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POLYNOMIAL INTERPOLATION IN NONDIVISION ALGEBRAS*

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Abstract. Algorithms for two types of interpolation polynomials in nondivision algebras are presented. One is based on the Vandermonde matrix, and the other is close to the Newton interpolation scheme. Examples are taken from $\mathbb{R}^4$-algebras. In the Vandermonde case, necessary and sufficient conditions for the existence of interpolation polynomials are given for commutative algebras. For noncommutative algebras, a conjecture is proposed. This conjecture is true for equidistant nodes. It is shown that the Newton form of the interpolation polynomial exists if and only if all node differences are invertible. Several numerical examples are presented.

Key words. interpolation polynomials in nondivision algebras, Vandermonde-type polynomials in nondivision algebras, Newton-type polynomials in nondivision algebras, numerical examples of interpolation polynomials in nondivision algebras of $\mathbb{R}^4$

AMS subject classifications. 15A66, 1604, 41A05, 65D05

1. Introduction.

The aim of this paper is to provide an algorithm for solving the polynomial interpolation problem in nondivision algebras and to study the conditions under which such an algorithm works. As examples of nondivision algebras, we mainly consider one of the eight $\mathbb{R}^4$-algebras, where $\mathbb{R}$ denotes the field of real numbers. An algebra in general is the vector space $\mathbb{R}^N$ equipped with an additional associative multiplication $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ which has a unit, usually abbreviated by 1. For further information, see a book by Garling [2]. These algebras are also called *geometric algebras*[1]. The names and the algebraic rules for the eight algebras in $\mathbb{R}^4$ can be found in a paper by Janovská and Opfer [4]. The algebras are abbreviated by the following symbols: (The names are included in parentheses. They where given by Hamilton in 1843 to $\mathbb{H}$, by Cockle[2] in 1849 to $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm tes}, \mathbb{H}_{\rm cotes}$ [1], and by Schmeikal in 2014 to the remaining four [11])

$$\mathbb{H}(\text{quaternions}),\ \mathbb{H}_{\rm coq}(\text{coquaternions or split-quaternions}),\ \mathbb{H}_{\rm tes}(\text{tessarines}),$$
$$\mathbb{H}_{\rm cotes}(\text{cotessarines}),\ \mathbb{H}_{\rm nec}(\text{nectarines}),\ \mathbb{H}_{\rm con}(\text{conectarines}),$$
$$\mathbb{H}_{\rm tan}(\text{tangerines}),\ \mathbb{H}_{\rm cotan}(\text{cotangerines}).$$

Note that $\mathbb{H}_{\rm tes}$, $\mathbb{H}_{\rm cotes}$, $\mathbb{H}_{\rm tan}$, $\mathbb{H}_{\rm cotan}$ are commutative. In general we use the notation $\mathcal{A}$ for one of these algebras. The problem under study will be called *interpolation problem*, and we consider two types of them. One is named the *Vandermonde interpolation problem* and the other the *Newton interpolation problem*. Both rely on a set $(x_k, f_k) \in \mathcal{A} \times \mathcal{A}$ of *data*, for $1 \le k \le n+1$, where the elements x_k are referred to as *nodes* and the elements f_k are referred to as *values*. The minimum requirement for the nodes is that they are pairwise distinct.

We need a notion of similarity. For this purpose it is useful to introduce the simple notation

$$a = (a_1, a_2, a_3, a_4), \quad a \in \mathcal{A},$$

for elements from $\mathbb{R}^4$-algebras. The first component a_1 of a is called the *real part of* a and is denoted by $a_1 = \Re(a)$. An element of the form $(a_1, 0, 0, 0)$ is named real, and the real elements of $\mathcal{A}$ can be identified with $\mathbb{R}$.

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[1]Wikipedia: Geometric algebra http://en.wikipedia.org/wiki/Geometric_algebra
[2]http://www.oocities.org/cocklebio/

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DEFINITION 1.1. *(1) Two elements $a, b \in \mathcal{A}$ are called similar, denoted by $a \sim b$, if there is an invertible $h \in \mathcal{A}$ such that*

$$b = h^{-1}ah.$$

The set of elements which are similar to a fixed $a \in \mathcal{A}$ is called the similarity class of a, denoted by $[a]$ and formally defined by

$$[a] := \{b : b = h^{-1}ah \textit{ for all invertible } h \in \mathcal{A}\}.$$

(2) Let $\mathcal{A}$ be one of the four noncommutative algebras. Define the conjugate of a by

$$\overline{a} = \mathrm{conj}(a) = (a_1, -a_2, -a_3, -a_4), \qquad \textit{and set} \qquad \mathrm{abs}_2(a) := a\overline{a}.$$

An early paper on the topic of similarity in connection with quaternions was already presented in 1936 by Wolf [12]. It is clear that similarity is an equivalence relation. Definition 1.1 part (1) applied to a commutative algebra $\mathcal{A}$ yields $[a] = \{a\}$, thus, in this case, the similarity class consists of only one element, which means that two elements in a commutative algebra are similar if and only if they are identical. Now by results of the already cited paper [4], it is easily possible to identify similar elements and to characterize invertible elements in one of the noncommutative algebras.

