 4] can be replaced by this better constant $c_p$, i.e.,

$$\left\| \gamma(A+\gamma I)^{-1} A^p \right\| \le c_p \gamma^p \,, \qquad 0 < p \le 1 \,. \tag{2.2}$$

Now we turn to the case of adjoint source conditions: the following estimate will be essential for the improvement of the results in [5]. It is an immediate consequence of (1.2) that

$$\mathrm{Re}\,\langle\, (A+\gamma I)u, u \,\rangle \ge \gamma \left\| u \right\|^2$$

and hence that

$$\left\| \gamma(A+\gamma I)^{-1} u \right\|^2 \le \mathrm{Re}\,\langle\, \gamma(A+\gamma I)^{-1} u, u \,\rangle \;. \tag{2.3}$$

Let us now assume that $u$ satisfies the source condition (1.4) with $0 < p < \frac{1}{2}$. Noting that (1.5) and (2.2) are also valid with $A$ and $A^*$ interchanged, we obtain together with (2.3) and $(A^p)^* = (A^*)^p$ that

$$\begin{aligned} \left\| \gamma(A+\gamma I)^{-1}(A^*)^p v \right\|^2 &\le \mathrm{Re}\,\langle\, \gamma(A+\gamma I)^{-1}(A^*)^p v, (A^*)^p v \,\rangle \\ &= \mathrm{Re}\,\langle\, v, A^p \gamma(A^*+\gamma I)^{-1}(A^*)^p v \,\rangle \\ &\le e_p \left\| v \right\| \left\| \gamma(A^*+\gamma I)^{-1}(A^*)^{2p} v \right\| \le e_p c_{2p} \gamma^{2p} \left\| v \right\|^2 . \end{aligned}$$

Thus,

$$\left\| \gamma(A+\gamma I)^{-1}(A^*)^p \right\| \le (e_p c_{2p})^{\frac{1}{2}} \gamma^p .$$

The constant $(e_p c_{2p})^{\frac{1}{2}}$ is much smaller than the constant $2e_p$ in the estimate of [5, Proposition 8], especially when $p$ is close to $\frac{1}{2}$.

In the next theorem we slightly improve the rate (1.7) for $p = \frac{1}{2}$ and prove the rate (1.8) for $\frac{1}{2} < p \le 1$.

THEOREM 2.1. *Let problem* (1.1) *have a solution $u$ satisfying the source condition* (1.4) *for some $\frac{1}{2} \le p \le 1$.*

*If $p = \frac{1}{2}$ and $\gamma(\delta) \sim \delta^{\frac{2}{3}} |\ln \delta|^{-\frac{1}{3}}$, then we obtain the rate*

$$\left\| u - u^\delta_{\gamma(\delta)} \right\| = O\left( (\delta |\ln \delta|)^{\frac{1}{3}} \right).$$

*If $\frac{1}{2} < p \le 1$ and $\gamma(\delta) \sim \delta^{\frac{2}{3}}$, then we obtain the rate*

$$\left\| u - u^\delta_{\gamma(\delta)} \right\| = O\left( \delta^{\frac{1}{3}} \right).$$

*Proof.* Let us first consider the case $p = \frac{1}{2}$. Assuming that $0 < \varepsilon \le \frac{1}{2}$, then (1.5), (2.2), and (2.3) imply that

$$\begin{aligned} \left\| \gamma(A+\gamma I)^{-1}(A^*)^{\frac{1}{2}} v \right\|^2 &\le \mathrm{Re}\left\langle\, v, A^\varepsilon A^{\frac{1}{2}-\varepsilon} \gamma(A^*+\gamma I)^{-1}(A^*)^{\frac{1}{2}} v \,\right\rangle \\ &\le e_{\frac{1}{2}-\varepsilon} \left\| A^\varepsilon \right\| \left\| v \right\| \left\| \gamma(A^*+\gamma I)^{-1}(A^*)^{1-\varepsilon} v \right\| \\ &\le e_{\frac{1}{2}-\varepsilon} c_{1-\varepsilon} \left\| A^\varepsilon \right\| \gamma^{1-\varepsilon} \left\| v \right\|^2 . \end{aligned}$$

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Together with (1.6) and (2.1) we thus obtain that

$$\left\|\gamma(A+\gamma I)^{-1}(A^*)^{\frac{1}{2}}\right\| \leq \left(\frac{16}{\pi^2}\cot\frac{\pi\varepsilon}{2}\,\|A\|^{\varepsilon}\,\gamma^{1-\varepsilon}\right)^{\frac{1}{2}}.$$

Since it is trivial to show that

$$0 < x\cot x < 1\,, \qquad 0 < x < \frac{\pi}{2}\,,$$

we further get the estimate

$$\left\|\gamma(A+\gamma I)^{-1}(A^*)^{\frac{1}{2}}\right\| \leq \left(\frac{32}{\pi^3}\,\|A\|^{\varepsilon}\,\varepsilon^{-1}\gamma^{1-\varepsilon}\right)^{\frac{1}{2}}.$$

This bound is minimized for $\varepsilon = \ln^{-1}\frac{\|A\|}{\gamma}$ if $\gamma < \|A\|\exp(-2)$ and for $\varepsilon = \frac{1}{2}$ otherwise. Therefore, we finally arrive at

$$\left\|\gamma(A+\gamma I)^{-1}(A^*)^{\frac{1}{2}}\right\| \leq \left(\frac{32}{\pi^3}\exp(1)\gamma\ln\frac{\|A\|}{\gamma}\right)^{\frac{1}{2}}, \qquad \gamma < \|A\|\exp(-2)\,.$$

This together with (1.3) and $\gamma(\delta) \sim \delta^{\frac{2}{3}}|\ln\delta|^{-\frac{1}{3}}$ yields the desired rate.

Let us now consider the case $\frac{1}{2} < p \leq 1$. Then (1.5) (note that $1-p < \frac{1}{2}$), (2.1), (2.2) ($p = 1$), and (2.3) imply that

$$\begin{aligned}\left\|\gamma(A+\gamma I)^{-1}(A^*)^p v\right\|^2 &\leq \operatorname{Re}\left\langle v, A^{2p-1}A^{1-p}\gamma(A^*+\gamma I)^{-1}(A^*)^p v\right\rangle \\ &\leq e_{1-p}\left\|A^{2p-1}\right\|\|v\|\left\|\gamma(A^*+\gamma I)^{-1}A^*v\right\| \\ &\leq e_{1-p}\left\|A^{2p-1}\right\|\gamma\,\|v\|^2\,.\end{aligned}$$

Thus,

$$\left\|\gamma(A+\gamma I)^{-1}(A^*)^p\right\| \leq \left(e_{1-p}\left\|(A)^{2p-1}\right\|\gamma\right)^{\frac{1}{2}}.$$

This together with (1.3) and $\gamma(\delta) \sim \delta^{\frac{2}{3}}$ yields the desired rate. □

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