Electronic Transactions on Numerical Analysis.
Volume 44, pp. 401–442, 2015.

ISSN 1068–9613.

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AN OVERVIEW OF MULTILEVEL METHODS WITH AGGRESSIVE COARSENING AND MASSIVE POLYNOMIAL SMOOTHING*

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Abstract. We review our two-level and multilevel methods with aggressive coarsening and polynomial smoothing. These methods can be seen as a less expensive and more flexible (in the multilevel case) alternative to domain decomposition methods. The polynomial smoothers employed by the reviewed methods consist of a sequence of Richardson iterations and can be performed using up to n processors, where n is the size of the considered matrix, thereby allowing for a higher level of parallelism than domain decomposition methods.

Key words. multigrid, aggressive coarsening, optimal convergence result

AMS subject classifications. 65F10,65M55

1. Introduction. In general, domain decomposition methods (see, e.g., [1, 2, 3, 4, 10, 12, 16] and for a comprehensive list of references see the monograph [18]) can be, in a multigrid terminology ([5, 13, 26]), viewed as two-level multigrid methods with a small coarse space and a massive smoother that uses local subdomain solvers. The size of the subdomains and the size of the coarse space are closely (and inversely) related, which has important implications for the parallel scalability of such methods. From both a theoretical and practical viewpoint, domain decomposition methods are sought whose convergence rate is independent or nearly independent of the mesh size and the subdomain size. Some of the most successful types of domain decomposition methods—substructuring methods such as BDDC [17], FETI [12], and FETI-DP [11] class methods and overlapping Schwarz methods—satisfy this property. There are, however, several disadvantages common to these domain decomposition algorithms when it comes to parallel scalability, and conflicting objectives are encountered. The local subdomain solvers are, especially for 3D problems, relatively expensive and scale unfavorably with respect to the subdomain size. Since only a small number of processors can be used efficiently by each subdomain solver, massive parallelism may also be limited. Both of these issues lead to a preference for small subdomains. However, using small enough subdomains gives rise to an undesirably large coarse-space problem, which destroys parallel scalability.

The goal of this paper is to review our two-level and multilevel methods with aggressive coarsening (i.e., leading to small coarse problems) and massive smoothing that do not have the above disadvantages. Instead of smoothers that are based on local subdomain solvers, we use special polynomial smoothers that are based on the properties of Chebyshev polynomials and are a sequence of Richardson iterations. In our most general result, assuming that the mesh size on the level l can be characterized by h_l and employing a carefully designed polynomial smoother as a multigrid relaxation process, we prove (for a general multigrid V-cycle) a convergence result independent of the ratio h_{l+1}/h_l, provided that the degree of our smoother is greater than or equal to Ch_{l+1}/h_l. A Richardson iteration can be performed in parallel using n processors, where n is the size of the considered matrix. Thus, our methods are asymptotically much less expensive than the domain decomposition methods that use direct subdomain solvers and allow for a finer-grained parallelism.

*Received July 19, 2014. Accepted May 19, 2015. Published online August 21, 2015. Recommended by M. Donatelli.

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It is generally believed that when assuming no regularity, it is not useful to perform more than $O(1)$ multigrid smoothing steps. One of the key objectives of this paper is to demonstrate that this is not the case. Our earlier methods are based on smoothed aggregation and show a significant acceleration compared to the use of $O(1)$ smoothing steps if the number of multigrid smoothing steps is approximately equal to the number of prolongator smoothing steps. Finally, our most general result is multilevel and fully independent of the smoothing of the prolongator.

This review paper covers a relatively large time span of research and is organized as follows. Four methods (frameworks) are described in four key sections. Two-level methods (Sections 3 and 4) are based on the smoothed-aggregation concept. It is shown that it makes sense to perform a number of multigrid smoothing iterations that is approximately equal to the number of *prolongator smoothing* steps. The method reviewed in Section 3 shows a significant acceleration if multiple prolongator smoothing and multigrid smoothing steps are performed, but it does not achieve a fully optimal convergence, that is, independent of both the fine resolution characterized by the mesh size h and the coarse-space resolution characterized by the subdomain size H for the cost of $O(H/h)$ multigrid smoothing steps. On the other hand, the two-level method of Section 4 is fully optimal. Section 5 reviews a two-level method that is fully optimal and also allows for a multilevel extension. A convergence rate independent of the resolutions of the first two levels is established. (That is, aggressive coarsening between level 1 and level 2 can be performed and is fully compensated by the smoothing). The method presented here is in spirit close to smoothed aggregation. The most general framework is described and analyzed in Section 6. A general (abstract) multigrid method with our polynomial smoother is considered. We stress that the analyzed method does not have to be based on smoothed aggregation. As stated above, for the cost of a number of multigrid smoothing iterations that is greater than or equal to ch_{l+1}/h_l, a convergence rate independent of the resolutions h_l on all levels is proved. The convergence estimates are confirmed numerically in Section 7. The results are summarized in Section 8.

2. Two-level variational multigrid.

The solution of a system of linear algebraic equations

$$Ax = \mathbf{f}, \tag{2.1}$$

where A is a symmetric positive definite $n \times n$ matrix that arises from a finite element discretization of an elliptic boundary value problem, is considered. To define a variational two-level method, two algorithmic ingredients are required: a linear *prolongator*, $P : \mathbb{R}^m \to \mathbb{R}^n$, $m < n$, and a *smoothing procedure*. In this paper, polynomial smoothers that can be expressed as a sequence of Richardson iterations

$$\mathbf{x} \leftarrow (I - \omega A)\,\mathbf{x} + \omega \mathbf{f} \tag{2.2}$$

are considered. Note that a particular case of interest is that of a small coarse space, that is, $m \ll n$. Let ν_1 and ν_2 be integers. A variational two-level method proceeds as follows:

ALGORITHM 1.
1. For $i = 1, \ldots, \nu_1$ do $\mathbf{x} \leftarrow (I - \alpha_i A)\,\mathbf{x} + \alpha_i \mathbf{f}$.
2. Set $\mathbf{d} = A\mathbf{x} - \mathbf{f}$.
3. Restrict $\mathbf{d}_2 = P^T \mathbf{d}$.
4. Solve the coarse problem $A_2 \mathbf{v} = \mathbf{d}_2$, where $A_2 = P^T AP$,
5. Correct $\mathbf{x} \leftarrow \mathbf{x} - P\mathbf{v}$,
6. For $i = 1, \ldots, \nu_2$ do $\mathbf{x} \leftarrow (I - \beta_i A)\,\mathbf{x} + \beta_i \mathbf{f}$.

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Throughout Sections 3 and 4, the *tentative prolongator* p is assumed to be constructed by a generalized unknowns aggregation method [22]. This is strictly because of considerations of algorithmic complexity. A simple example of a (non-generalized) aggregation method is presented below in Example 2.1. For a standard finite element discretization of a scalar elliptic problem, the generalized aggregation method coincides (up to a scaling) with a standard unknowns aggregation method [22]. The resulting prolongator p is an orthogonal matrix [22].

The final prolongator P is obtained by polynomial smoothing [19, 20, 22, 23, 24]

$$P = Sp, \quad S = (I - \omega_1 A) \cdots (I - \omega_\nu A), \tag{2.3}$$

where ν is a positive integer. The coefficients α_i, β_i, and ω_i are chosen carefully and kept in a close relationship. The tentative prolongator p is responsible for the approximation properties of the coarse-space $\mathrm{Range}(P)$. The *prolongator smoother* S enforces smoothness of the coarse-space functions.

EXAMPLE 2.1. Consider a one-dimensional Laplace equation discretized on a uniform mesh that consists of $n = mN$ nodes. A simple unknowns aggregation prolongator can be constructed as follows. Let the nodes be numbered in the usual way from left to right. The aggregates are formed as disjoint sets of N consecutive nodes, i.e.,

$$\begin{aligned} \{\mathcal{A}_i\}_{i=1}^m = \big\{ &\{1, 2, \dots, N\}, \{N+1, N+2, \dots, 2N\}, \dots, \\ &\{(m-1)N+1, \dots, mN-1, mN\} \big\}. \end{aligned} \tag{2.4}$$

The corresponding prolongator is given by

$$p_{ij} = \begin{cases} 1 & \text{iff } \; i \in \mathcal{A}_j, \\ 0 & \text{otherwise}, \end{cases} \tag{2.5}$$

that is, the j-th column is created by restricting a vector of ones onto the j-th aggregate with zeroes elsewhere. In matrix form,

$$p = \begin{bmatrix} 1 & & & \\ \vdots & & & \\ 1 & & & \\ & 1 & & \\ & \vdots & & \\ & 1 & & \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \\ & & & 1 \\ & & & \vdots \\ & & & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} \mathcal{A}_1 \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} \mathcal{A}_2 \\ \vphantom{\begin{matrix}\vdots\\ \vdots\\ \vdots\end{matrix}} \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} \mathcal{A}_m \end{matrix} . \tag{2.6}$$

The action of the prolongator corresponds to a disaggregation of the j-th $\mathbb{R}^m$-variable into N $\mathbb{R}^n$-variables forming the aggregate $\mathcal{A}_j$. Thus, p can be thought of as a discrete piecewise constant interpolation. The prolongator becomes an orthogonal matrix by the scaling

$$p \leftarrow \frac{1}{\sqrt{N}} p \, .$$

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For scalar problems (such as Example 2.1), the columns of the prolongator p have a disjoint nonzero structure. This can also be viewed as the discrete basis functions of the coarse-space $\mathrm{Range}(p)$ having disjoint supports. For non-scalar elliptic problems, several fine-level vectors are restricted to each of the aggregates. For example, for a discretization of the equations of linear elasticity, six rigid-body modes are restricted to each of the aggregates giving rise to six columns with the same nonzero structure. Such a set of columns is labeled a *super-column* and the corresponding set of coarse-level degrees of freedom (each associated with one column) a *super-node*. The super-columns have a disjoint nonzero structure corresponding to the disjoint nonzero structure of the aggregates. Thus, in general, it is assumed that the discrete coarse-space basis functions (columns of the prolongator p) are non-overlapping unless they belong to the same aggregate.

A key assumption to prove convergence of a two-level method is that the prolongator satisfies the *weak approximation condition*

$$\forall \mathbf{e} \in \mathbb{R}^n \, \exists \mathbf{v} \in \mathbb{R}^m : \; \|\mathbf{e} - p\mathbf{v}\|^2 \le \frac{C_A}{\varrho(A)} \left(\frac{H}{h}\right)^2 \|\mathbf{e}\|_A^2 \, . \tag{2.7}$$

Here, h is a characteristic element size of the fine-level discretization (assuming the quasi-uniformity of the mesh), and H is a characteristic diameter of the aggregates (understood as a set of finite element nodal points). For a scalar elliptic second-order problem, (2.7) was proved in [22]. For the case of linear elasticity in 3D, the reader is referred to [23]. A simple example is given below.

EXAMPLE 2.2. To illustrate the verification of property (2.7), consider a one-dimensional model example: for a given $f \in L_2((0,1))$, find $u \in H_0^1((0,1))$ such that

$$(u', v')_{L_2((0,1))} = (f, v)_{L_2((0,1))}, \qquad \forall v \in H_0^1((0,1)).$$

The model problem is discretized by P1-elements on a uniform grid of mesh size $h = \frac{1}{mN+1}$. The discretization leads to a system (2.1) of $n = mN$ linear algebraic equations. Each equation corresponds to one unconstrained node in the interval $(0,1)$. The tentative prolongator p is based on an aggregation of groups of N neighboring nodes as depicted in Figure 2.1 and described in Example 2.1. This yields (2.5) or equivalently the matrix form (2.6).

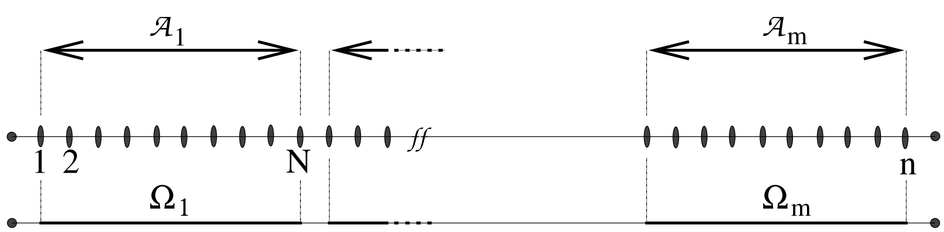

FIG. 2.1. *Aggregates of N nodes.*

Consider a finite element interpolator $\Pi_h : \mathbf{x} = (x_1, \ldots, x_n)^T \mapsto \sum_i x_i \varphi_i$. The tentative prolongator p applied to a vector $\mathbf{v} = (v_1, \ldots, v_m)^T$ returns the value v_i in all the variables of the aggregate $\mathcal{A}_i$. Hence, for the corresponding finite element function we have

$$\Pi_h p \mathbf{v} = v_i \qquad \text{on } \Omega_i, \tag{2.8}$$

where Ω_i is the subinterval of $(0,1)$ corresponding to the aggregate $\mathcal{A}_i$; see Figure 2.1.

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Let H denote the length of the intervals Ω_i. The verification of (2.7) consists in a straightforward application of the scaled Poincaré inequality on the subintervals Ω_i,

$$\min_{q\in\mathbb{R}}\|u-q\|_{L_2(\Omega_i)} \le CH|u|_{H^1(\Omega_i)}, \qquad \forall u \in H^1(\Omega_i), \tag{2.9}$$

and the equivalence of the discrete and continuous L_2-norms for finite element functions [9],

$$ch^{-1}\|\Pi_h\mathbf{u}\|^2_{L_2(\Omega_i)} \le \|\mathbf{u}\|^2_{l_2(\Omega_i)} \equiv \sum_{j\in\mathcal{A}_i} u_j^2 \le Ch^{-1}\|\Pi_h\mathbf{u}\|^2_{L_2(\Omega_i)}\,. \tag{2.10}$$

From (2.8), (2.9), and (2.10), it follows that for every $\mathbf{u} \in \mathbb{R}^n$, there exists a coarse-level vector $\mathbf{v} = (v_1,\dots,v_m)^T$ such that

$$\begin{aligned}
\|\mathbf{u}-p\mathbf{v}\|^2 &= \sum_{i=1}^m \|\mathbf{u}-p\mathbf{v}\|^2_{l_2(\Omega_i)} \le Ch^{-1}\sum_{i=1}^m \|\Pi_h\mathbf{u}-v_i\|^2_{L_2(\Omega_i)}\\
&= Ch^{-1}\sum_{i=1}^m \min_{q_i\in\mathbb{R}}\|\Pi_h\mathbf{u}-q_i\|^2_{L_2(\Omega_i)} \le C\frac{H^2}{h}\sum_{i=1}^m |\Pi_h\mathbf{u}|^2_{H^1(\Omega_i)}\\
&\le C\frac{H^2}{h}|\Pi_h\mathbf{u}|^2_{H^1((0,1))}\,.
\end{aligned}$$

Therefore, since $\varrho(A) \le Ch^{-1}$ (cf. [9]) and $|\Pi_h\mathbf{u}|_{H^1((0,1))} = \|\mathbf{u}\|_A$, it follows that there exists a mapping $Q_C : \mathbb{R}^n \to \mathrm{Range}(p)$ such that

$$\|\mathbf{u}-Q_C\mathbf{u}\| \le C\frac{H}{h\sqrt{\varrho(A)}}\|\mathbf{u}\|_A, \qquad \forall \mathbf{u}\in\mathbb{R}^n.$$

This shows (2.7).

The proof can be easily extended to multiple dimensions as well as to the case of unstructured domains Ω_i as long as the domains are shape regular (the constant in the scaled Poincaré inequality is uniformly bounded). This requirement can be also weakened by encapsulating the domains Ω_i into balls with bounded overlap [22].

The constant in the weak approximation condition (2.7) depends on the ratio $\frac{H}{h}$. As a result, the convergence of the straightforward two-level method depends on the same ratio. More specifically, assuming (2.7), the variational two-level method with the prolongator p and a single Jacobi post-smoothing step converges with the rate of convergence

$$\|E\mathbf{e}\|^2_A \le \left(1 - C\left(\frac{h}{H}\right)^2\right)\|\mathbf{e}\|^2_A, \tag{2.11}$$

where E is the error propagation operator of the method. The objective of the methods reviewed in this paper is to eliminate this dependence of the convergence of a two-level method on the ratio $\frac{H}{h}$ with a minimal possible cost. Domain decomposition methods strive toward the same goal. A typical domain decomposition method can be viewed as a two-level multigrid method with a small coarse space whose local resolution corresponds to the subdomain size and a block-smoother that uses direct subdomain solvers. The subdomain solvers are relatively expensive. Here, methods with a much lower cost that also open the room for a higher level of fine-grain parallelism are described and analyzed.

A two-level method is labeled *optimal* if, for a second-order elliptic problem discretized on a mesh with the mesh size h, it yields a small, sparse coarse space characterized by the diameter H and an H/h-independent rate of convergence for the cost of $O(H/h)$ elementary

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TABLE 2.1
Fine-level cost of an optimal two-level multigrid method and a domain decomposition method based on direct (banded and sparse) subdomain solvers. The coarse-level setup and solve are not included.

Method	Space dim. d	Subdomain problem size	Bandwidth	Total cost
DD+banded	2	$O((H/h)^2)$	$O(H/h)$	$O(n(H/h)^2)$
DD+banded	3	$O((H/h)^3)$	$O((H/h)^2)$	$O(n(H/h)^4)$
DD+sparse	2	$O((H/h)^2)$	–	$O(n(H/h))$
DD+sparse	3	$O((H/h)^3)$	–	$O(n(H/h)^3)$
Optimal TMG	any d	–	–	$O(n(H/h))$

smoothing steps. Generally, a Richardson iteration sweep given by (2.2) is considered as an elementary smoothing step. One cannot possibly expect a better result, i.e., a coarse-space size independent convergence with fewer than $O(H/h)$ smoothing steps since that many steps are needed to establish *essential communication* within the discrete distance $O(H/h)$, that is, the continuous distance $O(H)$.

An optimal two-level method is significantly less expensive than a domain decomposition (DD) method based on local subdomain solvers. A DD method needs to solve $(1/H)^d$ subdomain linear problems of size $O((H/h)^d)$. This is reflected in a comparison of the complexity of the optimal two-level method shown in Table 2.1. Whether a direct banded or a direct sparse solver is used for the Cholesky decomposition of the local matrices in the DD method, the cost of the optimal two-level method is significantly lower in three dimensions. Furthermore, an optimal two-level method is more amenable to massive parallelism. In this method, the smoothing using $O(H/h)$ Richardson sweeps (2.2), which constitutes a bottleneck of the entire procedure, can be performed using up to n processors. On the other hand, in a DD method, subdomain-level- and therefore much coarser-grained parallelism is natural, where typically only $O(m)$ processors can be utilized, where m is the number of subdomains.