THEOREM 1.2. *Let $a, b \in \mathcal{A}\backslash\mathbb{R}$ and let $\mathcal{A}$ be one of the four noncommutative $\mathbb{R}^4$ algebras. Then $a \sim b$ if and only if*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b), \tag{1.1}$$

where $\mathrm{abs}_2(a)$ is a real quantity with $\mathrm{abs}_2(a) \neq 0$ if and only if a is invertible and

$$a^{-1} = \frac{\overline{a}}{\mathrm{abs}_2(a)} \qquad \textit{if} \qquad \mathrm{abs}_2(a) \neq 0.$$

For $\mathrm{abs}_2(a)$ there is the formula

$$\mathrm{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \textit{for } a \in \mathbb{H}, \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \textit{for } a \in \mathbb{H}_{\mathrm{coq}}, \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \textit{for } a \in \mathbb{H}_{\mathrm{nec}}, \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \textit{for } a \in \mathbb{H}_{\mathrm{con}}, \end{cases}$$

and the following property holds:

$$\mathrm{abs}_2(ab) = \mathrm{abs}_2(ba) = \mathrm{abs}_2(a)\mathrm{abs}_2(b).$$

Proof. See [4]. □

The imposed condition for (1.1) that $a, b \notin \mathbb{R}$ can be omitted for $\mathbb{H}$ but not for the other three noncommutative algebras. This can be easily seen by the example $a = (1, 0, 0, 0)$, $b = (1, 3, 3, 0)$ for $\mathbb{H}_{\mathrm{coq}}$ and $\mathbb{H}_{\mathrm{nec}}$ and by $a = (1, 0, 0, 0)$, $b = (1, 3, 0, 3)$ for $\mathbb{H}_{\mathrm{con}}$. The condition (1.1) is valid for a, b, but $a = (a_1, 0, 0, 0)$ is not similar to any nonreal element. In a corresponding theorem for coquaternions by Pogoruy and Rodríguez-Dagnino [9], this condition is missing.

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2. The Vandermonde approach. Given a *polynomial* p of degree $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers) in the form

$$p(x) := \sum_{j=1}^{n+1} a_j x^{j-1}, \quad x, a_j \in \mathcal{A}, \quad 1 \le j \le n+1, \tag{2.1}$$

and pairwise distinct *nodes*

$$x_k \in \mathcal{A}, \quad 1 \le k \le n+1, \tag{2.2}$$

such that

$$x_k - x_{k+1} \text{ are invertible for all } 1 \le k \le n, \tag{2.3}$$

and *values*

$$f_k \in \mathcal{A}, \quad 1 \le k \le n+1, \tag{2.4}$$

with the requirement that

$$p(x_k) = f_k, \quad 1 \le k \le n+1. \tag{2.5}$$

We use this notation because it is convenient in some programming languages like MATLAB. The cases with $n = 1, n = 2$, or $n = 3$ are called *linear, quadratic*, and *cubic*, respectively. The condition (2.3) for the nodes augments the standard condition that the nodes are pairwise distinct. In a nondivision algebra, two distinct nodes do not necessarily have the property that their difference is invertible as can be seen below. The requirement given in (2.5) leads to a linear system in the algebra $\mathcal{A}$ with $n+1$ unknowns and equations defined by the Vandermonde matrix. Therefore we call this approach the Vandermonde approach.

LEMMA 2.1. *The interpolation problem defined by* (2.1)–(2.5) *does not necessarily have a solution, and if there is a solution, then it may not be unique.*

Proof. Let there be two solutions p, q to the interpolation problem. Then $(p - q)(x_k) = 0$, for $1 \le k \le 1 + n$, which implies that the difference polynomial $p - q$ of degree n has $n + 1$ zeros. However, the polynomials considered here do not necessarily obey the Haar condition according to which any polynomial of degree n with more than n zeros vanishes identically. See [4, 8] also for the case that the Vandermonde matrix is singular. □

The algebra $\mathbb{H}$ of quaternions is a division algebra (i.e., the only noninvertible element is the zero element) and the problem under consideration has been solved for $\mathbb{H}$ by Lam [6] with extensions by Lam and Leroy [7]. The result reads as follows:

THEOREM 2.2. *The interpolation problem in $\mathbb{H}$ as stated above has a unique solution if and only if the nodes obey the following rule: no three of them belong to the same similarity class.*

Proof. By Lam [6]. □

DEFINITION 2.3. *Let there be $n + 1$ pairwise distinct nodes $x_k \in \mathcal{A}$, $1 \le k \le n + 1$. If there does not exist a subset of three nodes which belong to the same similarity class, then we say that the nodes satisfy the Lam condition.*

The Lam condition is satisfied if the underlying algebra $\mathcal{A}$ is commutative or if the number of nodes is at most two. In the algebra of quaternions $\mathbb{H}$, two distinct nodes have the property that their difference is invertible. This is not true for the other $\mathbb{R}^4$-algebras. Therefore the Lam condition is not powerful enough to guarantee the solvability of the interpolation problem for the other algebras.

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2.1. The linear and the quadratic cases. We treat the linear and quadratic cases individually in order to obtain some information for the general case. The linear interpolation problem can be written as

$$a_1 + a_2 x_1 = f_1,$$
$$a_1 + a_2 x_2 = f_2.$$

By subtracting the second equation from the first, we obtain

$$a_2(x_1 - x_2) = f_1 - f_2,$$

with the solution

$$a_2 = (f_1 - f_2)(x_1 - x_2)^{-1}, \quad a_1 = f_1 - a_2 x_1.$$

COROLLARY 2.4. *The linear interpolation problem in any algebra $\mathcal{A}$ has a unique solution if and only if the difference of the two nodes $x_1 - x_2$ is invertible.*

Let $n = 2$. Then (2.5) implies

$$a_1 + a_2 x_1 + a_3 x_1^2 = f_1, \tag{2.6}$$
$$a_1 + a_2 x_2 + a_3 x_2^2 = f_2, \tag{2.7}$$
$$a_1 + a_2 x_3 + a_3 x_3^2 = f_3. \tag{2.8}$$

We introduce some notation also for later use:

$$\begin{aligned}
g_1(j,k) &:= x_k^{j-1}, && 1 \le j,k \le 3,\\
\varphi_1(k) &:= f_k, && 1 \le k \le 3,\\
g_2(3,k) &:= \left(g_1(3,k) - g_1(3,k+1)\right)(x_k - x_{k+1})^{-1}, && 1 \le k \le 3,\\
\varphi_2(k) &:= \left(\varphi_1(k) - \varphi_1(k+1)\right)(x_k - x_{k+1})^{-1}, && 1 \le k \le 2,\\
\varphi_3(1) &:= \left(\varphi_2(1) - \varphi_2(2)\right)\left(g_2(3,1) - g_2(3,2)\right)^{-1}.
\end{aligned}$$

Subtracting equation (2.7) from (2.6) and (2.8) from (2.7) yields

$$\sum_{j=2}^{3} a_j (x_k^{j-1} - x_{k+1}^{j-1}) = f_k - f_{k+1}, \qquad 1 \le k \le 2.$$

Multiplying each of the two equations by $(x_k - x_{k+1})^{-1}$ from the right, for $1 \le k \le 2$, yields

$$a_2 + a_3 g_2(3,k) = \varphi_2(k), \qquad 1 \le k \le 2. \tag{2.9}$$

By subtracting and multiplying again, we obtain the final solution

$$a_3 = \varphi_3(1).$$

If a_3 is known, then a_2 can be computed from (2.9) and a_1 from (2.6).

There are two critical steps. In order for the quadratic interpolation problem to have a solution, it is necessary and sufficient that

$$\text{(i): } (x_1 - x_2)^{-1},\ (x_2 - x_3)^{-1}, \quad \text{and} \quad \text{(ii): } \left(g_2(3,1) - g_2(3,2)\right)^{-1}$$

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exist. For the second part (ii), we define

$$f(x_1,x_2,x_3) := g_2(3,1) - g_2(3,2) \\ = (x_1^2 - x_2^2)(x_1 - x_2)^{-1} - (x_2^2 - x_3^2)(x_2 - x_3)^{-1}. \tag{2.10}$$

The central question is whether f is invertible. Here we have to distinguish between the commutative and the noncommutative algebras.

THEOREM 2.5. *Let $\mathcal{A}$ be one of the commutative algebras. Then the quadratic interpolation problem has a unique solution if and only if the three differences $x_1 - x_2$, $x_2 - x_3$, and $x_1 - x_3$ are invertible.*

Proof. Because of commutativity, we have $f(x_1,x_2,x_3) = x_1 - x_3$. □

We use the following lemma.