3. Naive beginning. In this section, a two-level method of [14] is described and analyzed. Even though it does not have a fully optimal convergence, it still has a practical value. For the cost of $O(H/h)$ smoothing steps in the smoothed prolongator and $O(H/h)$ standard multigrid smoothing steps, the rate of convergence of the straightforward method (2.11) is improved as

$$\|E\mathbf{e}\|_A^2 \leq \left(1 - C\frac{h}{H}\right)\|\mathbf{e}\|_A^2.$$

When the algorithm is used as a conjugate gradient method preconditioner, the convergence improves to

$$\|E\mathbf{e}\|_A^2 \leq \left(1 - C\sqrt{\frac{h}{H}}\right)\|\mathbf{e}\|_A^2. \tag{3.1}$$

Thus, when the coarse-space characteristic diameter H is increased 10 times (making the coarse problem 100 times smaller for a two-dimensional problem and 1000 times smaller for a three-dimensional problem) and the number of smoothing steps is increased 10 times, the expression $\sqrt{H/h}$ in the estimate (3.1) grows only by a factor of $\sqrt{10} = 3.162$.

On the theoretical side, the method justifies the use of $O(H/h)$ multigrid smoothing steps. It was generally believed that under the regularity-free assumption (2.7), it is impossible to improve the convergence by adding more pre- or post-smoothing steps. One of the main objectives of this paper is to show that this is not the case. The results in Sections 3 and 4

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justify using as many pre- or post-smoothing steps as there are *prolongator smoothing steps* in (2.3). In Section 6 it is proven that, for a general multilevel method, it makes sense to use $O(H/h)$ smoothing steps even for a method that does not use prolongator smoothing.

The two-level method of [14] uses the components described in the previous section—a tentative prolongator p obtained by the generalized unknowns aggregation, a smoothed prolongator $P = Sp$ as in (2.3) with the following choice

$$\omega_i = \frac{\omega}{\varrho(A)}, \qquad i = 1, \dots, \nu, \ \omega \in (0, 2),$$

and a post-smoother given by step 6 of Algorithm 1 with

$$\nu_2 = \nu + 1, \quad \beta_i = \frac{\omega}{\varrho(A)}, \qquad i = 1, \dots, \nu + 1.$$

The number of steps is assumed to satisfy the condition $\nu \geq C\frac{H}{h}$, and $\nu \approx \frac{1}{2}\frac{H}{h}$ is a recommended value. Here, H is a characteristic diameter of the aggregates. We will also consider a symmetrized version of this method in which the pre-smoother (step 1 of Algorithm 1) and the post-smoother coincide.

Under reasonable assumptions on the aggregates, the prolongator smoothing by S establishes natural overlaps of the coarse-space basis functions (columns of P) and leads to a sparse coarse-level matrix $P^T AP$ ([14]). This is illustrated in the following example.

EXAMPLE 3.1. Consider the discretization of the scalar elliptic problem given in Example 2.2, i.e., the weak form of a one-dimensional Poisson equation on $(0, 1)$ with homogeneous Dirichlet boundary conditions and its discretization by P1-elements on a uniform mesh. The resulting stiffness matrix A is tridiagonal and its pattern follows, aside from the Dirichlet boundary condition, the three-point scheme

$$\frac{1}{h}\left[-1, 2, -1\right].$$

Let the aggregates be N consecutive nodes as given by (2.4) and illustrated in Figure 2.1, the tentative prolongator be given by (2.5), the prolongator smoother be

$$S = \left(I - \frac{\omega}{\varrho(A)}A\right)^{\lfloor N/2 \rfloor},$$

and the final prolongator $P = Sp$. The standard finite element interpolator Π_h can be written as

$$\Pi_h : \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{i=1}^{n} x_i \varphi_i,$$

where $\{\varphi_i\}_{i=1}^n$ is the standard P1- (piecewise-linear) finite element basis. Then, interpolants of the coarse-space vectors given by the prolongator p are

$$\varphi_i^2 = \Pi_h p \mathbf{e}_i, \qquad i = 1, \dots, m,$$

where $\mathbf{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^m$ and m is the number of aggregates (see the top of Figure 3.1). Similarly, continuous basis functions corresponding to the smoothed prolongator are given by

$$\varphi_i^{2,S} = \Pi_h P \mathbf{e}_i = \Pi_h \left(I - \frac{\omega}{\varrho(A)}A\right)^{\lfloor N/2 \rfloor} p \mathbf{e}_i, \qquad i = 1, \dots, m.$$

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The overlap of the supports of the unsmoothed basis functions is only minimal as depicted in Figure 3.1 (top). Each smoothing step, i.e., the multiplication by $I - \omega/\varrho(A)A$, adds to the support one element on each side. Since $H = (N-1)h$, the recommended value implies $\nu \approx \frac{1}{2}H/h = (N-1)/2 \approx \lfloor N/2 \rfloor$. Figure 3.1 (bottom) shows that for aggregates of $N = 7$ nodes, the recommended value is $\nu \approx 3$, and the smoothing steps then establish a natural overlap of the smoothed basis functions. Finally, the coarse-level matrix $A_2 = P^T A P$ satisfies

$$\begin{aligned}\{(A_2)_{ij}\}_{i,j=1}^m &= \{(AP\mathbf{e}_i, P\mathbf{e}_j)\}_{i,j=1}^m \\ &= \left\{(\nabla \Pi_h P\mathbf{e}_i, \nabla \Pi_h P\mathbf{e}_j)_{L_2((0,1))}\right\}_{i,j=1}^m \\ &= \left\{(\nabla \varphi_i^{2,S}, \nabla \varphi_j^{2,S})_{L_2((0,1))}\right\}_{i,j=1}^m .\end{aligned}$$

The entry $(A_2)_{ij}$ can be nonzero only if the supports of the basis functions $\varphi_i^{2,S}$ and $\varphi_j^{2,S}$ overlap. Thus, the coarse-space matrix is sparse. This is true with the recommended choice of ν under reasonable conditions on the aggregates in general.

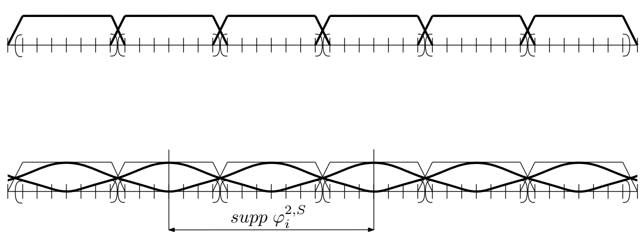

FIG. 3.1. *Non-smoothed (above) and smoothed (below) continuous basis functions.*

The convergence proof below takes advantage of the observation that the next smoothing iteration is never more efficient than the preceding one. Consider a linear smoothing iteration method with the error propagation operator M that is A-symmetric and a nonzero error vector $\mathbf{e}$ such that $M\mathbf{e} \neq \mathbf{0}$. Comparing the contraction ratios in the A-norm of one iteration to the next, it holds that

$$q_1 \equiv \frac{\|M\mathbf{e}\|_A}{\|\mathbf{e}\|_A} \leq q_2 \equiv \frac{\|M(M\mathbf{e})\|_A}{\|M\mathbf{e}\|_A}. \tag{3.2}$$

Indeed, the A-symmetry of M and the Cauchy-Schwarz inequality imply

$$\|M\mathbf{e}\|_A^2 = \langle AM\mathbf{e}, M\mathbf{e}\rangle = \langle AM(M\mathbf{e}), \mathbf{e}\rangle \leq \|M(M\mathbf{e})\|_A \|\mathbf{e}\|_A.$$

Dividing the above estimate by $\|M\mathbf{e}\|_A\|\mathbf{e}\|_A$ yields (3.2). The observation can be applied recursively to infer that the effect of $\nu + 1$ smoothing iterations can be estimated using the effect of the least efficient last smoothing iteration: assume for the time being that $M^i\mathbf{e} \neq \mathbf{0}$, for $i = 0, \ldots, \nu + 1$. Thus, for the A-symmetric error propagation operator M, (3.2) gives

$$\frac{\|M^i\mathbf{e}\|_A}{\|M^{i-1}\mathbf{e}\|_A} \leq \frac{\|M^{\nu+1}\mathbf{e}\|_A}{\|M^\nu\mathbf{e}\|_A}, \qquad i = 1, \ldots, \nu,$$

that is,

$$\frac{\|M^{\nu+1}\mathbf{e}\|_A}{\|\mathbf{e}\|_A} = \frac{\|M^{\nu+1}\mathbf{e}\|_A}{\|M^\nu\mathbf{e}\|_A}\frac{\|M^\nu\mathbf{e}\|_A}{\|M^{\nu-1}\mathbf{e}\|_A}\cdots\frac{\|M\mathbf{e}\|_A}{\|\mathbf{e}\|_A} \leq \left(\frac{\|M^{\nu+1}\mathbf{e}\|_A}{\|M^\nu\mathbf{e}\|_A}\right)^{\nu+1}. \tag{3.3}$$

If $M^\nu\mathbf{e} = \mathbf{0}$, then $\|M^{\nu+1}\mathbf{e}\|_A/\|\mathbf{e}\|_A = 0$, and the estimate above is not needed.

It is well-known [6] that the error propagation operator of the variational two-level method with a prolongator P and $\nu{+}1$ post-smoothing iterations with the error propagation operator M is given by

$$E = M^{\nu+1}\left[I - P(P^TAP)^{-1}P^TA\right].$$

The following abstract lemma provides an estimate for $\|E\|_A$.

LEMMA 3.2. *Assume there is a constant $C > 0$ such that the tentative prolongator p satisfies*

$$\text{for all fine } \mathbf{u}\in\mathbb{R}^n, \text{there is a coarse } \mathbf{v}\in\mathbb{R}^m: \quad \|\mathbf{u}-p\mathbf{v}\| \le \frac{C}{\sqrt{\varrho(A)}}\|\mathbf{u}\|_A. \tag{3.4}$$

Then the variational two-level method with the smoothed prolongator $P = M^\nu p$ and $\nu+1$ post-smoothing steps with the error propagation operator $M = I - \omega/\varrho(A)A$ satisfies

$$\left\|M^{\nu+1}\left[I - P(P^TAP)^{-1}P^TA\right]\right\|_A^2 \le \left[1-\frac{\omega(2-\omega)}{C^2}\right]^{\nu+1}. \tag{3.5}$$

Moreover, for the symmetrized version, the following estimate holds

$$\left\|M^{\nu+1}\left[I - P(P^TAP)^{-1}P^TA\right]M^{\nu+1}\right\|_A^2 \le \left[1-\frac{\omega(2-\omega)}{C^2}\right]^{2(\nu+1)}. \tag{3.6}$$

Proof. The matrix $P(P^TAP)^{-1}P^TA$ is an A-orthogonal projection onto $\mathrm{Range}(P)$. The complementary projection $Q = I - P(P^TAP)^{-1}P^TA$ is an A-orthogonal projection onto $\mathrm{Range}(P)^{\perp_A}$, where the symbol $\mathrm{Range}(P)^{\perp_A}$ denotes an A-orthogonal complement of $\mathrm{Range}(P)$. It holds that

$$\mathrm{Range}(Q) = \mathrm{Range}(P)^{\perp_A} = \mathrm{Ker}(P^TA). \tag{3.7}$$

The operator norm's submultiplicativity, (3.7), and the fact that $\|Q\|_A = 1$, since Q is an A-orthogonal projection, give

$$\|M^{\nu+1}Q\|_A \le \sup_{\mathbf{x}\in\mathrm{Ker}(P^TA)\setminus\{\mathbf{0}\}} \frac{\|M^{\nu+1}\mathbf{x}\|_A}{\|\mathbf{x}\|_A}\|Q\|_A \le \sup_{\mathbf{x}\in\mathrm{Ker}(P^TA)\setminus\{\mathbf{0}\}} \frac{\|M^{\nu+1}\mathbf{x}\|_A}{\|\mathbf{x}\|_A}. \tag{3.8}$$

Note that the above estimate originates in [6]. The key idea of the proof follows. Since $P = M^\nu p$, where M is a polynomial in A, M and A commute, hence $\mathrm{Ker}(P^TA)$ can be also characterized as

$$\mathrm{Ker}(P^TA) = \mathrm{Ker}((M^\nu p)^TA) = \mathrm{Ker}(p^TM^\nu A) = \mathrm{Ker}(p^TAM^\nu).$$

Thus, $M^\nu\mathbf{x} \in \mathrm{Ker}(p^TA)$ for $\mathbf{x}\in\mathrm{Ker}(P^TA)$. This fact together with the estimates (3.3) and (3.8) yield

$$\begin{aligned}\|M^{\nu+1}Q\|_A &\le \sup_{\mathbf{x}\in\mathrm{Ker}(P^TA)\setminus\{\mathbf{0}\}} \frac{\|M^{\nu+1}\mathbf{x}\|_A}{\|\mathbf{x}\|_A} \le \sup_{\mathbf{x}\in\mathrm{Ker}(P^TA)\setminus\{\mathbf{0}\}} \left(\frac{\|M^{\nu+1}\mathbf{x}\|_A}{\|M^\nu\mathbf{x}\|_A}\right)^{\nu+1} \\ &= \sup_{\mathbf{x}\in\mathrm{Ker}(p^TA)\setminus\{\mathbf{0}\}} \left(\frac{\|M\mathbf{x}\|_A}{\|\mathbf{x}\|_A}\right)^{\nu+1} = \left(\sup_{\mathbf{x}\in\mathrm{Ker}(p^TA)\setminus\{\mathbf{0}\}} \frac{\|M\mathbf{x}\|_A}{\|\mathbf{x}\|_A}\right)^{\nu+1}.\end{aligned} \tag{3.9}$$

The rest of the proof is again standard [6]. The term

$$\sup_{\mathbf{x}\in \mathrm{Ker}(p^TA)\setminus\{\mathbf{0}\}} \frac{\|M\mathbf{x}\|_A}{\|\mathbf{x}\|_A}, \qquad M = I - \frac{\omega}{\varrho(A)}A,$$

is estimated using an orthogonality trick from the proof of Céa's lemma. First, for any $\mathbf{x} \in \mathbb{R}^n$, it holds that

$$\begin{aligned}
\|M\mathbf{x}\|_A^2 &= \left\langle A\left(I - \frac{\omega}{\varrho(A)}A\right)\mathbf{x}, \left(I - \frac{\omega}{\varrho(A)}A\right)\mathbf{x}\right\rangle \\
&= \|\mathbf{x}\|_A^2 - 2\frac{\omega}{\varrho(A)}\langle A\mathbf{x}, A\mathbf{x}\rangle + \left(\frac{\omega}{\varrho(A)}\right)^2 \langle A^2\mathbf{x}, A\mathbf{x}\rangle \\
&\le \|\mathbf{x}\|_A^2 - 2\frac{\omega}{\varrho(A)}\langle A\mathbf{x}, A\mathbf{x}\rangle + \frac{\omega^2}{\varrho(A)}\langle A\mathbf{x}, A\mathbf{x}\rangle \\
&= \left[1 - \frac{\omega(2-\omega)}{\varrho(A)}\left(\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|_A}\right)^2\right]\|\mathbf{x}\|_A^2. \qquad (3.10)
\end{aligned}$$

Thus, the smoother is (for $\omega \approx 1$) efficient provided that

$$\frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|_A^2} \approx \varrho(A).$$

Next, let $\mathbf{x} \in \mathrm{Range}(p)^{\perp_A} = \mathrm{Ker}(p^TA)$. The assumption (3.4) guarantees the existence of $\mathbf{v}$ such that

$$\|\mathbf{x}\|_A^2 = \langle A\mathbf{x}, \mathbf{x}\rangle = \langle A\mathbf{x}, \mathbf{x} - p\mathbf{v}\rangle \le \|A\mathbf{x}\|\,\|\mathbf{x} - p\mathbf{v}\| \le \|A\mathbf{x}\|\frac{C}{\sqrt{\varrho(A)}}\|\mathbf{x}\|_A,$$

which follows along the lines of the proof of Céa's lemma and using the Cauchy-Schwarz inequality. Dividing the above inequality by $\frac{C}{\sqrt{\varrho(A)}}\|\mathbf{x}\|_A^2$, squaring the result, and substituting into (3.10) yields

$$\|M\mathbf{x}\|_A^2 \le \left[1 - \frac{\omega(2-\omega)}{C^2}\right]\|\mathbf{x}\|_A^2, \qquad \forall \mathbf{x} \in \mathrm{Range}(p)^{\perp_A}.$$

This and (3.9) give the result (3.5).

Since Q is an A-orthogonal projection, it hold that $Q^2 = Q$, and therefore it follows that $M^{\nu+1}QM^{\nu+1} = M^{\nu+1}Q^2M^{\nu+1}$ and

$$\begin{aligned}
\frac{\|M^\nu QM^\nu\mathbf{e}\|_A^2}{\|\mathbf{e}\|_A^2} &= \frac{\|M^\nu Q^2M^\nu\mathbf{e}\|_A^2}{\|\mathbf{e}\|_A^2} \le \|M^\nu\mathbf{e}\|_A^2\frac{\|QM^\nu\mathbf{e}\|_A^2}{\|\mathbf{e}\|_A^2} \\
&= \|M^\nu Q\|_A^2\frac{\langle AQM^\nu\mathbf{e}, QM^\nu\mathbf{e}\rangle}{\|\mathbf{e}\|_A^2} = \|M^\nu Q\|_A^2\frac{\langle AQM^\nu\mathbf{e}, M^\nu\mathbf{e}\rangle}{\|\mathbf{e}\|_A^2} \\
&= \|M^\nu Q\|_A^2\frac{\langle AM^\nu QM^\nu\mathbf{e}, \mathbf{e}\rangle}{\|\mathbf{e}\|_A^2} \le \|M^\nu Q\|_A^2\frac{\|M^\nu QM^\nu\mathbf{e}\|_A\|\mathbf{e}\|_A}{\|\mathbf{e}\|_A^2} \\
&= \|M^\nu Q\|_A^2\frac{\|M^\nu QM^\nu\mathbf{e}\|_A}{\|\mathbf{e}\|_A}.
\end{aligned}$$

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Dividing the above inequality by $\|M^\nu Q M^\nu \mathbf{e}\|_A / \|\mathbf{e}\|_A$ yields

$$\|M^\nu Q M^\nu\|_A \leq \|M^\nu Q\|_A^2. \tag{3.11}$$

The symmetrized result (3.6) follows from this estimate and (3.5), thus completing the proof. □

The estimate for the symmetrized method, i.e., that with the error propagation operator $E_{sym} = M^{\nu+1}\left[I - P(P^T AP)^{-1}P^T A\right] M^{\nu+1}$, is up to a constant the same as that of the nonsymmetric one with the error propagation operator $E = M^{\nu+1}\left[I - P(P^T AP)^{-1}P^T A\right]$. However, the symmetrized method can be used as a conjugate gradient method preconditioner. For $\nu = O(H/h)$, the estimate (3.6) and the assumption (2.7) give

$$\|M^{\nu+1}\left[I - P(P^T AP)^{-1}P^T A\right] M^{\nu+1}\|_A^2$$
$$\leq \left[1 - \frac{\omega(2-\omega)}{C^2_{(3.4)}}\right]^{2(\nu+1)} \leq \left[1 - \frac{\omega(2-\omega)}{C_1}\left(\frac{h}{H}\right)^2\right]^{C_2 \frac{H}{h}} \leq 1 - \frac{\omega(2-\omega)}{C_3}\frac{h}{H}.$$

If the symmetrized method with the error propagation operator E_{sym} is used as a conjugate gradient method preconditioner, the resulting condition number of the preconditioned system is

$$\text{cond} \leq \frac{1}{1 - \|E_{sym}\|_A} = O\left(\frac{H}{h}\right),$$

and the convergence of the resulting method is guided by the factor

$$\frac{\sqrt{\text{cond}} - 1}{\sqrt{\text{cond}} + 1} = 1 - \frac{1}{C_4}\sqrt{\frac{h}{H}}.$$

4. A fully optimal two-level method. In this section, an optimal two-level method of [23] is described and analyzed. In this method, the final prolongator is given by $P = Sp$, where S is a prolongator smoother and p is a tentative prolongator given by the generalized unknowns aggregation. The prolongator smoother is a polynomial in A that is an error propagation operator of an A-non-divergent smoother. Denote

$$A_S = S^2 A.$$

Two related algorithms are considered, namely a nonsymmetric and an A_S-symmetric variant. The nonsymmetric algorithm proceeds as follows:

ALGORITHM 2 (nonsymmetric).