LEMMA 2.6. *Let $z, h \in \mathcal{A}$, and let $\mathcal{A}$ be one of the noncommutative algebras. Then*

$$hz^k - z^k h = c_k(hz - zh), \qquad c_k \in \mathbb{R}, \quad \textit{for all } k \in \mathbb{N}.^3 \tag{2.11}$$

Proof. In all four noncommutative algebras, the formula

$$z^2 = -\mathrm{abs}_2(z) + 2\Re(z)z =: b_2 + c_2 z$$

holds, which implies (multiply by z and use the formula for z^2 again)

$$z^k = b_k + c_k z, \qquad b_k, c_k \in \mathbb{R}, \quad \text{for all } k \in \mathbb{N}.$$

(For more details and formulas for computing the constants b_k, c_k, see [4, p. 138] or [5, p. 247].) Thus, $z^k - c_k z = b_k \in \mathbb{R}$, and therefore, $h(z^k - c_k z) = (z^k - c_k z)h$ for all $h \in \mathcal{A}$. Rearranging terms yields (2.11). □

In the next lemma we gather some information about the consequences of the violation of the Lam condition.

LEMMA 2.7. *Let $\mathcal{A}$ be one of the noncommutative algebras, and let $x, y \in \mathcal{A}$ such that $x - y$ is invertible and $x \sim y$. Then*

$$(x^k - y^k)(x - y)^{-1} = c_k \in \mathbb{R}, \qquad \textit{for all } k \in \mathbb{N} \textit{ and all } y \in [x]. \tag{2.12}$$

Proof. By similarity we have $y = h^{-1}xh$. We multiply equation (2.11) form the left by h^{-1} and use the fact that the real number c_k commutes with all algebra elements and obtain $x^k - y^k = c_k(x - y)$ from which (2.12) follows. □

This lemma states that in all four noncommutative algebras $\mathcal{A}$, there are real numbers $c_k, k \in \mathbb{N}$, associated to all equivalence classes $[a]$. This also implies that under the conditions of Lemma 2.7, we have $f(x,y,z) = 0$, where f is defined in (2.10). Therefore, the Lam condition excludes the case $f(x,y,z) = 0$ but not necessarily the case that $f(x,y,z)$ is not invertible.

THEOREM 2.8. *Let $\mathcal{A}$ be one of the noncommutative algebras. Then the quadratic interpolation problem has a unique solution if in addition to $x_1 - x_2, x_2 - x_3$ being invertible, the quantity $f(x_1,x_2,x_3)$ defined in (2.10) is invertible.*

Proof. This follows from the above derivation of the highest coefficient a_3. The definition of $\varphi_3(1)$ depends on the invertibility of $f(x_1,x_2,x_3)$. □

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3 According to some experiments, this is possibly also true for other algebras $\mathcal{A}$ than $\mathbb{R}^4$-algebras when $\mathbb{R}$ is replaced by the center $\mathcal{C}_\mathcal{A}$ of $\mathcal{A}$. The center of an algebra is the set whose elements commute with all algebra elements.

EXAMPLE 2.9. Let

$$x_1 = (2,8,4,9), \qquad x_2 = (8,5,5,1), \qquad x_3 = (4,0,2,1),$$
$$f_1 = (1,2,1,1), \qquad f_2 = (8,6,3,5), \qquad f_3 = (1,2,4,0).$$

Note that $x_1 - x_3 = (-2,8,2,8)$ is not invertible in $\mathbb{H}_{\mathrm{coq}}$. Nevertheless, this problem has a solution in $\mathbb{H}_{\mathrm{coq}}$, which, shortened to four digits, is given by

$$a_1 = (357.1411, 479.8347, 185.6411, 567.8347),$$
$$a_2 = (-86.1452, -141.9758, -65.7823, -152.5806),$$
$$a_3 = (5.2460, 10.0121, 5.1411, 10.0202).$$

This example shows that the Vandermonde interpolation problem is not invariant under permutation of the nodes and values. In the above example, swapping x_1 and x_2 and f_1 and f_2, respectively, will render the problem unsolvable.

2.2. The general case. The technique to find the solution of a polynomial interpolation problem of degree n has been sketched already for the quadratic case. It essentially requires a triangulation of the underlying Vandermonde matrix. The general form of this procedure will be summarized in the following theorem.

THEOREM 2.10. *In order to solve the polynomial interpolation problem as stated in* (2.1)–(2.5)*, one has to perform the following steps: define*

$$g_1(j,k) := x_k^{j-1}, \qquad 1 \le j,k \le n+1, \tag{2.13}$$
$$\varphi_1(k) := f_k, \qquad 1 \le k \le n+1, \tag{2.14}$$
$$g_\ell(j,k) := \left(g_{\ell-1}(j,k) - g_{\ell-1}(j,k+1)\right)\left(g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)\right)^{-1}, \tag{2.15}$$
$$\varphi_\ell(k) := \big(\varphi_{\ell-1}(k) - \varphi_{\ell-1}(k+1)\big)\big(g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)\big)^{-1}, \tag{2.16}$$
$$\ell \le 2 \le n+1, \quad 1 \le k \le n-\ell+2, \quad \ell+1 \le j \le n+1,$$

where we assume that all inverses of $g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)$ occurring in (2.15) *and* (2.16) *exist. Then*

$$a_\ell + \sum_{j=\ell+1}^{n+1} a_j g_\ell(j,k) = \varphi_\ell(k), \qquad 1 \le \ell \le n+1, \quad 1 \le k \le n-\ell+2, \tag{2.17}$$

and

$$a_{n+1} = \varphi_{n+1}(1). \tag{2.18}$$

If a_{n+1} is known by (2.18)*, then we compute the coefficients $a_n, a_{n-1}, \ldots, a_1$ by back substitution using formula* (2.17) *with $k = 1$.*

Proof. We start with the representation

$$a_1 + \sum_{j=2}^{n+1} a_j x_k^{j-1} = f_k, \qquad 1 \le k \le n+1,$$

which is the same as

$$a_1 + \sum_{j=2}^{n+1} a_j g_1(j,k) = \varphi_1(k), \qquad 1 \le k \le n+1. \tag{2.19}$$

Subtracting the equation for $k+1$ from that for k in (2.19), multiplying the difference by $(x_k - x_{k+1})^{-1}$ (assuming that this is possible), and applying (2.15) and (2.16) yields

$$a_2 + \sum_{j=3}^{n+1} a_j g_2(j,k) = \varphi_2(k).$$

Having arrived at (2.17), we can use induction to show that

$$a_{\ell+1} + \sum_{j=\ell+2}^{n+1} a_j g_{\ell+1}(j,k) = \varphi_{\ell+1}(k). \qquad \square$$

Let us assume that we have solved an interpolation problem successfully. How to judge the quality of the computed coefficients $a_1, a_1, \ldots, a_{n+1}$? One possibility is to compute $\tilde{f}_k := p(x_k)$ and compare these values with the given ones f_k for all $1 \le k \le n+1$. For test purposes it is a good idea to choose all components of f_k as integers. Then the errors are the deviation of $\tilde{f}_k$ from being integer. This can be measured in the maximum norm by arranging each $\tilde{f}_k$ and f_k into a real $4(n+1)$ column vector and then forming $e := \max|\mathrm{col}(\tilde{f}_k) - \mathrm{col}(f_k)|$ as a measure for the *error*, where col indicates a rearrangement into a column vector.

EXAMPLE 2.11. Let $n = 3$ and

$$\begin{aligned} x_1 &= (2,8,4,9), & x_2 &= (8,5,5,1), & x_3 &= (4,0,2,1), & x_4 &= (9,9,4,4),\\ f_1 &= (1,2,1,1), & f_2 &= (8,6,3,5), & f_3 &= (1,2,4,0), & f_4 &= (3,9,3,1). \end{aligned}$$

These data were chosen randomly. There are no solutions in $\mathbb{H}_{\rm cotes}$ or $\mathbb{H}_{\rm nec}$ because $x_1 - x_2$ is not invertible in $\mathbb{H}_{\rm cotes}$ and $x_2 - x_3$ is not invertible in $\mathbb{H}_{\rm nec}$. For all other cases, the interpolation problem has a solution as given in the following tables, shortened to 4 digits.