- Repeat
  1. Pre-smooth $\mathbf{x} \leftarrow \mathcal{S}(\mathbf{x}, \mathbf{f})$, where $\mathcal{S}(.,.)$ is a single step of a linear iteration with the error propagation operator S.
  2. Solve $P^T AP\mathbf{v} = P^T(A\mathbf{x} - \mathbf{f})$.
  3. Update $\mathbf{x} \leftarrow \mathbf{x} - P\mathbf{v}$.
  4. Post-smooth

  $$\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)}A_S\right)\mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)}S^2\mathbf{f} \tag{4.1}$$

  until convergence in the A_S-norm.
- Post-process $\mathbf{x} \leftarrow \mathcal{S}(\mathbf{x}, \mathbf{f})$.

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By reordering the terms, the step (4.1) can be performed more efficiently as

$$\mathbf{x} \leftarrow \mathbf{x} - \frac{\omega}{\bar{\varrho}(A_S)} S^2 (A\mathbf{x} - \mathbf{f}).$$

The symmetric variant modifies the first step of Algorithm 2 to make it A_S-symmetric:
ALGORITHM 3 (A_S-symmetric).

- Repeat
  1. Pre-smooth
     (a) $\mathbf{x} \leftarrow \mathcal{S}(\mathbf{x}, \mathbf{f})$,
     (b) $\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)} A_S\right) \mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)} S^2 \mathbf{f}$,
     (c) $\mathbf{x} \leftarrow \mathcal{S}(\mathbf{x}, \mathbf{f})$.
  2. Solve $P^T A P \mathbf{v} = P^T (A\mathbf{x} - \mathbf{f})$.
  3. Update $\mathbf{x} \leftarrow \mathbf{x} - P\mathbf{v}$.
  4. Post-smooth

$$\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)} S^2 A\right) \mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)} S^2 \mathbf{f}$$

  until convergence in the A_S-norm.
- Post-process $\mathbf{x} \leftarrow \mathcal{S}(\mathbf{x}, \mathbf{f})$.

In order to justify the use of the above algorithms, the well-posedness of their coarse-level problems needs to be inspected. In summary, the coarse-level problems of Algorithms 2 and 3 have a solution for all relevant right-hand sides, and the prolongated and smoothed coarse-level problem solution does not depend on the choice of the pseudo-inverse since

$$\operatorname{Ker}(p^T SASp) = \operatorname{Ker}(Sp).$$

In more detail, it is clear from Algorithms 2 and 3 that all admissible right-hand sides have the form

$$(Sp)^T AS\mathbf{e} = p^T SAS\mathbf{e}, \qquad \mathbf{e} \in \mathbb{R}^n.$$

Such right-hand sides are orthogonal to the possible kernel of the coarse-level matrix since for $\mathbf{u} \in \operatorname{Ker}(p^T SASp) = \operatorname{Ker}(Sp)$, it holds that

$$\langle p^T SAS\mathbf{e}, \mathbf{u} \rangle = \langle AS\mathbf{e}, Sp\mathbf{u} \rangle = 0.$$

Thus, the coarse-level problem has a solution. This solution is unique up to a component that belongs to $\operatorname{Ker}(p^T S^2 Ap) = \operatorname{Ker}(Sp)$. Thus, the prolongated and smoothed solution (that is, $P\mathbf{v} = Sp\mathbf{v}$) is unique and independent of the choice of the pseudoinverse.

Denote $S' = I - \frac{\omega}{\bar{\varrho}(A_S)} A_S$. By a direct computation, the error propagation operator of Algorithm 2 can be written as

$$\begin{aligned} E &= S'(I - P(P^T AP)^+ P^T A)S = S'(I - Sp((Sp)^T ASp)^+ (Sp)^T A)S \\ &= S'S(I - p((Sp)^T ASp)^+ (Sp)^T AS) = S'S(I - p(p^T A_S p)^+ p^T A_S) \end{aligned} \tag{4.2}$$

since A commutes with S. Similarly, the error propagation operator of Algorithm 3 is

$$E_{sym} = ESS' = S'S(I - p(p^T A_S p)^+ p^T A_S) SS'. \tag{4.3}$$

Since S and S' are symmetric and commute with A_S, making them also A_S-symmetric, and $I - p(p^T A_S p)^+ p^T A_S$ is an A_S-symmetric projection, the error propagation operator E_{sym} is also A_S-symmetric.

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The error $\mathbf{e} \in \operatorname{Ker}(S)$ is eliminated by the first smoothing step of Algorithms 2 and 3. Therefore, it is sufficient to establish convergence bounds for the errors $\mathbf{e} \perp \operatorname{Ker}(S)$.

THEOREM 4.1. *Let $\omega \in [0,2]$, $\bar{\varrho}(A_S) \geq \varrho(A_S)$ be an upper bound and S be a polynomial in A that satisfies $\varrho(S) \leq 1$. Assume there exists a constant $C > 0$ such that: for every fine vector $\mathbf{u} \in \mathbb{R}^n$, there is a coarse vector $\mathbf{v} \in \mathbb{R}^m$ that satisfies*

$$\|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C}{\bar{\varrho}(A_S)} \|\mathbf{u}\|_A^2. \tag{4.4}$$

Then the error propagation operator E of one iteration of Algorithm 2 and the error propagation operator E_{sym} of one iteration of Algorithm 3 satisfy

$$\|E\|_{A_S}^2 \leq q(C,\omega), \tag{4.5}$$

and

$$\|E_{sym}\|_{A_S} \leq q(C,\omega), \tag{4.6}$$

respectively, where

$$q(C,\omega) = 1 - \frac{\omega(2-\omega)}{C + \omega(2-\omega)}.$$

Here, the A_S-operator norm is defined as

$$\|B\|_{A_S} = \max_{\mathbf{x} \in \mathbb{R}^n \cap (\operatorname{Ker}(S))^\perp \setminus \{\mathbf{0}\}} \frac{\|B\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}}.$$

Furthermore, for any positive integer ν, it holds that

$$\|SE^\nu\|_A^2 \leq q^\nu(C,\omega) \qquad \text{and} \qquad \|SE_{sym}^\nu\|_A \leq q^\nu(C,\omega). \tag{4.7}$$

REMARK 4.2. The function $q(.,\omega)$ in Theorem 4.1 is minimized by $\omega = 1$, which gives $q(C,1) = 1 - \frac{1}{C+1}$.

REMARK 4.3. The estimates (4.7) justify the use of the post-processing step in Algorithms 2 and 3, respectively.

Proof of Theorem 4.1. First consider the analysis of Algorithm 2. It is easy to verify that

$$Q = I - p(p^T A_S p)^+ p^T A_S$$

is an A_S-orthogonal projection onto $T \equiv (\operatorname{Range}(p))^{\perp_{A_S}}$. Hence, $\|Q\|_{A_S} = 1$. Using (4.2) yields

$$\begin{aligned}\|E\|_{A_S} = \|S'SQ\|_{A_S} &\leq \max_{\mathbf{x} \in T \cap (\operatorname{Ker}(S))^\perp \setminus \{\mathbf{0}\}} \frac{\|S'S\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}} \|Q\|_{A_S} \\ &= \max_{\mathbf{x} \in T \cap (\operatorname{Ker}(S))^\perp \setminus \{\mathbf{0}\}} \frac{\|S'S\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}}.\end{aligned}$$

Since S, S', and A_S are polynomials in A, they commute, and S and S' are A_S-symmetric. Furthermore, since $\varrho(S) \leq 1$ and $\varrho(S') \leq 1$, it holds that

$$\|S'S\mathbf{x}\|_{A_S} \leq \|S'\mathbf{x}\|_{A_S} \qquad \text{and} \qquad \|S'S\mathbf{x}\|_{A_S} \leq \|S\mathbf{x}\|_{A_S}.$$

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Therefore,

$$\|E\|_{A_S} = \max_{\mathbf{x}\in T\cap(\mathrm{Ker}(S))^\perp\setminus\{\mathbf{0}\}} \frac{\|S'S\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}} \le \max_{\mathbf{x}\in T\cap(\mathrm{Ker}(S))^\perp\setminus\{\mathbf{0}\}} \min\left\{\frac{\|S\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}}, \frac{\|S'\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}}\right\}. \tag{4.8}$$

It is shown below that, for $\mathbf{x} \in T \cap (\mathrm{Ker}(S))^\perp$, when the smoother S fails to reduce the error in the A_S-norm ($\|S\mathbf{x}\|_{A_S}/\|\mathbf{x}\|_{A_S} \approx 1$), the performance of the smoother S' is improved. More specifically, for all $\mathbf{x} \in T \cap (\mathrm{Ker}(S))^\perp$, $\mathbf{x} \neq \mathbf{0}$,

$$\frac{\|S'\mathbf{x}\|^2_{A_S}}{\|\mathbf{x}\|^2_{A_S}} \le 1 - \frac{\omega(2-\omega)}{C}\frac{\|S\mathbf{x}\|^2_{A_S}}{\|\mathbf{x}\|^2_{A_S}}. \tag{4.9}$$

First, the term $\|S'\mathbf{x}\|^2_{A_S}$ is estimated as follows

$$\begin{aligned}
\|S'\mathbf{x}\|^2_{A_S} &= \left\|(I - \frac{\omega}{\bar\varrho(A_S)}A_S)\mathbf{x}\right\|^2_{A_S} \\
&= \|\mathbf{x}\|^2_{A_S} - 2\frac{\omega}{\bar\varrho(A_S)}\|A_S\mathbf{x}\|^2 + \left(\frac{\omega}{\bar\varrho(A_S)}\right)^2\|A_S\mathbf{x}\|^2_{A_S} \\
&\le \|\mathbf{x}\|^2_{A_S} - 2\frac{\omega}{\bar\varrho(A_S)}\|A_S\mathbf{x}\|^2 + \frac{\omega^2}{\bar\varrho(A_S)}\|A_S\mathbf{x}\|^2 \\
&= \|\mathbf{x}\|^2_{A_S}\left(1 - \frac{\omega(2-\omega)}{\bar\varrho(A_S)}\frac{\|A_S\mathbf{x}\|^2}{\|\mathbf{x}\|^2_{A_S}}\right).
\end{aligned} \tag{4.10}$$

Recall that $A_S = AS^2$ and A and S commute. Hence $\|S^2\mathbf{x}\|^2_A = \|S\mathbf{x}\|^2_{A_S}$, and

$$\frac{\|A_S\mathbf{x}\|^2}{\|\mathbf{x}\|^2_{A_S}} = \frac{\|A_S\mathbf{x}\|^2}{\|S^2\mathbf{x}\|^2_A}\frac{\|S^2\mathbf{x}\|^2_A}{\|\mathbf{x}\|^2_{A_S}} = \frac{\|AS^2\mathbf{x}\|^2}{\|S^2\mathbf{x}\|^2_A}\frac{\|S\mathbf{x}\|^2_{A_S}}{\|\mathbf{x}\|^2_{A_S}}. \tag{4.11}$$

Next, a lower bound for the first fraction on the right-hand side of (4.11) is established. To this end, an orthogonality trick known from the proof of Céa's lemma is used. First, denote

$$T = (\mathrm{Range}(p))^{\perp_{A_S}} = \mathrm{Ker}(p)^T A_S = \mathrm{Ker}(p)^T AS^2.$$

Hence,

$$\forall \mathbf{x} \in T: \ S^2\mathbf{x} \in \mathrm{Ker}(p)^T A = (\mathrm{Range}(p))^{\perp_A}.$$

Thus, for any $\mathbf{x} \in T$, using assumption (4.4) with $\mathbf{u} = S^2\mathbf{x}$ and the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|S^2\mathbf{x}\|^2_A = \langle AS^2\mathbf{x}, S^2\mathbf{x}\rangle &= \langle AS^2\mathbf{x}, S^2\mathbf{x} - p\mathbf{v}\rangle \le \|AS^2\mathbf{x}\|\|S^2\mathbf{x} - p\mathbf{v}\| \\
&\le \frac{C^{1/2}}{\bar\varrho(A_S)^{1/2}}\|AS^2\mathbf{x}\|\|S^2\mathbf{x}\|_A.
\end{aligned}$$

Dividing both sides by $\|S^2\mathbf{x}\|^2_A$ yields the desired coercivity bound,

$$\frac{\|AS^2\mathbf{x}\|}{\|S^2\mathbf{x}\|_A} \ge \sqrt{\frac{\bar\varrho(A_s)}{C}}, \qquad \forall \mathbf{x} \in T \cap (\mathrm{Ker}(S))^\perp. \tag{4.12}$$

Combining (4.12), (4.11), and (4.10) proves (4.9).

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By substituting (4.9) into (4.8) and taking into account that $0 \le \|S\mathbf{x}\|_{A_S}^2/\|\mathbf{x}\|_{A_S}^2 \le 1$, the final estimate (4.5) is obtained as follows

$$\|E\|_{A_S}^2 \le \max_{\mathbf{x}\in T\cap(\mathrm{Ker}(S))^\perp\setminus\{\mathbf{0}\}} \min\left\{\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}, 1-\frac{\omega(2-\omega)}{C}\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}\right\}$$
$$\le \max_{t\in[0,1]} \min\left\{t, 1-\frac{\omega(2-\omega)}{C}t\right\} = 1-\frac{\omega(2-\omega)}{C+\omega(2-\omega)}\,.$$

Equation (4.3) shows that the error propagation operator of Algorithm 3 is given by $E_{sym}=S'SQSS'$. Then, since Q is an A_S-orthogonal projection, (3.11) yields

$$\|E_{sym}\|_{A_S} \le \|E\|_{A_S}^2,$$

proving(4.6).

Finally, using $\varrho(S)\le 1$ and (4.5), we have for any $\mathbf{x}\perp \mathrm{Ker}(S)$ that

$$\frac{\|SE^\nu\mathbf{x}\|_A}{\|\mathbf{x}\|_A} = \frac{\|E^\nu\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_A} = \frac{\|E^\nu\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}}\frac{\|\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_A} \le \frac{\|E^\nu\mathbf{x}\|_{A_S}}{\|\mathbf{x}\|_{A_S}} \le q^{\nu/2}(C,\omega),$$

proving the first part of (4.7). The second part is obtained from (4.6) analogously. ∎

In view of Theorem 4.1, a smoothing polynomial S is desired that is an error propagation operator of an A-non-divergent smoother and makes $\varrho(S^2A)$ as small as possible. A polynomial smoother $S=\mathrm{pol}(A)$ with such a minimizing property is described below. Specifically, the polynomial smoother satisfies

$$\varrho(S^2A) \le \frac{\varrho(A)}{(1+2\mathrm{deg}(S))^2}, \qquad \varrho(S)\le 1. \tag{4.13}$$

LEMMA 4.4 ([7]). *For any $\varrho>0$ and integer $d>0$, there is a unique polynomial p of degree d that satisfies the constraint $p(0)=1$ and minimizes the quantity*

$$\max_{0\le\lambda\le\varrho} p^2(\lambda)\lambda\,.$$

This polynomial is given by

$$p(\lambda) = \left(1-\frac{\lambda}{r_1}\right)\cdots\left(1-\frac{\lambda}{r_d}\right), \quad r_k=\frac{\varrho}{2}\left(1-\cos\frac{2k\pi}{2d+1}\right), \tag{4.14}$$

and satisfies

$$\max_{0\le\lambda\le\varrho} p^2(\lambda)\lambda = \frac{\varrho}{(2d+1)^2}, \qquad \textit{and} \tag{4.15}$$
$$\max_{0\le\lambda\le\varrho} |p(\lambda)| = 1. \tag{4.16}$$

Proof. If $p^2(\lambda)\lambda$ can be written as the linearly transformed Chebyshev polynomial of order $2n+1$,

$$p^2(\lambda)\lambda = q(\lambda) = \frac{c}{2}(1-T_{2n+1}(1-2\lambda/L)), \qquad c=\max_{0\le\lambda\le L} p^2(\lambda)\lambda, \tag{4.17}$$

then it follows from minimax properties of Chebyshev polynomials that p is the sought polynomial. The zeros of $q(\lambda)$ are the points λ where $T_{2n+1}(1-2\lambda/L)=1$, that is,

$1-2\lambda/L = \cos\frac{2k\pi}{2n+1}$. The value $k=0$ gives the simple root $\lambda = 0$ of q, while $k = 1,\dots,n$, yield double roots of q given by (4.14). This proves that p is indeed the polynomial (4.14).

To prove (4.15), we determine the quantity c in (4.17) from the condition that $p(0) = 1$. This condition implies that the linear term of $p^2(\lambda)\lambda$ is one, hence from (4.17) it follows that $1 = (p^2(\lambda)\lambda)'(0) = (c/L)T'_{2n+1}(1)$. Therefore, $c = L/T'_{2n+1}(1) = L/(2n+1)^2$.

It remains to prove (4.16). First, (4.16) is equivalent to $p^2(\lambda) \leq 1$, for all λ, $0 \leq \lambda \leq 1$. Using (4.17), this is equivalent to

$$\frac{L(1-T_{2n+1}(1-2\lambda/L))}{2\lambda T'_{2n+1}(1)} \leq 1, \qquad \text{for all } \lambda,\ 0 \leq \lambda \leq L. \tag{4.18}$$

Using the substitution $1-2\lambda/L = x$, (4.18) becomes by a simple manipulation

$$T_{2n+1}(x) \geq 1 + T'_{2n+1}(1)(x-1), \qquad \text{for all } x,\ -1 \leq x \leq 1,$$

which is the well-known fact that the graph of a Chebyshev polynomial lies above its tangent at $x = 1$. ◻

We choose S to be the polynomial (4.14) in A with $\varrho = \varrho(A)$. Using the spectral mapping theorem, (4.15), and (4.16), we get (4.13).