- Solution for $\mathbb{H}$ with error $= 2.5757 \cdot 10^{-14}$:

$$\begin{aligned} a_1 &= (-6.4416, -15.2697, 8.2518, 2.6443),\\ a_2 &= (1.0192, 4.8057, 0.9450, -3.7386),\\ a_3 &= (-0.0542, -0.0930, -0.4554, 0.4117),\\ a_4 &= (-0.0063, -0.0076, 0.0215, 0.0002). \end{aligned}$$

- Solution for $\mathbb{H}_{\rm coq}$ with error $= 5.4001 \cdot 10^{-13}$ ($x_1 - x_3$ is not invertible):

$$\begin{aligned} a_1 &= (176.1447, 257.9025, 114.5842, 290.1670),\\ a_2 &= (-69.1053, -115.1597, -55.6326, -122.4325),\\ a_3 &= (10.2252, 14.3371, 6.0045, 16.4766),\\ a_4 &= (-0.5421, -0.4238, -0.0404, -0.6820). \end{aligned}$$

- Solution for $\mathbb{H}_{\rm tes}$ (commutative) with error $= 3.7303 \cdot 10^{-14}$:

$$\begin{aligned} a_1 &= (-5.1033, 9.7931, -5.4347, 5.3327),\\ a_2 &= (1.0124, -2.5193, 0.8486, -1.9091),\\ a_3 &= (-0.1439, -0.0969, 0.2835, 0.4170),\\ a_4 &= (0.0535, 0.0014, -0.0606, -0.0041). \end{aligned}$$

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• Solution for $\mathbb{H}_{\rm con}$ with error $= 6.3594 \cdot 10^{-13}$ ($x_1 - x_3$ is not invertible):

$$\begin{aligned} a_1 &= (-1.4124, -6.9135, 11.4218, -16.4555),\\ a_2 &= (9.9449, 0.5466, -3.5760, 10.3660),\\ a_3 &= (-2.9882, 0.6470, 1.0686, -1.9385),\\ a_4 &= (0.1220, -0.0208, -0.0261, 0.1024). \end{aligned}$$

• Solution for $\mathbb{H}_{\rm tan}$ (commutative) with error $= 3.4195 \cdot 10^{-14}$:

$$\begin{aligned} a_1 &= (-23.9102, -17.9102, 3.6414, 1.6414),\\ a_2 &= (6.6223, 4.1439, -4.5867, -3.8300),\\ a_3 &= (-0.2334, -0.0737, 0.528, 0.4998),\\ a_4 &= (-0.0036, -0.0038, 0.0198, -0.0454). \end{aligned}$$

• Solution for $\mathbb{H}_{\rm cotan}$ (commutative) with error $= 5.4179 \cdot 10^{-14}$:

$$\begin{aligned} a_1 &= (2.7916, 46.4053, -41.7540, 4.0301),\\ a_2 &= (0.4118, -14.6794, 14.4364, -0.4932),\\ a_3 &= (0.1728, 1.5803, -1.4896, 0.1744),\\ a_4 &= (-0.0166, -0.0503, 0.0482, -0.0125). \end{aligned}$$

There is one general result for interpolation polynomials in commutative algebras.

THEOREM 2.12. *Let $\mathcal{A}$ be one of the commutative algebras. Then the Vandermonde interpolation polynomial of degree n exists if and only if for the underlying nodes $x_1, x_2, \ldots, x_{n+1}$, the node differences*

$$x_k - x_{\ell+k-1}, \qquad 2 \le \ell \le n+1, \quad 1 \le k \le n - \ell + 2 \tag{2.20}$$

are invertible.

Proof. We prove that in the formulas (2.13)–(2.16) for finding the interpolation polynomial, the factors involving a computation of an inverse can be expressed as follows:

$$g_{\ell-1}(\ell, k) - g_{\ell-1}(\ell, k+1) = x_k - x_{\ell+k-1}. \tag{2.21}$$

For $\ell = 2$ the statement follows directly because $g_1(2,k) - g_1(2,k+1) = x_k - x_{k+1}$ for $1 \le k \le n$. Now for $\ell = 3$,

$$\begin{aligned} g_2(3,k) - g_2(3,k+1) &= \big(g_1(3,k) - g_1(3,k+1)\big)\big(g_1(2,k) - g_1(2,k+1)\big)^{-1}\\ &\quad - \big(g_1(3,k+1) - g_1(3,k+2)\big)\big(g_1(2,k+1) - g_1(2,k+2)\big)^{-1}\\ &= (x_k^2 - x_{k+1}^2)(x_k - x_{k+1})^{-1} - (x_{k+1}^2 - x_{k+2}^2)(x_{k+1} - x_{k+2})^{-1}\\ &= (x_k + x_{k+1}) - (x_{k+1} + x_{k+2}) = x_k - x_{k+2}, \quad 1 \le k \le n-1. \end{aligned}$$

For $\ell = 4$ we compute $g_3(4,k) - g_3(4,k+1)$. The first part is

$$\begin{aligned} g_3(4,k) &= \big(g_2(4,k) - g_2(4,k+1)\big)\big(g_2(3,k) - g_2(3,k+1)\big)^{-1}\\ &= \big(g_2(4,k) - g_2(4,k+1)\big)\big(x_k - x_{k+2}\big)^{-1}. \end{aligned}$$

We continue with the first factor

$$\begin{aligned} &g_2(4,k) - g_2(4,k+1)\\ &\quad = \big(g_1(4,k) - g_1(4,k+1)\big)\big(g_1(2,k) - g_1(2,k+1)\big)^{-1}\\ &\qquad - \big(g_1(4,k+1) - g_1(4,k+2)\big)\big(g_1(2,k+1) - g_1(2,k+2)\big)^{-1} \end{aligned}$$

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$$
\begin{aligned}
&= (x_k^3 - x_{k+1}^3)(x_k - x_{k+1})^{-1} - (x_{k+1}^3 - x_{k+2}^3)(x_{k+1} - x_{k+2})^{-1} \\
&= (x_k^2 + x_k x_{k+1}) - (x_{k+1}x_{k+2} + x_{k+2}^2) = x_k^2 - x_{k+2}^2 + x_{k+1}(x_k - x_{k+2}) \\
&= (x_k - x_{k+2})(x_k + x_{k+1} + x_{k+2}).
\end{aligned}
$$

Thus, the first and second parts are given by

$$
g_3(4,k) = x_k + x_{k+1} + x_{k+2}, \qquad g_3(4,k+1) = x_{k+1} + x_{k+2} + x_{k+3},
$$

and as final result, their difference is $x_k - x_{k+3}$, for $1 \le k \le n-2$, as desired. For general ℓ, induction with respect to ℓ can be used to prove (2.21). □

A simple count reveals that the list of node differences in (2.20) contains all possible differences $x_{k_1} - x_{k_2}$ with $1 \le k_1 < k_2 \le n+1$.

COROLLARY 2.13. *Let $\mathcal{A}$ be one of the commutative algebras. Then the Vandermonde interpolation polynomial exists if and only if all node differences are invertible.*

In $\mathbb{H}$, the field of quaternions, the Lam condition is equivalent to the condition that all quadratic interpolation problems on three arbitrarily selected, pairwise distinct nodes have a solution.

DEFINITION 2.14. Let there be $n+1 \ge 3$ pairwise distinct nodes in one of the algebras $\mathcal{A}$. If for all subsets of three nodes, the quadratic interpolation problem has a solution, then we say that the nodes obey the *extended Lam condition*.

CONJECTURE 2.15. *Let the extended Lam condition be valid. Then the Vandermonde interpolation polynomial exists.*

The conjecture is apparently true for commutative algebras. It is also true for equidistant nodes which have the form $x_{k+1} := (\ell k + k_0)\xi$, $0 \le k \le n$, where ξ is a fixed, invertible algebra element and ℓ, k_0 are fixed integers with $\ell \ne 0$. The standard case is $\ell = k_0 = 1$. It is not difficult to show that equidistant nodes always lead to an invertible element f as defined in (2.10). Thus, the extended Lam condition is valid, and the decisive quantity $g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)$ required in (2.15), (2.16) is always an integer multiple of ξ. Thus, it is invertible, and the Vandermonde interpolation polynomial exists in this case.

3. The Newton approach. The Vandermonde approach has the advantage of dealing with a polynomial in standard form (2.1), but it also has several disadvantages. Just by looking at the nodes, it is difficult to judge whether an interpolation polynomial exists for data from nondivision algebras. And if it exists, the Vandermonde approach may lead to numerically bad results since it has been known for a long time (see Gautschi [3]) that the Vandermonde matrix in standard form has a very high condition number. There are also several papers to circumvent this difficulty by various means. One example is a paper by Reichel and Opfer [10]. Though the present author is not aware of investigations about the condition number for Vandermonde matrices with entries from nondivision algebras, it cannot be expected that the condition number is smaller than for the standard case.

The interpolation problem to be treated here has the following setting: given elements of an algebra $\mathcal{A}$ as data (x_k, f_k), $1 \le k \le n+1$, representing the nodes and values, respectively, with the minimal requirement that the nodes are pairwise distinct. Wanted is a polynomial p of degree n in the form

$$
p(x) := \sum_{j=1}^{n+1} a_j p_{j-1}(x), \quad x, a_j \in \mathcal{A}, \quad 1 \le j \le n+1, \ \text{where}
$$

$$
p_0(x) := 1, \quad \text{for all } x \in \mathcal{A}, \qquad p_j(x) := \prod_{k=1}^{j} (x - x_k), \quad 1 \le j \le n,
$$

and the requirement that

$$p(x_k) = f_k, \qquad 1 \le k \le n+1.$$

We call this problem the *Newton interpolation problem*.