THEOREM 4.5. *Assume that the tentative prolongator p satisfies* (2.7)*. Let the final prolongator be constructed as $P = Sp$, where the prolongator smoother is $S = p(A)$ and $p(\cdot)$ is the polynomial* (4.14) *with $\rho = \varrho(A)$. Assume the degree of S satisfies*

$$\deg(S) \geq c\frac{H}{h}, \tag{4.19}$$

and $\omega \in (0,2)$. Then, Algorithms 2 and 3 converge with

$$q(\omega) = 1 - \frac{1}{C(\omega)},$$

where $C(\omega) > 0$ is a constant independent of both h and H (cf. Theorem 4.1).

Proof. The statement (4.13) and the assumption (4.19) give

$$\varrho(S^2A) \leq \bar{\varrho}(S^2A) \equiv \frac{\varrho(A)}{(1+2\deg(S))^2} \leq C\left(\frac{h}{H}\right)^2 \varrho(A).$$

Substituting this into (2.7) yields

$$\forall \mathbf{e} \in \mathbb{R}^n\ \exists \mathbf{v} \in \mathbb{R}^m : \quad \|\mathbf{e} - p\mathbf{v}\|^2 \leq \frac{C}{\bar{\varrho}(S^2A)}\|\mathbf{e}\|_A^2.$$

The result now follows from Theorem 4.1. ◻

5. An alternative with a multilevel extension. The optimal two-level framework described in Section 4 requires the exact solution of the coarse-level problem. In this section, we describe a method that is a modification of the one given in [15]. The framework we use here allows for employing more than two levels with aggressive coarsening between the first two of them.

Let us consider first the two-level case. The content of this section is based on the following key observation: assume that for a particular error $\mathbf{e}$, the following *weak approximation condition* holds,

$$\text{for the error } \mathbf{e}\text{, there exists a coarse } \mathbf{v} \text{ such that} \quad \|\mathbf{e} - p\mathbf{v}\| \leq \frac{C}{\sqrt{\rho(A)}}\|\mathbf{e}\|_A. \tag{5.1}$$

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Then, assuming a reasonable smoother, a single iteration of a two-level method (with an error propagation operator E) satisfies

$$\|E\mathbf{e}\|_A \leq \left(1 - \frac{1}{q(C)}\right) \|\mathbf{e}\|_A.$$

Here $q(C) \geq 1$ is a monotonous function growing with growing C. Thus we observe that the two-level error estimate holds, so to speak, point by point. If the weak approximation condition is satisfied for a particular error $\mathbf{e}$, then the error estimate gives the contraction number for this particular error. In what follows, we will refer to this property of the estimate as a *pointwise convergence property*. This is by no means easy to prove. We will provide such a proof, based on [5], in Lemma 5.4. The convergence proof of [5] satisfies, under some restrictions when certain arguments are avoided, the pointwise convergence property.

Our first goal is to guarantee a coarse-space size independent rate of convergence for a two-level method with $O(H/h)$ smoothing iterations. The convergence of a two-level method is guided by condition (5.1). One obvious way how to make (5.1) weaker (and thereby allowing for a small coarse space) is to reduce $\rho(A)$. This can be achieved by solving instead a transformed problem

$$S^2 A\mathbf{x} = S^2 \mathbf{f}, \tag{5.2}$$

where $S = (I - \alpha_1 A) \cdots (I - \alpha_d A)$ is a polynomial in A such that $\varrho(S) \leq 1$ and $\varrho(S^2 A) \ll \rho(A)$. In other words, we use a polynomial S being an error propagation operator of a non-divergent polynomial iterative method such that $\varrho(S^2 A) \ll \rho(A)$. In this case, however, the weak approximation condition (5.1) becomes,

$$\|\mathbf{e} - p\mathbf{v}\| \leq \frac{C}{\sqrt{\rho(S^2 A)}} \|\mathbf{e}\|_{S^2 A}. \tag{5.3}$$

The norm on the right-hand side is generally smaller than the A-norm. For a given continuous problem to be solved, such a condition is difficult to verify. The goal of this section is to prove uniform convergence of a two-level method assuming the weak approximation condition with a *reduced spectral bound* $\varrho(S^2 A)$ and *the original* A-norm on the right-hand side, that is,

$$\|\mathbf{e} - p\mathbf{v}\| \leq \frac{C}{\sqrt{\rho(S^2 A)}} \|\mathbf{e}\|_A. \tag{5.4}$$

In fact, assuming that

$$\varrho(S^2 A) \leq C \left(\frac{h}{H}\right)^2 \varrho(A),$$

(which was proved in Section 4 assuming deg$(S) \geq cH/h$), condition (5.4) follows from our general assumption (2.7). To be able to prove uniform convergence under assumption (5.4) (that is, to prove (5.3) based on (5.4)), we have to compensate for the loss of the following coercivity condition:

$$\|\mathbf{e}\|_{S^2 A} \geq q \|\mathbf{e}\|_A, \qquad 0 < q < 1. \tag{5.5}$$

Here, the number $q \in (0, 1)$ is the threshold we choose. Let the reader imagine, for example, $q = 0.5$. For a given error vector $\mathbf{e} \in \mathbb{R}^n$, we consider two possible cases: 1. the coercivity (5.5) holds, or 2. the coercivity (5.5) fails. Given a threshold $q \in (0, 1)$, assume (5.5) holds

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for a specific error $\mathbf{e}$. Then the weak approximation condition (5.3) follows from (5.4) with $C_{(5.3)} = C_{(5.4)}/q$. In this case, the two-level procedure for the transformed problem (5.2) is efficient. Let us take a look at the opposite case. If $\|\mathbf{e}\|_{S^2A} \leq q\|\mathbf{e}\|_A,\ q < 1$, then equivalently $\|S\mathbf{e}\|_A \leq q\|\mathbf{e}\|_A$. In other words the loss of coercivity (5.5) makes the iterative method with the error propagation operator S efficient. Summing up, if the coercivity (5.5) holds then (5.3) follows from (5.4). In case of loss of coercivity, the iterative method with the error propagation operator S is efficient. So, under (5.4) either the two-level method for solving the transformed problem (5.2) or the iterative method with an error propagation operator S is efficient.

Thus, we will base our method on two basic elements: 1. solving the transformed problem (5.2) by a two-level method, or 2. employing the outer iteration for solving the original problem $A\mathbf{x} = \mathbf{f}$ using the iterative method with an error propagation operator S. Recall that

$$S = (I - \alpha_1 A)\cdots(I - \alpha_\nu A), \qquad \alpha_1, \ldots, \alpha_\nu \in \mathbb{R}^n,$$

and

$$\varrho(S) \leq 1. \tag{5.6}$$

We start with an investigation of the nonsymmetric algorithm. The estimate for the symmetric version will later follow by a trivial argument. In what follows, we will prove a general convergence bound for the following algorithm:

ALGORITHM 4.

1. Perform a single iteration of the multilevel method for solving the problem

$$A_S\mathbf{x} = \mathbf{f}_S, \qquad A_S = S^2 A, \quad \mathbf{f}_S = S^2\mathbf{f}. \tag{5.7}$$

2. For $i = 1, \ldots, d$ do $\mathbf{x} \leftarrow (I - \alpha_i A)\mathbf{x} + \alpha_i\mathbf{f}$.

Note that in the context of a multilevel method, the matrix A_S is never constructed and only its action is evaluated.

5.1. Analysis of the nonsymmetric algorithm. The error propagation operator E of Algorithm 4 has the form

$$E = SE_S, \tag{5.8}$$

where E_S is an error propagation operator corresponding to step 1.

THEOREM 5.1. *For every $t \in [0, 1]$, define a set*

$$V(t) = \{\mathbf{v} \in \mathbb{R}^n \ :\ \|\mathbf{v}\|_{A_S} \geq t\|\mathbf{v}\|_A\}. \tag{5.9}$$

Then

$$\|E\|_A \leq \sup_{t\in[0,1]} \left\{ t \sup_{\mathbf{v}\in V(t)} \frac{\|E_S\mathbf{v}\|_{A_S}}{\|\mathbf{v}\|_{A_S}} \right\}. \tag{5.10}$$

REMARK 5.2. Note that S is a polynomial in A such that $\varrho(S) \leq 1$, hence $\|\cdot\|_{A_S} \leq \|\cdot\|_A$, and $V(t)$ is a set of vectors $\mathbf{v} \in \mathbb{R}^n$ such that the norm equivalence

$$t\|\mathbf{v}\|_A \leq \|\mathbf{v}\|_{S^2A} \leq \|\mathbf{v}\|_A,$$

holds. ($V(t) \equiv \{\mathbf{v} \in \mathbb{R}^n :\ \|\mathbf{v}\|_{A_S} \geq t\|\mathbf{v}\|_A\} = \{\mathbf{v} \in \mathbb{R}^n :\ t\|\mathbf{v}\|_A \leq \|\mathbf{v}\|_{A_S} \leq \|\mathbf{v}\|_A\}$.) Therefore, for vectors $\mathbf{u} \in V(t)$, the desired weak approximation condition (5.3) follows from the weakened approximation condition (5.4) with a constant $C_1 = C_2/t$.

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Proof of Theorem 5.1. Clearly, since $E = SE_S$ (see (5.8)), it holds for every $\mathbf{e} \in \mathbb{R}^n$ that

$$\frac{\|E\mathbf{e}\|_A}{\|\mathbf{e}\|_A} = \frac{\|SE_S\mathbf{e}\|_A}{\|S\mathbf{e}\|_A} \cdot \frac{\|S\mathbf{e}\|_A}{\|\mathbf{e}\|_A} = \frac{\|E_S\mathbf{e}\|_{A_S}}{\|\mathbf{e}\|_{A_S}} \cdot \frac{\|\mathbf{e}\|_{A_S}}{\|\mathbf{e}\|_A}. \tag{5.11}$$

Let us set

$$t = \frac{\|\mathbf{e}\|_{A_S}}{\|\mathbf{e}\|_A}.$$

Then, trivially, $\mathbf{e} \in V(t)$. Hence by (5.11) and $t \in [0,1]$ it follows that

$$\frac{\|E\mathbf{e}\|_A}{\|\mathbf{e}\|_A} \leq \frac{\|\mathbf{e}\|_{A_S}}{\|\mathbf{e}\|_A} \cdot \frac{\|E_S\mathbf{e}\|_{A_S}}{\|\mathbf{e}\|_{A_S}} \leq t \sup_{\mathbf{v}\in V(t)} \frac{\|E_S\mathbf{v}\|_{A_S}}{\|\mathbf{v}\|_{A_S}} \leq \max_{t\in[0,1]} \left\{ t \sup_{\mathbf{v}\in V(t)} \frac{\|E_S\mathbf{v}\|_{A_S}}{\|\mathbf{v}\|_{A_S}} \right\},$$

proving (5.10). ∎

We further investigate the case when step 1 of Algorithm 4 is given by the following nonsymmetric two-level procedure:

ALGORITHM 5. Let $\bar{\varrho}(A_S)$ be an upper bound of $\varrho(A_S)$.

1. Perform $\mathbf{x} \leftarrow \mathbf{x} - p(p^T A_S p)^+ p^T(\mathbf{f}_S - A_S\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$, $p : \mathbb{R}^m \to \mathbb{R}^n$ is a full-rank linear mapping, $m < n$, and $\mathbf{f}_S = S^2\mathbf{f}$. Here, the symbol $+$ denotes the pseudoinverse.
2. Perform the following smoothing procedure:

$$\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)} A_S\right)\mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)}\mathbf{f}_S, \qquad \omega \in (0,1). \tag{5.12}$$

REMARK 5.3. Algorithm 5 is implemented as follows (assuming the coarse-level matrix is regular): first, the auxiliary smoothed prolongator $p_S = Sp = (I - \alpha_1 A)\cdots(I - \alpha_\nu A)p$ is evaluated. This is done by setting $p^0 = p$ and evaluating

$$p^i = (I - \omega_i A)p^{i-1}, \quad i = 1, \dots, \nu.$$

Then we set $p_S = p^\nu$. Further, we calculate the coarse-level matrix $A_2^S = (p_S)^T A p_S$. Note that $A_2^S = p^T A_S p = p^T S^2 A p$. We decompose the matrix A_2^S by a Cholesky decomposition and, in each iteration, evaluate the action of the inverse of $A_2^S = p^T A_S p$ in the usual way by double backward substitution.

Note that the application that we have in mind deal with $S = \mathrm{pol}(A)$ having degree about $\frac{1}{2}\frac{H}{h}$ and a prolongator p given by generalized aggregation. Here, h is the resolution of a finite element mesh and H the characteristic diameter of the aggregates. In Example 3.1 we have demonstrated on a model example that such a construction leads to a sparse coarse-level matrix $p^T S^2 A p$.

The error propagation operator of Algorithm 5 is

$$E = \left(I - \frac{\omega}{\bar{\varrho}(A_S)} A_S\right)(I - p(p^T A_S p)^+ p^T A_S).$$

The following lemma is a two-level variant of the multilevel estimate of [5]. Its proof is basically a routine simplification of the original, where certain technicalities have been taken care of. Unlike the original convergence theorem of [5], this lemma gives an estimate satisfying the *pointwise convergence property*. Further, our lemma avoids the assumption of $\|\cdot\|_A$-boundedness of the interpolation operator $Q : \mathbf{u} \in \mathbb{R}^n \mapsto p\mathbf{v} \in \mathrm{Range}(p)$ in (5.4), that is, this

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lemma gives the estimate based solely on the weak approximation condition (and of course, the assumption on the smoother). Note that the simplest error estimate for a two-level method, that is, the estimate by Achi Brandt in [6] based on the orthogonality trick of Céa, does not satisfy the pointwise convergence property.

LEMMA 5.4. *Let A be a symmetric positive semidefinite matrix, $\bar{\varrho}(A) \geq \varrho(A)$, and p an $n \times m$ full-rank matrix with $m < n$. Further, let R be a symmetric positive definite $n \times n$ matrix such that*

$$K \equiv I - R^{-1}A \tag{5.13}$$

is positive semidefinite in the A-inner product. (Note that from the symmetry of R, the A-symmetry of K follows.) We assume there is a constant $C_R > 0$ such that for all $\mathbf{w} \in \mathbb{R}^n$ it holds that

$$\frac{1}{\bar{\varrho}(A)}\|\mathbf{w}\|^2 \leq C_R(R^{-1}\mathbf{w}, \mathbf{w}). \tag{5.14}$$

Let $V \subset \mathbb{R}^n$. Assume further there is a constant $C_A = C_A(V) > 0$ such that for every $\mathbf{u} \in V$, there exists a vector $\mathbf{v} \in \mathbb{R}^m$ such that

$$\|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C_A}{\bar{\varrho}(A)}\|\mathbf{u}\|_A^2. \tag{5.15}$$

Then for every error $\mathbf{e} \in V$ and the error propagation operator of the nonsymmetric two-level method,

$$E = K[I - p(p^TAp)^+p^TA] = (I - R^{-1}A)[I - p(p^TAp)^+p^TA],$$

it holds that

$$\|E\mathbf{e}\|_A^2 \leq \left(1 - \frac{1}{1 + C_AC_R}\right)\|\mathbf{e}\|_A^2. \tag{5.16}$$

Proof. We start with introducing some notations. First, let P_2 be an A-orthogonal projection onto $\mathrm{Range}(p)$. Further, we set

$$T = I - K = R^{-1}A, \qquad E_2 = I - P_2;$$

see (5.13). Note that

$$E = (I - T)E_2. \tag{5.17}$$

Let $\mathbf{u} \in V$. Then (5.15) holds for $\mathbf{u}$. Our goal is to prove that

$$\|E\mathbf{u}\|_A^2 \leq \left(1 - \frac{1}{C}\right)\|\mathbf{u}\|_A^2,$$

where C is a positive constant dependent on C_A and C_R. Instead, we will prove an equivalent estimate

$$(\mathbf{u}, \mathbf{u})_A \leq C[(\mathbf{u}, \mathbf{u})_A - (E\mathbf{u}, E\mathbf{u})_A]. \tag{5.18}$$

Our first step will be to establish

$$(\mathbf{u}, \mathbf{u})_A - (E\mathbf{u}, E\mathbf{u})_A \geq (P_2\mathbf{u}, \mathbf{u}) + (TE_2\mathbf{u}, E_2\mathbf{u})_A. \tag{5.19}$$

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Clearly, we have that $I = E_2 + P_2$. Since P_2 is an A-orthogonal projection, the decomposition $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_1 = E_2\mathbf{w} = (I - P_2)\mathbf{w}$, $\mathbf{w}_2 = P_2\mathbf{w}$ is A-orthogonal, that is, taking into account that $(P_2\mathbf{u}, P_2\mathbf{u})_A = (P_2\mathbf{u}, \mathbf{u})_A$,

$$(\mathbf{u}, \mathbf{u})_A = (E_2\mathbf{u}, E_2\mathbf{u})_A + (P_2\mathbf{u}, \mathbf{u})_A. \tag{5.20}$$

Further, we notice that from (5.17) it follows that $E_2 = TE_2 + E$. Using this identity, (5.17), and the fact that $T = R^{-1}A$ is A-symmetric, we have

$$\begin{aligned}
(E_2\mathbf{u}, E_2\mathbf{u})_A &= ((TE_2 + E)\mathbf{u}, (TE_2 + E)\mathbf{u})_A \\
&= (E\mathbf{u}, E\mathbf{u})_A + 2(TE_2\mathbf{u}, E\mathbf{u})_A + (TE_2\mathbf{u}, TE_2\mathbf{u})_A \\
&= (E\mathbf{u}, E\mathbf{u})_A + 2(TE_2\mathbf{u}, (I - T)E_2\mathbf{u})_A + (TE_2\mathbf{u}, TE_2\mathbf{u})_A \\
&= (E\mathbf{u}, E\mathbf{u})_A + 2(T(I - T)E_2\mathbf{u}, E_2\mathbf{u})_A + (T^2E_2\mathbf{u}, E_2\mathbf{u})_A \\
&= (E\mathbf{u}, E\mathbf{u})_A + (T(I - T)E_2\mathbf{u}, E_2\mathbf{u})_A + ([T(I - T) + T^2]E_2\mathbf{u}, E_2\mathbf{u})_A \\
&= (E\mathbf{u}, E\mathbf{u})_A + (T(I - T)E_2\mathbf{u}, E_2\mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A.
\end{aligned}$$

By assumption, $K = I - R^{-1}A = I - T$ is positive semidefinite in the A-inner product. Further, since R^{-1} is positive definite, $R^{-1}A$ is positive semidefinite in the A-inner product. Hence, it follows that the spectrum of $T = R^{-1}A$ is contained in $[0, 1]$ and $T(I - T)$ is A-positive semidefinite. Therefore, $(T(I - T)E_2\mathbf{u}, E_2\mathbf{u})_A \geq 0$, and the previous identity becomes

$$(E_2\mathbf{u}, E_2\mathbf{u})_A \geq (E\mathbf{u}, E\mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A. \tag{5.21}$$

Now, (5.20) and (5.21) yield

$$(\mathbf{u}, \mathbf{u})_A \geq (E\mathbf{u}, E\mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A + (P_2\mathbf{u}, \mathbf{u})_A,$$

proving (5.19).

Using (5.19), we will now establish the inequality

$$(\mathbf{u}, \mathbf{u})_A \leq C[(P_2\mathbf{u}, \mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A], \tag{5.22}$$

where $C = C(C_A, C_R) > 0$. Note that from (5.22), the desired result (5.18) and in turn also the final statement (5.16) follow. Indeed, since $P_2 = I - E_2$ and E_2 is an A-orthogonal projection, we have

$$\begin{aligned}
(P_2\mathbf{u}, \mathbf{u})_A + (TE_2\mathbf{u}, \mathbf{u})_A &= (\mathbf{u}, \mathbf{u})_A - (E_2\mathbf{u}, \mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A \\
&= (\mathbf{u}, \mathbf{u})_A - (E_2\mathbf{u}, E_2\mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A \\
&= (\mathbf{u}, \mathbf{u})_A - ((I - T)E_2\mathbf{u}, E_2\mathbf{u})_A \\
&= (\mathbf{u}, \mathbf{u})_A - (E\mathbf{u}, E_2\mathbf{u})_A.
\end{aligned}$$

Further, since $I - T = I - R^{-1}A$ is an A-symmetric operator with the spectrum contained in the interval $[0, 1]$, we also have

$$(E\mathbf{u}, E\mathbf{u})_A = ((I - T)E_2\mathbf{u}, (I - T)E_2\mathbf{u})_A \leq ((I - T)E_2\mathbf{u}, E_2\mathbf{u})_A = (E\mathbf{u}, E_2\mathbf{u})_A.$$

Thus,

$$(P_2\mathbf{u}, \mathbf{u})_A + (TE_2\mathbf{u}, \mathbf{u})_A \leq (\mathbf{u}, \mathbf{u})_A - (E\mathbf{u}, E\mathbf{u})_A,$$

and (5.18) follows from (5.22). Thus, it remains to establish (5.22).

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Let $Q_2 : \mathbf{u} \in V \mapsto p\mathbf{v},\ \mathbf{v} \in \mathbb{R}^m$, be an operator fulfilling (5.15), that is, $Q_2 : \mathbf{u} \mapsto p\mathbf{v}$, $\mathbf{u} \in V$, $\mathbf{v} \in \mathbb{R}^m$, such that

$$\|\mathbf{u} - p\mathbf{v}\|^2 = \|(I - Q_2)\mathbf{u}\|^2 \leq \frac{C_A}{\bar{\varrho}(A)}\|\mathbf{u}\|_A^2.$$

Since $E_2 = I - P_2$, we have $I = E_2 + P_2$, and therefore

$$\begin{aligned}(\mathbf{u},\mathbf{u})_A &= (P_2\mathbf{u},\mathbf{u})_A + (E_2\mathbf{u},\mathbf{u})_A \\ &= (P_2\mathbf{u},\mathbf{u})_A + (E_2\mathbf{u},(I-Q_2)\mathbf{u})_A + (E_2\mathbf{u},Q_2\mathbf{u})_A \\ &= (P_2\mathbf{u},\mathbf{u})_A + (E_2\mathbf{u},(I-Q_2)\mathbf{u})_A,\end{aligned} \tag{5.23}$$

because it holds that $(E_2\mathbf{u}, Q_2\mathbf{u})_A = 0$, which is a consequence of $\mathrm{Range}(Q_2) = \mathrm{Range}(p)$ and $\mathrm{Range}(E_2) = \mathrm{Range}(I - P_2) = \mathrm{Range}(p)^{\perp_A}$. By the Cauchy-Schwarz inequality and assumptions (5.15) and (5.14), we get

$$\begin{aligned}(E_2\mathbf{u},(I-Q_2)\mathbf{u})_A &\leq \|AE_2\mathbf{u}\|\|(I-Q_2)\mathbf{u}\| \\ &\leq C_A^{1/2}\bar{\varrho}(A)^{-1/2}\|\mathbf{u}\|_A\|AE_2\mathbf{u}\| \\ &\leq (C_AC_R)^{1/2}\|\mathbf{u}\|_A(R^{-1}E_2\mathbf{u}, AE_2\mathbf{u})^{1/2} \\ &= (C_AC_R)^{1/2}\|\mathbf{u}\|_A(TE_2\mathbf{u}, AE_2\mathbf{u})^{1/2}.\end{aligned}$$

By substituting the above estimate into (5.23), using the Cauchy-Schwarz inequality $a_1b_1 + a_2b_2 \leq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}$, and the inequality $(P_2\mathbf{u},\mathbf{u})_A \leq (\mathbf{u},\mathbf{u})_A$, we get

$$\begin{aligned}(\mathbf{u},\mathbf{u})_A &\leq (P_2\mathbf{u},\mathbf{u})_A + (C_AC_R)^{1/2}\|\mathbf{u}\|_A(TE_2\mathbf{u}, AE_2\mathbf{u}) \\ &\leq (\mathbf{u},\mathbf{u})_A^{1/2}(P_2\mathbf{u},\mathbf{u})_A^{1/2} + (C_AC_R)^{1/2}\|\mathbf{u}\|_A(TE_2\mathbf{u}, AE_2\mathbf{u}) \\ &\leq \|\mathbf{u}\|_A\left(1\cdot(P_2\mathbf{u},\mathbf{u})^{1/2} + (C_AC_R)^{1/2}(TE_2\mathbf{u},\mathbf{u})_A)_A^{1/2}\right) \\ &\leq \|\mathbf{u}\|_A(1 + C_AC_R)^{1/2}[(P_2\mathbf{u},\mathbf{u})_A + (TE_2\mathbf{u}, E_2\mathbf{u})_A]^{1/2},\end{aligned}$$

proving (5.22) with $C = 1 + C_AC_R$. Using (5.19), the estimate (5.18) follows from (5.22), and in turn the statement (5.16). □

Next we analyze the nonsymmetric Algorithm 4 with step 1 given by Algorithm 5. Summing up, we analyze the following abstract method:

ALGORITHM 6. Let $\bar{\varrho}(A_S) \geq \varrho(A_S)$ be an available upper bound.

1. Perform $\mathbf{x} \leftarrow \mathbf{x} - p(p^TA_Sp)^+p^T(\mathbf{f}_S - A_S\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$, $p : \mathbb{R}^m \to \mathbb{R}^n$ is a full-rank linear mapping, $m < n$, and $\mathbf{f}_S = S^2\mathbf{f}$. Here the symbol $+$ denotes the pseudoinverse.
2. Perform the following smoothing procedure:

$$\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)}A_S\right)\mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)}\mathbf{f}_S, \qquad \omega \in (0,1).$$

3. For $i = 1,\ldots,d$ do $\mathbf{x} \leftarrow (I - \alpha_iA)\mathbf{x} + \alpha_i\mathbf{f}$. Here $(I - \alpha_1A)\cdots(I - \alpha_dA) = S$.

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THEOREM 5.5. *Let $\bar{\varrho}(A_S) \geq \varrho(A_S)$. Assume that there is a constant $C > 0$ such that*

$$\forall \mathbf{u} \in \mathbb{R}^n \, \exists \mathbf{v} \in \mathbb{R}^m : \quad \|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C}{\bar{\varrho}(A_S)} \|\mathbf{u}\|_A^2. \tag{5.24}$$

Then the error propagation operator E of the nonsymmetric Algorithm 4 with step 1 given by Algorithm 5 (that is, Algorithm 6) satisfies

$$\|E\|_A^2 \leq 1 - \frac{1}{1 + \frac{C}{\omega}} < 1.$$

Proof. In view of Theorem 5.1, we need to estimate

$$\|E\|_A \leq \sup_{t \in [0,1]} \left\{ t \sup_{\mathbf{u} \in V(t)} \frac{\|E_S \mathbf{u}\|_{A_S}}{\|\mathbf{u}\|_{A_S}} \right\},$$

where $V(t)$ is given by (5.9). By (5.24) and (5.9),

$$\forall \mathbf{u} \in V(t) \, \exists \mathbf{v} \in \mathbb{R}^n : \quad \|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C/t^2}{\bar{\varrho}(A_S)} \|\mathbf{u}\|_{A_S}^2. \tag{5.25}$$

Clearly, the smoother (5.12) can be written in the form $\mathbf{x} \leftarrow (I - R^{-1}A_S)\mathbf{x} + R^{-1}\mathbf{f}_S$ with the A-symmetric positive definite error propagation operator

$$K = I - R^{-1}A_S = I - \frac{\omega}{\bar{\varrho}(A_S)} A_S, \qquad \omega \in [0,1],$$

and therefore,

$$R^{-1} = \frac{\omega}{\bar{\varrho}(A_S)} I.$$

Thus, (5.14) is satisfied with a constant $C_R = 1/\omega$. By this, all assumptions of Lemma 5.4 applied to Algorithm 5 are satisfied for all $\mathbf{u} \in V(t)$ with constants $C_A = C/t^2$ (see (5.25)) and $C_R = 1/\omega$. Hence, by Lemma 5.4, choosing $V = V(t)$, it follows that

$$\sup_{\mathbf{u} \in V(t)} \frac{\|E_S \mathbf{u}\|_{A_S}^2}{\|\mathbf{u}\|_{A_S}^2} \leq 1 - \frac{1}{1 + \frac{C}{\omega t^2}}, \qquad \omega \in [0,1],$$

and Theorem 5.1 gives

$$\|E\|_A^2 \leq \sup_{t \in [0,1]} \left\{ t^2 \left(1 - \frac{1}{1 + \frac{C}{\omega t^2}} \right) \right\}.$$

It remains to evaluate the maximum of the function

$$\phi(t) = t^2 \left(1 - \frac{1}{1 + \frac{C}{\omega t^2}} \right), \quad t \in [0,1].$$

By inspecting the first and second derivative, we find that the function $\phi(t)$, $t \in \mathbb{R}$, does not attain its maximum in the interval $(0,1)$. Thus, the function $\phi(t)$, $t \in [0,1]$, attains its maximum for $t = 1$, proving our statement. ∎

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5.2. Analysis of the symmetrization. The error propagation operator of Algorithm 6 can be written as

$$E = SE_S = SS'Q, \qquad S' = I - \frac{\omega}{\bar{\varrho}(A_S)} A_S, \quad Q = I - p(p^T A_S p)^+ p^T A_S.$$

The operator Q is an A_S-orthogonal projection onto $(\mathrm{Range}(p))^{\perp_{A_S}}$. Our goal is to derive the operator $E_{sym} = E^* E$, where E^* is the A-adjoint operator. Since S and S' are polynomials in A, $A_S = S^2 A$, and Q is A_S-symmetric, we have for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned} \langle AE^* E\mathbf{x}, \mathbf{y}\rangle &= \langle AE\mathbf{x}, E\mathbf{y}\rangle = \langle ASS'Q\mathbf{x}, SS'Q\mathbf{y}\rangle \\ &= \langle A_S S'Q\mathbf{x}, S'Q\mathbf{y}\rangle = \langle A_S Q(S')^2 Q\mathbf{x}, \mathbf{y}\rangle = \langle AS^2 Q(S')^2 Q\mathbf{x}, \mathbf{y}\rangle. \end{aligned}$$

Thus, we conclude that

$$E_{sym} = S^2 Q(S')^2 Q.$$

One can easily see that the above error propagation operator corresponds to the following algorithm:

ALGORITHM 7. Let $\bar{\varrho}(A_S) \geq \varrho(A_S)$ be an available upper bound.

1. Perform $\mathbf{x} \leftarrow \mathbf{x} - p(p^T A_S p)^+ p^T (\mathbf{f}_S - A_S \mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$, $p : \mathbb{R}^m \to \mathbb{R}^n$ is a full-rank linear mapping, $m < n$, and $\mathbf{f}_S = S^2 \mathbf{f}$. Here the symbol $+$ denotes the pseudoinverse.
2. Perform twice the following smoothing procedure:
$$\mathbf{x} \leftarrow \left(I - \frac{\omega}{\bar{\varrho}(A_S)} A_S\right)\mathbf{x} + \frac{\omega}{\bar{\varrho}(A_S)} \mathbf{f}_S, \qquad \omega \in (0, 1).$$
3. Perform $\mathbf{x} \leftarrow \mathbf{x} - p(p^T A_S p)^+ p^T (\mathbf{f}_S - A_S \mathbf{x})$.
4. Perform twice: for $i = 1, \dots, d$ do $\mathbf{x} \leftarrow (I - \alpha_i A)\mathbf{x} + \alpha_i \mathbf{f}$. Here, $(I - \alpha_1 A) \cdots (I - \alpha_d A) = S$.

The following convergence bound for $\|E_{sym}\|_A$ is a trivial consequence of Theorem 5.5.

THEOREM 5.6. *Let $\bar{\varrho}(A_S) \geq \varrho(A_S)$. Assume there is a constant $C > 0$ such that*

$$\forall \mathbf{u} \in \mathbb{R}^n\ \exists \mathbf{v} \in \mathbb{R}^m : \ \ \|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C}{\bar{\varrho}(A_S)} \|\mathbf{u}\|_A^2.$$

Then the error propagation operator E_{sym} of the symmetric Algorithm 7 satisfies

$$\|E_{sym}\|_A = \|E\|_A^2 \leq 1 - \frac{1}{1 + \frac{C}{\omega}} < 1.$$

Proof. By $E_{sym} = E^* E$ and Theorem 5.5 we have

$$\langle AE_{sym}\mathbf{e}, \mathbf{e}\rangle = \|E\mathbf{e}\|_A^2 \leq \left(1 - \frac{1}{1 + \frac{C}{\omega}}\right) \|\mathbf{e}\|_A^2.$$

Since the operator E_{sym} is A-symmetric positive semidefinite, it immediately follows that

$$\|E_{sym}\|_A = \lambda_{\max}(E_{sym}) = \|E\|_A^2 \leq 1 - \frac{1}{1 + \frac{C}{\omega}},$$

completing the proof. ∎

THEOREM 5.7. *Assume the prolongator p satisfies the assumption (2.7) and the smoother S is given by $S = p(A)$, with $p(.)$ defined by (4.14), where $\varrho = \varrho(A)$. We set*

$$\bar{\varrho}(A_S) = \frac{\varrho(A)}{(1+2\text{deg(S)})^2}\,.$$

Assume that there is a constant $c > 0$ such that

$$\text{deg}(S) \geq c\frac{H}{h}\,.$$

Then the error propagation operator E of Algorithm 6 and the error propagation operator E_{sym} of Algorithm 7 satisfy

$$\|E_{sym}\|_A = \|E\|_A^2 \leq 1 - \frac{1}{C},$$

where $C > 1$ is independent of both h and H.

Proof. By assumption (2.7), we have

$$\forall \mathbf{e} \in \mathbb{R}^n\, \exists \mathbf{v} \in \mathbb{R}^m: \quad \|\mathbf{e} - p\mathbf{v}\|^2 \leq \frac{C_A}{\varrho(A)}\left(\frac{H}{h}\right)^2 \|\mathbf{e}\|_A^2. \tag{5.26}$$

Further, since we assume $\text{deg}(S) \geq c\frac{H}{h}$, the statement (4.13) gives

$$\varrho(S^2A) \leq \bar{\varrho}(S^2A) \equiv \frac{\varrho(A)}{(1+2\text{deg}(S))^2} \leq C\left(\frac{h}{H}\right)^2 \varrho(A).$$

Substituting the above estimate into (5.26) yields

$$\forall \mathbf{u} \in \mathbb{R}^n\, \exists \mathbf{v} \in \mathbb{R}^m: \quad \|\mathbf{u} - p\mathbf{v}\|^2 \leq \frac{C}{\bar{\varrho}(A_S)}\|\mathbf{u}\|_A^2.$$

The proof now follows from Theorem 5.6. ∎

5.3. A multilevel extension. In this section we investigate the case when step 1 of Algorithm 4 uses a multigrid solver for solving the transformed problem (5.7). Our goal is to prove the result independent of the first- and second-level resolution. Note that in this section we consider the nonsymmetric Algorithm 4. The extension to a symmetric algorithm is possible by following the arguments of Section 5.2.

We consider a standard variational multigrid with prolongators (full-rank linear mappings)

$$P_{l+1}^l : \mathbb{R}^{n_{l+1}} \to \mathbb{R}^{n_l}, \qquad n_1 = n,\; n_{l+1} < n_l, \quad l = 1, \ldots, L, \tag{5.27}$$

that is, multigrid with coarse-level matrices

$$A_l^S = (P_l^1)^T A_S P_l^1, \qquad P_l^1 = P_2^1 \cdots P_L^{l-1}, \quad P_1^1 = I,$$

and restrictions given by the transpose of the prolongators. Here, L denotes the number of levels. The particular case of interest is $n_2 \ll n_1$.