LEMMA 3.1. *Let the data $(x_k, 0)$ be given with the property that the differences $x_{k_1} - x_{k_2}$ of the nodes are invertible for all $1 \le k_1 < k_2 \le n+1$. Then the solution of the Newton interpolation problem is $p(x) = 0$ for all $x \in \mathcal{A}$.*

Proof. The hypothesis for the nodes implies that all differences $x_k - x_\ell, k \ne \ell$, are invertible. We show that $a_j = 0$, for $1 \le j \le n+1$. For this purpose, we insert $x_1, x_2, \ldots, x_{n+1}$ into p in this order and obtain $a_1 = p(x_1) = 0$, $p(x_2) = a_1 + a_2(x_2 - x_1) = 0$, which implies $a_2 = 0$ since $x_2 - x_1$ is invertible. Then, $p(x_3) = a_1 + a_2(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2) = 0$ implies $a_3 = 0$ since $(x_3 - x_1)(x_3 - x_2)$ is invertible. Thus, all a_j, for $1 \le j \le n+1$, vanish. ∎

This implies that the solution is unique.

COROLLARY 3.2. *Assume that all node differences $x_k - x_\ell$ for $k \ne \ell$ are invertible. Let there be two solutions p and q to the Newton interpolation problem. Then $p = q$.*

For the general problem we can state the following theorem.

THEOREM 3.3. *Let all node differences $x_k - x_\ell$ for $k \ne \ell$ be invertible. Then there is a unique Newton interpolation polynomial for the data $(x_k, f_k), 1 \le k \le n+1$. If one of the node differences $x_k - x_\ell$ for $k \ne \ell$ is not invertible, then no Newton interpolation polynomial exists.*

Proof. The assumption that all node differences are invertible, implies that

$$p_j(x_\ell) = \begin{cases} 0 & \text{for } \ell \le j, \\ \text{invertible} & \text{for } \ell > j. \end{cases}$$

Now,

$$p(x_1) = a_1 = f_1. \tag{3.1}$$

Assume that a_j are known for all $1 \le j \le \ell$. Then,

$$p(x_{\ell+1}) = \sum_{j=1}^{\ell} a_j p_{j-1}(x_{\ell+1}) + a_{\ell+1} p_\ell(x_{\ell+1}) = f_{\ell+1}.$$

This implies

$$a_{\ell+1} = \Big(f_{\ell+1} - \sum_{j=1}^{\ell} a_j p_{j-1}(x_{\ell+1})\Big)\big(p_\ell(x_{\ell+1})\big)^{-1}, \quad 1 \le \ell \le n, \tag{3.2}$$

and all coefficients are known. If one of the differences has no inverse, then there will be one ℓ such that $p_\ell(x_{\ell+1})$ has no inverse and formula (3.2) cannot be applied. ∎

EXAMPLE 3.4. We use the data from Example 2.11 and choose the commutative tessarine case $\mathcal{A} = \mathbb{H}_{\text{tes}}$. For commutative algebras, the two types of polynomials must coincide in the sense that they have the same values at all $x \in \mathcal{A}$, which does not imply that the coefficients are the same with the exception of the highest coefficient.

• Solution for $\mathbb{H}_{\text{tes}}$ (commutative) with error $= 1.7764 \cdot 10^{-15}$:

$$\begin{aligned} a_1 &= (1.0000, \ 2.0000, \ 1.0000, \ 1.0000) = f_1, \\ a_2 &= (0.1765, \ 0.2059, -0.3235, \ 0.7059), \\ a_3 &= (-0.0335, -0.0933, \ 0.0626, \ 0.1760), \\ a_4 &= (0.0535, \ 0.0014, -0.0606, -0.0041). \end{aligned}$$

Note that the error here is by a factor of 21 smaller than the corresponding error for $\mathbb{H}_{\rm tes}$ in Example 2.11. The polynomial value at $x := (1, 2, 3, 4)$ is

$$(6.458660398875651, 4.787370206864643, 1.650198860414113, 4.589677899172335).$$

It differs from the corresponding value for the Vandermonde polynomial with the same data by at most two digits in the last two positions. The coefficient a_4 coincides in all computed digits with the corresponding coefficient a_4 from the Vandermonde polynomial.

It would be of interest to derive an error expansion for growing degrees n for both types of interpolation. However, this may be a topic for another paper.

4. Concluding remarks. Though we have chosen examples from $\mathbb{R}^4$ algebras, the two algorithms given in (2.13)–(2.18) and in (3.1), (3.2) are valid for all types of algebras. An easy way of implementing them is to use the "overloading technique" offered by MATLAB.

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