For the sake of brevity we assume simple Richardson pre- and post-smoothers

$$\mathbf{x}_l \leftarrow \left(I - \frac{\omega}{\varrho(A_l^S)}A_l^S\right)\mathbf{x}_l + \frac{\omega}{\varrho(A_l^S)}\mathbf{f}_l, \qquad \mathbf{x}_l, \mathbf{f}_l \in \mathbb{R}^{n_l}, \tag{5.28}$$

on all levels $l = 1, \dots, L-1$. We define nested coarse spaces and associated norms,

$$U_l = \text{range}(P_l^1), \qquad \|\,.\,\|_l : P_l^1 \mathbf{x} \mapsto (\mathbf{x}^T \mathbf{x})^{1/2}, \qquad \mathbf{x} \in \mathbb{R}^{n_l},\ l = 1, \dots, L.$$

THEOREM 5.8 ([5]). *Let E_S be the error propagation operator of a variational multigrid V- or W-cycle algorithm for solving the problem (5.7) with the prolongators (5.27) and pre- and/or post-smoothers (5.28) on all levels $l = 1, \dots, L-1$. Let $V \subset U_1$. We assume that there are constants $C_1, C_2 > 0$ and linear mappings $Q_l : U_1 \to U_l$, $l = 2, \dots, L$, $Q_1 = I$, such that for every $\mathbf{u} \in V$,*

$$\|(Q_l - Q_{l+1})\mathbf{u}\|_l^2 \le \frac{C_1}{\varrho(A_l^S)} \|\mathbf{u}\|_{A_S}^2, \qquad l = 1, \dots, L-1, \tag{5.29}$$

and

$$\|Q_l \mathbf{u}\|_{A_S}^2 \le C_2 \|\mathbf{u}\|_{A_S}^2, \qquad l = 1, \dots, L. \tag{5.30}$$

Then for every $\mathbf{u} \in V$,

$$\|E_S \mathbf{u}\|_{A_S}^2 \le \left(1 - \frac{1}{CL}\right) \|\mathbf{u}\|_{A_S}^2, \qquad \text{where } C = \left(1 + C_2^{1/2} + \left(\frac{C_1}{\omega}\right)^{1/2}\right)^2.$$

Proof. The proof follows by minor modifications of the original one of [5]. ∎

THEOREM 5.9. *Let E be the error propagation operator of Algorithm 4 with E_S (the error propagation operator corresponding to step 1 of Algorithm 4) given by variational multigrid as described in this section. Assume that there are positive constants C_a , C_s and linear mappings Q_l, $l = 1, \dots, L$, $Q_1 = I$, such that for every $\mathbf{u} \in U_1$,*

$$\|(Q_l - Q_{l+1})\mathbf{u}\|_l^2 \le \frac{C_a}{\varrho(A_l^S)} \|\mathbf{u}\|_A^2, \qquad l = 1, \dots, L-1, \tag{5.31}$$

and

$$\|S Q_l \mathbf{u}\|_A^2 \le C_s \|\mathbf{u}\|_A^2, \qquad \forall \mathbf{u} \in U_1. \tag{5.32}$$

Then

$$\|E\|_A^2 = \|S E_S\|_A^2 \le q < 1,$$

where q depends only on C_a, C_s, and ω in (5.28). The value of q is given by the formula

$$q = \sup_{t \in [0,1]} \left\{ t^2 \left(1 - \frac{1}{C(t)L}\right) \right\}, \quad C(t) = \left(1 + \left(\frac{C_s}{t^2}\right)^{1/2} + \left(\frac{C_t}{t^2 \omega}\right)^{1/2}\right)^2. \tag{5.33}$$

Proof. By Theorem 5.1, we have

$$\|E\|_A \le \sup_{t \in [0,1]} \left\{ t \sup_{\mathbf{u} \in V(t)} \frac{\|E_S \mathbf{u}\|_{A_S}}{\|\mathbf{u}\|_{A_S}} \right\},$$

where E_S is the error propagation operator of variational multigrid for solving the transformed problem (5.2). From the definition of the set $V(t)$ and $\varrho(S) \le 1$, we have,

$$t\|\mathbf{u}\|_A \le \|\mathbf{u}\|_{A_S} \le \|\mathbf{u}\|_A, \qquad \forall \mathbf{u} \in V(t). \tag{5.34}$$

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Hence, for $\mathbf{u} \in V(t)$, (5.34), (5.31), and (5.32) imply (5.29) and (5.30) with $C_1 = C_a/t^2$ and $C_2 = C_s/t^2$. Then by Theorem 5.8, choosing $V = V(t)$,

$$\sup_{\mathbf{u}\in V(t)} \frac{\|E_S\mathbf{u}\|_{A_S}^2}{\|\mathbf{u}\|_{A_S}^2} \le \left(1 - \frac{1}{C(t)L}\right), \quad \text{where } C(t) = \left(1 + \left(\frac{C_s}{t^2}\right)^{1/2} + \left(\frac{C_t}{t^2\omega}\right)^{1/2}\right)^2,$$

and by Theorem 5.1,

$$\|E\|_A^2 \le \sup_{t\in[0,1]} \left\{ t^2 \left(1 - \frac{1}{C(t)L}\right)\right\},$$

proving (5.33). It is easy to see that the above supremum is smaller than one and depending only on C_a,C_s, and ω. □

In what follows we verify the assumptions of the theory developed in this section for a model example. In particular, we prove that our multilevel method converges independently of the first- and second-level resolution.

We consider the following model example: let $\Omega = (0,1)\times(0,1)$. For a given $f \in L_2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \equiv (\nabla u, \nabla v)_{L_2(\Omega)} = (v,f)_{L_2(\Omega)}, \qquad \forall v \in H_0^1(\Omega).$$

We assume a system of nested regular triangulations τ_{h_l} with mesh sizes h_l, where

$$h_2 = Nh_1, \;\; h_{l+1} = 2h_l, \qquad l = 2,\dots,L-1,$$

and N, L are integers $N > 2$, $L \ge 2$. Let n_l be the number of interior vertices of τ_{h_l}. On each level we consider the standard P1 finite element basis $\{\varphi_i^l\}_{i=1}^{n_l}$ and denote the corresponding finite element space by V_{h_l}. We assume the standard scaling $\|\varphi_i^l\|_{L_\infty(\Omega)} = 1$. It is well known [9] that the corresponding stiffness matrices satisfy

$$\varrho \left\{a(\varphi_l^l, \varphi_j^l)\right\}_{i,j=1}^{n_l} \le C. \tag{5.35}$$

Here and in what follows, c, C are positive constants independent of the mesh size on any level and the number of levels. We assume that the prolongators $P_2^1, P_3^2, \dots, P_L^{L-1}$ are constructed by a natural embedding of coarse spaces,

$$\Pi_h P_l^1 \mathbf{e}_i^l = \varphi_i^l, \quad i = 1,\dots,n_l, \qquad \Pi_h : \mathbf{u} \in \mathbb{R}^{n_1} \mapsto \sum_{i=1}^{n_1} \mathbf{x}_i \varphi_i^1.$$

Here, $\mathbf{e}_i^l$ denotes the i-th canonical basis vector of $\mathbb{R}^{n_l}$.

REMARK 5.10. One can easily see that the following relations are equivalent to the above assumption on the prolongation operators,

$$\varphi_i^l = \Pi_h P_l^1 \mathbf{e}_i^l = \Pi_h P_{l-1}^1 P_l^{l-1} \mathbf{e}_i^l = \Pi_{h_{l-1}} P_l^{l-1} \mathbf{e}_i^l.$$

In other words,

$$\Pi_{h_{l-1}}^{-1} \varphi_i^l = P_l^{l-1} \mathbf{e}_i^l.$$

Thus, our assumption on the prolongators means that the i-th column of P_l^{l-1} (that is, $P_l^{l-1}\mathbf{e}_i^l$) is the representation of the finite element basis function φ_i^l with respect to the finer basis $\{\varphi_j^{l-1}\}_{j=1}^{n_{l-1}}$.

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If the smoother S is the identity, all coarse-level matrices A_l^S are finite element stiffness matrices

$$A_{h_l} = \left\{a(\varphi_i^l, \varphi_j^l)\right\}_{i,j=1}^{n_l}.$$

In a finite element stiffness matrices, a_{ij} can be nonzero only if the vertex j belongs to an element adjacent to the vertex i. The usage of a smoother S of degree 1 causes that the fill-in of all coarse-level matrices A_l^S increases; the entry a_{ij} of A_l^S, $l = 2, \ldots, L$, becomes nonzero if the vertex j of τ_{h_l} belongs to two layers of elements adjacent to the vertex i. We choose the smoother S of largest degree such that the coarse-level matrices $A_l^S, l > 1$, have such a pattern. It is routine to verify that such a degree is the nearest integer that is smaller or equal to $\frac{1}{2}h_2/h_1$. Then $\deg(S) \geq Ch_2/h_1$, and (4.13) together with (5.35) give

$$\varrho(A_1^S) = \varrho(A_S) \leq C\left(\frac{h_1}{h_2}\right)^2. \tag{5.36}$$

For $l > 1$, we have by (5.6) and (5.35),

$$\begin{aligned}\varrho(A_l^S) &= \sup_{\mathbf{u}\in\mathbb{R}^{n_l}} \frac{(P_l^1\mathbf{u})^T AS^2(P_l^1\mathbf{u})}{\mathbf{u}^T\mathbf{u}} \leq \sup_{\mathbf{u}\in\mathbb{R}^{n_l}} \frac{(P_l^1\mathbf{u})^T A(P_l^1\mathbf{u})}{\mathbf{u}^T\mathbf{u}}\\ &= \varrho((P_l^1)^T A P_l^1) = \varrho(A_{h_l}) \leq C.\end{aligned}$$

In what follows, we verify assumptions (5.31) and (5.32) of Theorem 5.9. We choose Q_l, $l = 2, \ldots, L$, so that $\Pi_h Q_l$ is the $L_2(\Omega)$-orthogonal projection onto V_{h_l}. The following are well-known properties of finite element functions ([9]):

$$ch_l\|\mathbf{u}\|_l \leq \|\Pi_h\mathbf{u}\|_{L_2(\Omega)} \leq Ch_l\|\mathbf{u}\|_l, \qquad \mathbf{u}\in U_l, \tag{5.37}$$

$$\|\Pi_h(I-Q_l)\mathbf{u}\|_{L_2(\Omega)} \leq Ch_l|\Pi_h\mathbf{u}|_{H^1(\Omega)}, \qquad \mathbf{u}\in U_1, \tag{5.38}$$

$$|\Pi_h Q_l\mathbf{u}|_{H^1(\Omega)} \leq C|\Pi_h\mathbf{u}|_{H^1(\Omega)}, \qquad \mathbf{u}\in U_1. \tag{5.39}$$

Clearly,

$$|\Pi_h\mathbf{u}|_{H^1(\Omega)}^2 = a(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}) = \|\mathbf{u}\|_A^2 = |\Pi_h\mathbf{u}|_{H^1(\Omega)}^2, \qquad \mathbf{u}\in U_1. \tag{5.40}$$

From (5.39), (5.40), and $\|S\|_A = \varrho(S) \leq 1$ (see (4.13)), it follows that

$$\|SQ_l\mathbf{u}\|_A \leq \|Q_l\mathbf{u}\|_A = |\Pi_h Q_l\mathbf{u}|_{H^1(\Omega)} \leq C|\Pi_h\mathbf{u}|_{H^1(\Omega)} \leq C\|\mathbf{u}\|_A,$$

proving (5.32). Using well-known properties of the $L_2(\Omega)$-orthogonal projections $\Pi_h Q_l$, we have,

$$\begin{aligned}\|\Pi_h(I-Q_{l+1})\mathbf{u}\|_{L_2(\Omega)}^2 &= \|\Pi_h(I-Q_l)\mathbf{u}\|_{L_2(\Omega)}^2 + \|\Pi_h(Q_l-Q_{l+1})\mathbf{u}\|_{L_2(\Omega)}^2\\ &\geq \|\Pi_h(Q_l-Q_{l+1})\mathbf{u}\|_{L_2(\Omega)}^2.\end{aligned}$$

Therefore, by (5.38),

$$\|\Pi_h(Q_l-Q_{l+1})\mathbf{u}\|_{L_2(\Omega)} \leq \|\Pi_h(I-Q_{l+1})\mathbf{u}\|_{L_2(\Omega)} \leq Ch_{l+1}|\Pi_h\mathbf{u}|_{H^1(\Omega)}.$$

Hence, by (5.37) and (5.40),

$$\|(Q_l-Q_{l+1})\mathbf{u}\|_l \leq C\frac{h_{l+1}}{h_l}\|\mathbf{u}\|_A.$$

The last inequality together with (5.36) give (5.31) for $l = 1$. The last inequality, (5.37), and $h_{l+1} = 2h_l$ prove (5.31) for $l = 2, \ldots, L-1$.

Since both assumptions (5.31) and (5.32) of Theorem 5.9 are satisfied, Algorithm 4 with step 2 given by the multigrid method described in this section converges with a rate of convergence independent of both the coarse- and fine-level resolutions.

6. A general multilevel result. In this section, we give a nearly optimal convergence result for general variational multigrid with aggressive coarsening and special polynomial smoothing. The results presented here are based on [21], and the smoother analysis is a minor generalization of the estimate in [8]. In this method, we allow for aggressive coarsening between any of two consecutive levels and prove a convergence rate independent of the degree of coarsening. In case of multigrid based on the hierarchy of nested quasiuniform meshes with characteristic resolution h_l, to guarantee a convergence result independent of the degree of coarsening (that is, independent of h_{l+1}/h_l), we need to perform on each level a number of Richardson smoothing steps that is greater than or equal to ch_{l+1}/h_l. Here, c is a positive constant independent of the level l.

6.1. General estimates based on the XZ-identity by Xu and Zikatanov. Consider the standard variational multigrid V-cycle in the form

$$\mathbf{x} \leftarrow (I - R_{mgm}^{-1}A)\mathbf{x} + R_{mgm}^{-1}\mathbf{f}, \qquad \mathbf{x}, \mathbf{f} \in \mathbb{R}^{n_1}, n_1 = n,$$

with injective linear prolongators

$$P_{l+1}^{l} : \mathbb{R}^{n_{l+1}} \to \mathbb{R}^{n_l}, \qquad n_{l+1} < n_l, \quad l = 1, \ldots, L-1,$$

that is, a multigrid algorithm with restriction operators given by the transpose of the prolongators, and coarse-level matrices given by

$$A_l = (P_l^{l-1})^T A_{l-1} P_l^{l-1} = (P_l^1)^T A P_l^1, \quad P_l^1 \equiv P_1^2 \ldots P_l^{l-1}, \quad A_1 = A.$$

We assume that the pre-smoothers are of the form

$$\mathbf{x} \leftarrow (I - R_l^{-1}A_l)\mathbf{x} + R_l^{-1}\mathbf{f}, \tag{6.1}$$

and the post-smoothers have the form

$$\mathbf{x} \leftarrow (I - R_l^{-T}A_l)\mathbf{x} + R_l^{-T}\mathbf{f}. \tag{6.2}$$

Here, $\mathbf{f} \in \mathbb{R}^{n_l}$ is a right-hand side and R_l is an invertible $n_l \times n_l$ matrix.

Our estimates are based on the following general XZ-identity (Xu and Zikatanov [26]). Its proof, in the form we use it, can be found in [25].

THEOREM 6.1. *Consider the symmetric variational V-cycle multigrid with the components as described at the beginning of this section. Assume the matrices $R_l^T + R_l - A_l$, $l = 1, \ldots, L-1$, are positive definite. Then the following identity holds:*

$$\begin{aligned} \langle A\mathbf{v}, \mathbf{v}\rangle &\leq \langle R_{mgm}\mathbf{v}, \mathbf{v}\rangle \\ &= \inf_{\{\mathbf{v}_l\}} \left\{ \|\mathbf{v}_L\|_{A_L}^2 + \sum_{l=1}^{L-1} \|R_l^T \mathbf{v}_l^f + A_l P_{l+1}^l \mathbf{v}_{l+1}\|^2_{(R_l^T + R_l - A_l)^{-1}} \right\}, \\ \mathbf{v}_1 &= \mathbf{v}, \quad \mathbf{v}_l^f \equiv \mathbf{v}_l - P_{l+1}^l \mathbf{v}_{l+1}. \end{aligned} \tag{6.3}$$

The infimum here is taken over the components $\{\mathbf{v}_l\}$ of all possible decompositions of $\mathbf{v} \in \mathbb{R}^n$ obtained as follows: starting with $\mathbf{v}_1 = \mathbf{v}$, for $l \geq 1$, $\mathbf{v}_l = \mathbf{v}_l^f + P_{l+1}^l \mathbf{v}_{l+1}$, i.e., choosing $\mathbf{v}_{l+1} \in \mathbb{R}^{n_{l+1}}$ arbitrary, we then let $\mathbf{v}_l^f = \mathbf{v}_l - P_{l+1}^l \mathbf{v}_{l+1}$.

REMARK 6.2. The requirement that the matrices $R_l + R_l^T - A_l$ are positive definite is equivalent to the smoothers (6.1) and (6.2) being A_l-convergent.

We define the hierarchy of coarse spaces $V_L \subset \ldots \subset V_2 \subset V_1$ and the associated norms $\|\cdot\|_{V_l},\ l = 1, \ldots, L$, by

$$V_l = \mathrm{Range}(P_l^1), \quad \|\cdot\|_{V_l} : P_l^1\mathbf{x} \mapsto \|\mathbf{x}\| \equiv \sqrt{\mathbf{x}^T\mathbf{x}}, \qquad l = 1, \ldots, L. \tag{6.4}$$

Further, we define

$$\lambda_{k,l} = \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\left\langle AP_k^1\mathbf{x}, P_k^1\mathbf{x}\right\rangle}{\|P_k^1\mathbf{x}\|_{V_l}^2}, \qquad k = 1, \ldots, L, 1 \le l \le k. \tag{6.5}$$

REMARK 6.3. Definition (6.5) allows the following interpretation: The spectral bound

$$\varrho(A_k) = \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\langle A_k\mathbf{x}, \mathbf{x}\rangle}{\|\mathbf{x}\|^2} = \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\left\langle AP_k^1\mathbf{x}, P_k^1\mathbf{x}\right\rangle}{\|P_k^1\mathbf{x}\|_{V_k}^2}$$

indicates the smoothness of the space V_k with respect to the norm $\|\cdot\|_{V_k}$. The quantity

$$\begin{aligned}\lambda_{k,l} &\equiv \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\left\langle AP_k^1\mathbf{x}, P_k^1\mathbf{x}\right\rangle}{\|P_k^1\mathbf{x}\|_{V_l}^2} = \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\left\langle AP_k^1\mathbf{x}, P_k^1\mathbf{x}\right\rangle}{\|P_l^1 P_k^l\mathbf{x}\|_{V_l}^2} \\ &= \sup_{\mathbf{x}\in\mathbb{R}^{n_k}\setminus\{\mathbf{0}\}} \frac{\left\langle AP_k^1\mathbf{x}, P_k^1\mathbf{x}\right\rangle}{\|P_k^l\mathbf{x}\|^2},\end{aligned} \tag{6.6}$$

for $l < k$, indicates the smoothness of the space V_k with respect to the finer space norm $\|\cdot\|_{V_l}$.

THEOREM 6.4. *Consider the symmetric variational V-cycle multigrid with the components as described at the beginning of this section. Let $\bar{\lambda}_{l+1,l} \ge \lambda_{l+1,l}$, $l = 1, \ldots, L-1$, be upper bounds. We assume the existence of linear mappings $Q_l : V_1 \to V_l$, $l = 1, \ldots, L$, $Q_1 = I$, such that for all finest-level vectors $\mathbf{v} \in V_1$ and all levels $l = 1, \ldots, L-1$, it holds that*

$$\|(Q_l - Q_{l+1})\mathbf{v}\|_{V_l} \le \frac{C_a}{\sqrt{\bar{\lambda}_{l+1,l}}}\|\mathbf{v}\|_A\,, \tag{6.7}$$

and for all levels $l = 1, \ldots, L$,

$$\|Q_l\|_A \le C_s. \tag{6.8}$$

Here C_a and C_s are positive constants independent of the level. Further, we assume that our smoothers, R_l, $l = 1, \ldots, L-1$, satisfy

$$R_l^T + R_l - A_l \ge \alpha\, A_l, \tag{6.9}$$

and the symmetrized smoothers $\bar{R}_l$ defined by

$$I - \bar{R}_l^{-1}A_l = (I - R_l^{-T}A)(I - R_l^{-1}A_l) \tag{6.10}$$

satisfy

$$\|\mathbf{v}\|_{\bar{R}_l}^2 \le \beta\left(\bar{\lambda}_{l+1,l}\|\mathbf{v}\|^2 + \|\mathbf{v}\|_{A_l}^2\right), \qquad \forall\mathbf{v} \in \mathbb{R}^{n_l}. \tag{6.11}$$

Then the resulting multigrid operator R_{mgm} is nearly spectrally equivalent to A, or, more precisely,

$$\langle A\mathbf{v}, \mathbf{v}\rangle \le \langle R_{mgm}\mathbf{v}, \mathbf{v}\rangle \le \left[C_s^2 + 2(L-1)\left(\beta(C_a^2 + 4C_s^2) + \frac{1}{\alpha}C_s^2\right)\right]\langle A\mathbf{v}, \mathbf{v}\rangle,$$

for all $\mathbf{v} \in V_1$.

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REMARK 6.5. Condition (6.9) represents a minimal requirement on the smoother that is usually not difficult to satisfy. Consider, for example, Richardson iteration with the error propagation operator $I - 1/\varrho(A_l)A_l$. Then, $R_l = R_l^T = \varrho(A_l)I$, and condition (6.9) is satisfied with $\alpha = 1$.

REMARK 6.6. The difference to the results based on the theory in [5] is in our use of the weak approximation condition (6.7). The original theory instead relied on the condition

$$\|(Q_l - Q_{l+1})\mathbf{v}\|_{V_l} \le \frac{C_a}{\sqrt{\varrho(A_l)}}\|\mathbf{v}\|_A, \tag{6.12}$$

and the approximation properties of the space V_{l+1} are thus measured against the smoothness of the space V_l (because of $\varrho(A_l)$). In typical applications, the approximation on the left-hand side of (6.12) is guided by h_{l+1}, while the spectral bound of A_l and the scaling of the $\|\cdot\|_{V_l}$-norm are guided by h_l. To prove (6.12), the ratio h_{l+1}/h_l has to be bounded, and the resolutions of the spaces V_l and V_{l+1} have to be comparable.

In our case, the approximation properties of the space V_{l+1} are measured against (the upper bound of)

$$\lambda_{l+1,l} \equiv \sup_{\mathbf{x}\in \mathrm{Range}(P^l_{l+1})\setminus\{\mathbf{0}\}} \frac{\langle A_l\mathbf{x},\mathbf{x}\rangle}{\|\mathbf{x}\|^2} \le \varrho(A_l),$$

that is, against the smoothness of the space V_{l+1} (measured with respect to the norm $\|\cdot\|_{V_l}$ used on the left-hand side of (6.7)). Therefore the resolutions of the spaces V_l and V_{l+1} do not have to be comparable. The current estimate thus allows us to prove a convergence result independent of the coarsening ratio. The cost of the uniform convergence result when the coarsening ratio becomes large ($\lambda_{l+1,l} \ll \varrho(A_l)$) is a stronger condition on the smoother which arises through the smoothing condition (6.11).

Proof of Theorem 6.4. It is easy to prove that the symmetrized smoothers $\bar{R}_l$ defined in (6.10) have the form

$$\bar{R}_l = R_l\left(R_l^T + R_l - A_l\right)^{-1} R_l^T. \tag{6.13}$$

We use the triangle inequality, the trivial inequality $(a+b)^2 \le 2(a^2+b^2)$, $a,b\in\mathbb{R}$, (6.13), and the assumption (6.9) to estimate

$$\begin{aligned}
&\sum_{l=1}^{L-1}\|R_l^T\mathbf{v}_l^f + A_lP^l_{l+1}\mathbf{v}_{l+1}\|^2_{(R_l^T+R_l-A_l)^{-1}}\\
&\quad\le 2\sum_{l=1}^{L-1}\|R_l\mathbf{v}_l^f\|^2_{(R_l^T+R_l-A_l)^{-1}} + 2\sum_{l=1}^{L-1}\|A_lP^l_{l+1}\mathbf{v}_{l+1}\|^2_{(R_l^T+R_l-A_l)^{-1}}\\
&\quad\le 2\sum_{l=1}^{L-1}\|\mathbf{v}_l^f\|^2_{\bar{R}_l} + \frac{2}{\alpha}\sum_{l=1}^{L-1}\|A_lP^l_{l+1}\mathbf{v}_{l+1}\|^2_{A_l^{-1}}\\
&\quad= 2\sum_{l=1}^{L-1}\|\mathbf{v}_l^f\|^2_{\bar{R}_l} + \frac{2}{\alpha}\sum_{l=1}^{L-1}\|P^l_{l+1}\mathbf{v}_{l+1}\|^2_{A_l}\\
&\quad= 2\sum_{l=1}^{L-1}\|\mathbf{v}_l^f\|^2_{\bar{R}_l} + \frac{2}{\alpha}\sum_{l=1}^{L-1}\|\mathbf{v}_{l+1}\|^2_{A_{l+1}}\\
&\quad= 2\sum_{l=1}^{L-1}\|\mathbf{v}_l^f\|^2_{\bar{R}_l} + \frac{2}{\alpha}\sum_{l=2}^{L}\|\mathbf{v}_l\|^2_{A_l}.
\end{aligned}\tag{6.14}$$

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Thus, based on (6.3) and (6.14), we conclude that in order to get the estimate for the mutual condition number of A and R_{mgm}, it is sufficient to find a *particular decomposition* $\mathbf{v} \mapsto \{\mathbf{v}_i\}$ such that

$$\|\mathbf{v}_L\|^2_{A_L} + 2\sum_{l=1}^{L-1} \|\mathbf{v}_l^f\|^2_{\bar{R}_l} + \frac{2}{\alpha}\sum_{l=2}^{L} \|\mathbf{v}_l\|^2_{A_l} \leq C\|\mathbf{v}\|^2_A. \tag{6.15}$$

Indeed, from (6.3), (6.14), and (6.15), it follows that

$$\langle A\mathbf{v}, \mathbf{v}\rangle \leq \langle R_{mgm}\mathbf{v}, \mathbf{v}\rangle \leq C\langle A\mathbf{v}, \mathbf{v}\rangle, \qquad \forall \mathbf{v} \in \mathbb{R}^n. \tag{6.16}$$

We write the linear mappings Q_l in the form $Q_l = P_l^1 \tilde{Q}_l$ and choose $\mathbf{v}_l = \tilde{Q}_l\mathbf{v}$. Hence,

$$\mathbf{v}_l^f = (\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}.$$

First, we have $\|\mathbf{v}_l\|_{A_l} = \|\tilde{Q}_l\mathbf{v}\|_{A_l} = \|P_l^1\tilde{Q}_l\mathbf{v}\|_A = \|Q_l\mathbf{v}\|_A$. Hence, by (6.8), we have

$$\|\mathbf{v}_L\|^2_{A_L} \leq C_s^2\|\mathbf{v}\|^2_A, \qquad \sum_{l=2}^{L} \|\mathbf{v}_l\|^2_{A_l} \leq C_s^2(L-1)\|\mathbf{v}\|^2_A. \tag{6.17}$$

Further, we estimate using assumptions (6.11), (6.7), (6.8), and definition (6.4),

$$\begin{aligned}
\|\mathbf{v}_l^f\|^2_{\bar{R}_l} &= \|(\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}\|^2_{\bar{R}_l} \\
&\leq \beta(\bar{\lambda}_{l+1,l}\|(\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}\|^2 + \|(\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}\|^2_{A_l}) \\
&= \beta(\bar{\lambda}_{l+1,l}\|P_l^1(\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}\|^2_{V_l} + \|P_l^1(\tilde{Q}_l - P^l_{l+1}\tilde{Q}_{l+1})\mathbf{v}\|^2_A) \\
&= \beta(\bar{\lambda}_{l+1,l}\|(Q_l - Q_{l+1})\mathbf{v}\|^2_{V_l} + \|(Q_l - Q_{l+1})\mathbf{v}\|^2_A) \\
&\leq \beta(\bar{\lambda}_{l+1,l}\|(Q_l - Q_{l+1})\mathbf{v}\|^2_{V_l} + 2(\|(Q_l\|^2_A + \|Q_{l+1})\mathbf{v}\|^2_A)) \\
&\leq \beta(C_a^2 + 4C_s^2)\|\mathbf{v}\|^2_A. \qquad (6.18)
\end{aligned}$$

Substituting (6.17) and (6.18) into the left-hand side of (6.15) yields (6.15) with

$$C = C_s^2 + 2(L-1)\left(\beta(C_a^2 + 4C_s^2) + \frac{1}{\alpha}C_s^2\right).$$

The proof is now completed by (6.16). □

6.2. The smoother and its analysis. Let A be a symmetric positive definite $n \times n$ matrix. We investigate the smoother with the error propagation operator

$$I - R^{-T}A = I - R^{-1}A = S^\gamma\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right), \tag{6.19}$$

where S is a polynomial in A satisfying $\varrho(S) \leq 1$ and $A_S = S^2A$, $\bar{\varrho}(A_S) \geq \varrho(A_S)$ is an available upper bound, and γ is a positive integer. The implementation of this smoother for a specific S we use can be found in Remark 6.10.

Clearly, for a smoother $\bar{A}$ introduced in (6.13), we have

$$I - \bar{R}^{-1}A \equiv (I - R^{-T}A)(I - R^{-1}A) = S^{2\gamma}\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)^2. \tag{6.20}$$

Therefore,

$$\bar{R}^{-1} = A^{-1}\left[I - \left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)^2 S^{2\gamma}\right]. \tag{6.21}$$

The following lemma states a key result about our smoother.

LEMMA 6.7. *Let A be a symmetric positive definite $n \times n$ matrix, S a polynomial in A satisfying $\varrho(S) \leq 1$ and $A_S = S^2 A$, $\bar{\varrho}(A_S) \geq \varrho(A_S)$ an available upper bound, and $\gamma > 0$ an integer. Then, for every $q \in (0,1)$, the symmetrized smoother $\bar{R}$ given by (6.21) satisfies*

$$\|\mathbf{x}\|_{\bar{R}}^2 \leq \frac{1}{1-q^{2\gamma}}\|\mathbf{x}\|_A^2 + \frac{\bar{\varrho}(A_S)}{q^2}\|\mathbf{x}\|^2, \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{6.22}$$

Proof. Let $\{v_i\}_{i=1}^n$ be eigenvectors of A. Since S is a polynomial in A, the eigenvectors v_i are also eigenvectors of S. We choose an arbitrary $q \in (0,1)$ and define

$$U_1 = \{\text{span}\{v_i\} : \lambda_i(S) \leq q\}, \qquad U_2 = \{\text{span}\{v_i\} : \lambda_i(S) > q\}.$$

First we prove that

$$\|\mathbf{x}\|_{\bar{R}}^2 \leq \frac{1}{1-q^{2\gamma}}\|\mathbf{x}\|_A^2, \qquad \forall \mathbf{x} \in U_1. \tag{6.23}$$

By (6.21) and $\varrho(I - 1/\bar{\varrho}(A_S)A_S) \leq 1$, we have for $\mathbf{x} \in U_1$

$$\begin{aligned}\langle \bar{R}^{-1}\mathbf{x},\mathbf{x}\rangle &= \langle A^{-1}\mathbf{x},\mathbf{x}\rangle - \left\langle A^{-1}\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)^2 S^{2\gamma}\mathbf{x},\mathbf{x}\right\rangle \\ &\geq \langle A^{-1}\mathbf{x},\mathbf{x}\rangle - \left\langle A^{-1}S^{2\gamma}\mathbf{x},\mathbf{x}\right\rangle \geq (1-q^{2\gamma})\langle A^{-1}\mathbf{x},\mathbf{x}\rangle.\end{aligned}$$

Since U_1 is an invariant subspace of both $\bar{R}$ and A and both $\bar{R}$ and A are symmetric positive definite on U_1, the above estimate gives (6.23).

Similarly, we prove

$$\|\mathbf{x}\|_{\bar{R}}^2 < \frac{\bar{\varrho}(A_S)}{q^2}\|\mathbf{x}\|^2, \qquad \forall \mathbf{x} \in U_2. \tag{6.24}$$

Let $\mathbf{x} \in U_2$. We estimate using the identity (6.21), $\varrho(S) \leq 1$, and $0 \leq I - 1/\bar{\varrho}(A_S)A_S \leq I$,

$$\begin{aligned}\langle \bar{R}^{-1}\mathbf{x},\mathbf{x}\rangle &= \langle A^{-1}\mathbf{x},\mathbf{x}\rangle - \left\langle A^{-1}\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)^2 S^{2\gamma}\mathbf{x},\mathbf{x}\right\rangle \\ &\geq \langle A^{-1}\mathbf{x},\mathbf{x}\rangle - \left\langle A^{-1}\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)\mathbf{x},\mathbf{x}\right\rangle \\ &= \frac{1}{\bar{\varrho}(A_S)}\langle S^2\mathbf{x},\mathbf{x}\rangle > \frac{q^2}{\bar{\varrho}(A_S)}\|\mathbf{x}\|^2.\end{aligned}$$

Since U_2 is an invariant subspace of $\bar{R}$, which is symmetric positive definite on U_2, the above estimate gives (6.24).

Clearly, the sets U_1 and U_2 form an orthogonal decomposition of $\mathbb{R}^n$, that is, there is a unique mapping

$$\mathbf{x} \in \mathbb{R}^n \mapsto [\mathbf{x}_1,\mathbf{x}_2] \in U_1 \times U_2 : \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad \langle \mathbf{x}_1,\mathbf{x}_2\rangle = 0.$$

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Since U_1 and U_2 are invariant subspaces of $\bar{R}$ and A, the decomposition is also $\bar{R}$- and A-orthogonal. Thus, using the above decomposition, (6.23), and (6.24), we can estimate

$$\|\mathbf{x}\|_{\bar{R}}^2 = \|\mathbf{x}_1\|_{\bar{R}}^2 + \|\mathbf{x}_2\|_{\bar{R}}^2 \leq \frac{1}{1-q^{2\gamma}}\|\mathbf{x}_1\|_A^2 + \frac{\bar{\varrho}(A_S)}{q^2}\|\mathbf{x}_2\|^2 \leq \frac{1}{1-q^{2\gamma}}\|\mathbf{x}\|_A^2 + \frac{\bar{\varrho}(A_S)}{q^2}\|\mathbf{x}\|^2,$$

proving (6.22). ∎

LEMMA 6.8. *Let $S = p(A)$, where p is the polynomial given by (4.14) with $\varrho = \bar{\varrho}(A)$. Here $\bar{\varrho}(A) \geq \varrho(A)$ is an available upper bound. Further, let γ be a positive integer, $A_S = S^2 A$, and*

$$\bar{\varrho}(A_S) = \frac{\bar{\varrho}(A)}{(1+\deg(S))^2}. \tag{6.25}$$

Then $\varrho(A_S) \leq \bar{\varrho}(A_S)$, and the symmetrized smoother $\bar{R}$ given by (6.20) satisfies for all $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_{\bar{R}}^2 \leq \left[\inf_{q\in(0,1)} \max\left\{\frac{1}{1-q^{2\gamma}}, \frac{1}{q^2}\right\}\right]\left(\|\mathbf{x}\|_A^2 + \frac{\bar{\varrho}(A)}{(1+2\deg(S))^2}\|\mathbf{x}\|^2\right). \tag{6.26}$$

Proof. First we prove $\varrho(A_S) \leq \bar{\varrho}(A_S)$. We estimate using $A_S = S^2 A$, $S = p(A)$, the spectral mapping theorem, $\bar{\varrho}(A) \geq \varrho(A)$, and (4.15),

$$\varrho(A_S) = \varrho(S^2 A) = \max_{t\in\sigma(A)} p^2(t)t \leq \max_{t\in[0,\bar{\varrho}(A)]} p^2(t)t = \frac{\bar{\varrho}(A)}{(1+2\deg(S))^2}.$$

The estimate $\bar{\varrho}(A_S) \geq \varrho(A_S)$ now follows by (6.25).

The main statement (6.26) is a consequence of the statement (6.22) in Lemma 6.7 and $\bar{\varrho}(A_S) \geq \varrho(A_S)$. ∎

REMARK 6.9. For $\gamma = 1$, using the minimizer $\hat{q} = 1/\sqrt{2}$, we have (see (6.26))

$$\min_{q\in(0,1)} \max\left\{\frac{1}{1-q^{2\gamma}}, \frac{1}{q^2}\right\} = 2.$$

Similarly, for $\gamma = 2$, using the minimizer $\hat{q} = \sqrt{\frac{-1+\sqrt{5}}{2}}$, we get

$$\min_{q\in(0,1)} \max\left\{\frac{1}{1-q^{2\gamma}}, \frac{1}{q^2}\right\} = \frac{2}{-1+\sqrt{5}} \doteq 1.618034. \tag{6.27}$$

REMARK 6.10 (Implementation of the smoother). To implement the action of S, we perform the sequence of Richardson sweeps

for $i = 1, \ldots, \deg(S)$ do
  $\mathbf{x} \leftarrow \mathbf{x} - \frac{\bar{\varrho}(A)}{r_i}(A\mathbf{x} - \mathbf{f})$,
  $r_i = \frac{\bar{\varrho}(A)}{2}\left(1 - \cos\frac{2i\pi}{2\deg(S)+1}\right)$;

see (4.14). The smoothing step with the linear part $I - \frac{1}{\bar{\varrho}(A_S)}A_S$, $A_S = S^2 A$, is then performed as

$$\mathbf{x} \leftarrow \mathbf{x} - \frac{1}{\bar{\varrho}(A_S)} S^2(A\mathbf{x} - \mathbf{f}).$$

6.3. The final abstract result. In this section we summarize the results of Section 6.1 and Section 6.2 in the form of a theorem.

THEOREM 6.11. *Let* $\bar{\lambda}_{l+1,l} \geq \lambda_{l+1,l}$ *and* $\bar{\varrho}(A_l) \geq \varrho(A_l),\ l = 1,\ldots,L-1,$ *be upper bounds. (The upper bounds* $\bar{\varrho}(A_l)$ *must be available.) We assume that there are linear mappings* $Q_l : V_1 \to V_l$, $l = 1,\ldots,L$, *that satisfy the assumptions* (6.7) *and* (6.8) *with positive constants* C_a *and* C_s *independent of the level* l. *Further, we assume that on each level, the linear part of both pre- and post-smoother is given by* (6.19) *with* $\gamma = 2$, $A = A_l$, $S = p(A_l)$, *where* $p(\cdot)$ *is the polynomial* (4.14) *with* $\varrho = \bar{\varrho}(A_l)$ *and its degree* d *satisfying*

$$d \geq C_{deg}\sqrt{\frac{\bar{\varrho}(A_l)}{\bar{\lambda}_{l,l+1}}}. \tag{6.28}$$

In (6.19)*, we use* $\bar{\varrho}(A_S)$ *given by* (6.25)*. Then, for every* $\mathbf{v} \in V_1$*, the equivalence*

$$\langle A\mathbf{v},\mathbf{v}\rangle \leq \langle R_{mgm}\mathbf{v},\mathbf{v}\rangle \leq \left[C_s^2 + 2(L-1)\left(\beta(C_a^2+4C_s^2)+\frac{1}{\alpha}C_s^2\right)\right]\langle A\mathbf{v},\mathbf{v}\rangle$$

holds with $\alpha = 1$ *and*

$$\beta = \frac{2}{-1+\sqrt{5}}\max\left\{1, \frac{1}{4C_{deg}^2}\right\}. \tag{6.29}$$

Proof. The proof consists in the verification of the assumptions of Theorem 6.4. The assumptions (6.7) and (6.8) are also assumptions of this theorem. Thus, it remains to find the bounds for α in (6.9) and β in (6.11).

To get the estimate for α, we have to verify (6.9) for a smoother with the linear part given by (6.19) with $\gamma = 2$. From (6.19), $\bar{\varrho}(A_S) \geq \varrho(A_S)$, $\varrho(S) \leq 1$, and the fact that S is a polynomial in A, we get

$$R^{-1} = R^{-T} = \langle A^{-1}\mathbf{x},\mathbf{x}\rangle - \left\langle A^{-1}\left(I - \frac{1}{\bar{\varrho}(A_S)}A_S\right)S^2\mathbf{x},\mathbf{x}\right\rangle \leq \langle A^{-1}\mathbf{x},\mathbf{x}\rangle, \quad \forall \mathbf{x}.$$

Here, $A_S = S^2A$ and $\bar{\varrho}(A_S) \geq \varrho(A_S)$ is a given upper bound based on $\bar{\varrho}(A) \geq \varrho(A)$ by (6.25). Thus, $R = R^T \geq A$ and (6.9) holds with $\alpha = 1$. To estimate β, we use Lemma 6.8 and assumption (6.28). By (6.26), (6.27), and (6.28), we have for any $\mathbf{x} \in \mathbb{R}^{n_l}$,

$$\begin{aligned}
\|\mathbf{x}\|^2_{\bar{R}_l} &\leq \frac{2}{-1+\sqrt{5}}\left(\|\mathbf{x}\|^2_{A_l} + \frac{\bar{\varrho}(A_l)}{(1+2\mathrm{deg}(S_l))^2}\|\mathbf{x}\|^2\right)\\
&\leq \frac{2}{-1+\sqrt{5}}\left(\|\mathbf{x}\|^2_{A_l} + \frac{\bar{\varrho}(A_l)}{4\mathrm{deg}^2(S_l)}\|\mathbf{x}\|^2\right)\\
&\leq \frac{2}{-1+\sqrt{5}}\left(\|\mathbf{x}\|^2_{A_l} + \frac{1}{4C_{deg}^2}\bar{\lambda}_{l+1,l}\|\mathbf{x}\|^2\right)\\
&\leq \frac{2}{-1+\sqrt{5}}\max\left\{1,\frac{1}{4C_{deg}^2}\right\}\left(\|\mathbf{x}\|^2_{A_l} + \bar{\lambda}_{l+1,l}\|\mathbf{x}\|^2\right),
\end{aligned}$$

proving (6.11) with β as in (6.29). ∎

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6.4. A model example. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a polygon or polytope. Consider a model elliptic problem with the H_0^1-equivalent form

$$\text{find } u \in H_0^1(\Omega): \quad a(u,v) = (f,v)_{L_2(\Omega)}, \qquad \forall v \in H_0^1(\Omega).$$

Here $a(\cdot,\cdot)$ is a bilinear form satisfying

$$c|u|^2_{H^1(\Omega)} \le a(u,u) \le C|u|^2_{H^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega),$$

and $f \in L_2(\Omega)$.

Assume a quasiuniform triangulation of Ω and a hierarchy of quasiuniform triangulations obtained by its successive refinement. The finest triangulation will be denoted by τ_{h_1}, the second finest by τ_{h_2}, etc. Here, h_l is a characteristic resolution of a triangulation τ_{h_l}. The case of interest is $h_l \ll h_{l+1}$. Let L denote the number of triangulations. The P1 finite element space corresponding to the triangulation τ_{h_l} will be denoted by V_{h_l} and its basis by $\{\varphi_i^l\}_{i=1}^{n_l}$. Realizing the Dirichlet constraints (i.e., by removing the basis functions corresponding to the vertices located at $\delta\Omega$) results in a hierarchy of nested spaces

$$H_0^1(\Omega) \supset V_{h_1} \supset V_{h_2} \supset \ldots \supset V_{h_L}.$$

The interpolation operators are defined in the usual way,

$$\Pi_{h_l} \ : \ \mathbf{x} \in \mathbb{R}^{n_l} \mapsto \sum_{i=1}^{n_l} x_i \varphi_i^l.$$

Let Q_{h_l} denote the $L_2(\Omega)$-orthogonal projection onto V_{h_l}. We will use the following well-known properties of the finite element functions [9]:

$$\|(I - Q_{h_l})u\|_{L_2(\Omega)} \le C h_l |u|_{H^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega), \tag{6.30}$$

$$|Q_{h_l} u|_{H^1(\Omega)} \le C|u|_{H^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega), \tag{6.31}$$

$$c\|\Pi_{h_l}\mathbf{x}\|^2_{L_2(\Omega)} \le h_l^d \|\mathbf{x}\|^2 \le C\|\Pi_{h_l}\mathbf{x}\|^2_{L_2(\Omega)}, \qquad \forall \mathbf{x} \in \mathbb{R}^{n_l}, \tag{6.32}$$

$$c h_l^{d-1} \le \varrho(A_l) \le C h_l^{d-2}. \tag{6.33}$$

As in Section 5.3, we define the prolongation operators P_{l+1}^l by the natural embedding of the spaces V_{h_l}, that is,

$$P_{l+1}^l = \Pi_{h_l}^{-1} \Pi_{h_{l+1}}. \tag{6.34}$$

Further we set

$$\tilde{Q}_l = \Pi_{h_l}^{-1} Q_{h_l} \Pi_{h_1}, \quad Q_l = P_l^1 \tilde{Q}_l, \qquad l = 1, \ldots, L. \tag{6.35}$$

Our goal is to verify the assumptions of Theorem 6.11 for the linear operators Q_l.

By (6.6),

$$\lambda_{l+1,l} = \sup_{\mathbf{x} \in \mathbb{R}^{n_{l+1}} \setminus \{\mathbf{0}\}} \frac{\langle A_{l+1}\mathbf{x}, \mathbf{x}\rangle}{\|P_{l+1}^l \mathbf{x}\|^2}. \tag{6.36}$$

Further, using the estimate (6.32) and definition (6.34), we get

$$\|P_{l+1}^l \mathbf{x}\|^2 \ge c h_l^{-d} \|\Pi_{h_l} P_{l+1}^l \mathbf{x}\|^2_{L_2(\Omega)} = c h_l^{-d} \|\Pi_{h_{l+1}} \mathbf{x}\|^2_{L_2(\Omega)} \ge c \left(\frac{h_{l+1}}{h_l}\right)^d \|\mathbf{x}\|^2.$$

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Substituting the above estimate into (6.36) and using (6.33) yields

$$\lambda_{l+1,l} \leq C\left(\frac{h_l}{h_{l+1}}\right)^d \varrho(A_{l+1}) \leq C h_l^d h_{l+1}^{-2}.$$

Thus, there is a positive constant C such that that

$$\bar{\lambda}_{l+1,l} \equiv C h_l^d h_{l+1}^{-2} \geq \lambda_{l+1,l}, \qquad \forall l = 1,\dots,L-1. \tag{6.37}$$

As $\varrho(\bar{A}_l) \geq \varrho(A_l)$, we take

$$\bar{\varrho}(A_l) = \max_{i=1,\dots,n_l} \sum_{j=1}^{n_l} |(A_l)_{ij}|.$$

This bound satisfies $c h_l^{d-2} \leq \bar{\varrho}(A_l) \leq C h_l^{d-2}$. Thus we conclude that

$$c\left(\frac{h_{l+1}}{h_l}\right)^2 \leq \frac{\bar{\varrho}(A_l)}{\bar{\lambda}_{l+1,l}} \leq C\left(\frac{h_{l+1}}{h_l}\right)^2. \tag{6.38}$$

As required by Theorem 6.11, we assume that on each level, the linear part of both pre- and post-smoother is given by (6.19) with $\gamma = 2$, $A = A_l$, $S = p(A_l)$, where $p(\cdot)$ is the polynomial (4.14) with $\varrho = \bar{\varrho}(A_l)$ and its degree d satisfying

$$d \geq C\frac{h_{l+1}}{h_l}. \tag{6.39}$$

In (6.19), $\bar{\varrho}(A_S)$ given by (6.25) is used. From (6.39) and (6.38), we have (6.28). The properties (6.7) and (6.8) are now consequences of (6.30), (6.31), (6.32), the definitions (6.35), (6.34), (6.4), and the estimate (6.37). Let $u = \Pi_{h_1}\mathbf{u}$, $\mathbf{u} \in V_1$. Using (6.31), (6.31), (6.32), and (5.40), we get

$$\begin{aligned}
\|(Q_l - Q_{l+1})\mathbf{u}\|_{V_l}^2 &= \|P_l^1(\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1})\mathbf{u}\|_{V_l}^2 = \|(\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1})\mathbf{u}\|^2 \\
&\leq C h_l^{-d} \|\Pi_{h_l}(\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1})\Pi_{h_1}^{-1} u\|_{L_2(\Omega)}^2 \\
&= C h_l^{-d} \|(\Pi_{h_l}\tilde{Q}_l - \Pi_{h_{l+1}}\tilde{Q}_{l+1})\Pi_{h_1}^{-1} u\|_{L_2(\Omega)}^2 \\
&= C h_l^{-d} \|(Q_{h_l} - Q_{h_{l+1}})u\|_{L_2(\Omega)}^2 = C h_l^{-d} \|(I - Q_{h_{l+1}})Q_{h_l} u\|_{L_2(\Omega)}^2 \\
&\leq C h_l^{-d} h_{l+1}^2 |Q_{h_l} u|_{H^1(\Omega)}^2 \leq C h_l^{-d} h_{l+1}^2 |u|_{H^1(\Omega)}^2 \leq \frac{C}{\bar{\lambda}_{l+1,l}} \|\mathbf{u}\|_A^2,
\end{aligned}$$

proving (6.7).

To prove (6.8), we estimate using (5.40),

$$\begin{aligned}
\|Q_l \mathbf{u}\|_A &\leq C|\Pi_{h_1} P_l^1 \tilde{Q}_l \Pi_{h_1}^{-1} u|_{H^1(\Omega)} = C|\Pi_{h_l}\tilde{Q}_l \Pi_{h_1}^{-1} u|_{H^1(\Omega)} \\
&= C|Q_{h_l} u|_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)} \leq C\|\mathbf{u}\|_A.
\end{aligned}$$

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We summarize our result in form of a theorem.

THEOREM 6.12. *Consider the model problem and a hierarchy of nested finite element spaces as described in this section. We assume the prolongators are given by* (6.34)*. Further, we assume that on each level, the linear part of both pre- and post-smoother is given by* (6.19) *with* $\gamma = 2$, $A = A_l$, $S = p(A_l)$*, where* $p(\cdot)$ *is the polynomial* (4.14) *with* $\varrho = \bar{\varrho}(A_l)$ *and its degree* d *satisfying*

$$d \geq C \frac{h_{l+1}}{h_l}.$$

In (6.19)*,* $\bar{\varrho}(A_S)$ *given by* (6.25) *is used. Then the resulting multigrid V-cycle converges in the* A*-norm, and its rate of convergence is estimated by*

$$1 - \frac{C}{L},$$

where the constant C *is independent of the resolutions* h_l, $l = 1, \dots, L$*.*

7. Numerical illustration. In this section, the most general multilevel smoothed aggregation method of Section 6 is chosen to demonstrate the efficiency of the strategy of aggressive coarsening combined with massive polynomial smoothing in general and to illustrate numerically the validity of the convergence bounds. Numerical experiments comparing the efficiency of all the presented methods (of increasing generality) and other domain decomposition methods, preferably on a parallel platform, are beyond the scope of this theoretical paper. This section closely follows [8].

A numerical solution of the Poisson equation is considered in two and three dimensions. For the former case, finite element discretizations using two successively refined unstructured meshes (labeled Mesh7 and Mesh8) of a square in 2D are used. A Dirichlet boundary condition is imposed at all 1924 and 3844 boundary nodes, respectively. For the 3D case, a mesh of a deformed cubic domain discretized using over 1.3 million degrees of freedom is considered. Dirichlet boundary conditions are imposed only on one of the boundary faces.

In all of the experiments, the smoothed aggregation method is used as a preconditioner in the conjugate gradient method. In particular, in the symmetric positive definite case that is considered here, the preconditioned conjugate gradient method is employed. The following stopping criterion is used

$$\sqrt{\frac{\mathbf{z}_k^T \mathbf{r}_k}{\mathbf{z}_0^T \mathbf{r}_0}} \leq \varepsilon \sqrt{\kappa},$$

where $\mathbf{r}_k = \mathbf{f} - A\mathbf{x}_k$ denotes the residual at iteration k, $\mathbf{z}_k = B^{-1}\mathbf{r}_k$ denotes the preconditioned residual at iteration k, κ is an estimate of the condition number of the preconditioned system, and $\varepsilon = 10^{-6}$.

The condition number estimates provided in the tables below are approximations obtained from $1/(1-\varrho)$ where ϱ is the value

$$\left(\sqrt{\frac{\mathbf{z}_{n_{it}}^T \mathbf{r}_{n_{it}}}{\mathbf{z}_0^T \mathbf{r}_0}} \right)^{1/n_{it}}.$$

Here, n_{it} is the iteration count k when the stopping criterion is reached. Aggressive coarsening is employed only between the first two levels—higher order polynomial degrees ν_1^P and ν_1^R are used for the smoothed prolongator and the relaxation method, respectively, whereas $\nu_k = 1$

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TABLE 7.1
Results for the 2D unstructured problem with 205 761 *degrees of freedom (Mesh7).*

1^{st} coarse level size	# levels	ν_1^P	ν_1^R	Setup time	Iter. time	# iterations	Cond. #	Oper. cmplx.
289	3	12	12	2.220	4.645	9	2.693	1.00282
289	3	12	11	2.232	4.287	9	2.917	1.00282
289	3	12	10	2.218	4.331	10	3.337	1.00282
289	3	9	11	1.597	4.658	10	3.578	1.00199
289	3	9	10	1.591	4.631	11	3.834	1.00199
289	3	9	9	1.607	4.233	11	4.276	1.00199
289	3	9	8	1.594	4.174	12	4.972	1.00199
2500	4	6	6	1.757	2.353	8	2.530	1.03879
2500	4	6	5	1.754	2.313	9	2.625	1.03879
2500	4	6	4	1.759	1.919	9	2.554	1.03879
2500	4	6	3	1.732	1.902	11	3.643	1.03879
22500	5	3	3	2.386	1.922	9	2.682	1.55218
22500	5	3	2	2.345	1.557	9	2.713	1.55218
22500	5	3	1	2.347	1.228	9	2.545	1.55218
22500	5	2	2	1.457	1.643	10	3.457	1.37477
22500	5	2	1	1.453	1.249	10	3.174	1.37477
22500	5	1	1	1.220	1.100	10	2.906	1.19465
17271	5	1	1	1.020	1.310	10	3.692	1.10914
17271	5	1	1	1.022	1.243	14	5.741	1.10914

for $k > 1$. Details of the aggregation procedure can be found in [8]. All experiments are carried out on a laptop equipped with a 2 GHz Intel Core2 Duo P7350 CPU and 4 GB of RAM.

The results for the two-dimensional problems are summarized in Tables 7.1 and 7.2. The results for the three-dimensional problem are tabulated in Table 7.3. The first column in the tables reports the size of the first coarse-level problem. The last column reports the operator complexity, a standard algebraic multigrid measure defined as the sum of the numbers of nonzero entries of the matrices at all coarsening levels divided by the number of nonzero entries of the fine-grid matrix. The last 2 lines in the tables correspond to a standard SA solver with the default aggregation of [24] and the Gauss-Seidel and Jacobi relaxation, respectively, used on all levels with $\nu_k = 1$.

The tables clearly demonstrate that the multilevel smoothed aggregation method with aggressive coarsening performs as expected. In particular, the condition number and the number of iterations are well controlled when the problem size or the first-level coarse-space size are varied. This confirms the uniform convergence bounds proved in the previous section. The setup and iterative solution time rise as the degree of coarsening is increased. This is due to the fact that the experiments are performed on a serial architecture. The aggressive coarsening and massive smoothing is the right choice on a massively parallel machine that can exploit the parallelism enabled by the additive nature of polynomial smoothers. The results for the Gauss-Seidel smoother are provided for comparison only; the polynomial smoothers and the (scaled) Jacobi smoother are straightforward to parallelize and hence of more practical interest.

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TABLE 7.2
Results for the 2D unstructured problem with 821 121 degrees of freedom (Mesh8).

1st coarse level size	# levels	ν_1^P	ν_1^R	Setup time	Iter. time	# iterations	Cond. #	Oper. cmplx.
144	3	30	30	17.876	46.878	9	3.210	1.00028
144	3	30	25	17.669	42.745	10	4.059	1.00028
144	3	30	20	17.669	41.570	12	5.582	1.00028
1156	4	13	13	9.728	21.462	9	2.889	1.00324
1156	4	13	12	9.743	20.038	9	2.891	1.00324
1156	4	13	11	9.700	18.432	9	3.035	1.00324
1156	4	13	10	9.647	18.460	10	3.527	1.00324
10201	4	6	6	7.178	11.344	9	3.048	1.04092
10201	4	6	5	7.121	9.781	9	3.023	1.04092
10201	4	6	4	7.087	8.284	9	3.020	1.04092
10201	4	6	3	7.049	8.135	11	3.652	1.04092
10201	4	6	2	7.132	8.621	15	6.757	1.04092
10201	5	2	2	5.930	7.741	11	3.900	1.38440
10201	5	2	1	5.930	5.899	11	3.571	1.38440
10201	5	1	1	4.798	7.741	10	3.072	1.19946
90434	5	1	1	4.891	6.655	11	5.076	1.1787
90434	5	1	1	4.950	6.669	15	9.007	1.1787

TABLE 7.3
Results for the 3D problem with 1 367 631 degrees of freedom.

1st coarse level	# levels	ν_1^P	ν_1^R	Setup time	It. time	# iterations	Cond. #	Oper. Cmplx.
64	3	8	8	30.174	30.799	7	2.118	1.00003
64	3	8	7	30.132	31.044	8	2.453	1.00003
64	3	8	6	30.059	30.131	9	3.045	1.00003
64	3	8	5	30.047	31.231	11	4.074	1.00003
64	3	6	6	23.425	29.974	9	3.192	1.00003
64	3	6	5	23.359	31.061	11	4.117	1.00003
1680	4	4	4	27.344	17.737	7	2.266	1.00159
1680	4	4	3	27.339	14.569	7	1.927	1.00159
1680	4	4	2	27.499	14.178	9	2.557	1.00159
1680	4	4	1	27.441	15.475	14	6.172	1.00159
1680	4	3	3	20.595	18.053	9	2.673	1.00111
1680	4	3	2	20.620	14.005	9	2.644	1.00111
1680	4	3	1	20.585	14.978	14	6.159	1.00111
46248	4	2	2	44.717	11.461	6	1.560	1.14223
46248	4	2	1	44.426	8.494	6	1.492	1.14223
46248	4	1	1	22.701	8.551	7	1.935	1.04375
51266	4	1	1	27.214	16.372	7	2.394	1.10982
51266	4	1	1	27.106	11.467	10	3.449	1.10982

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8. Concluding remarks. We started with a smoothed aggregation method based on a two-level method with aggressive coarsening that justifies the usage of multiple smoothing steps under regularity-free conditions. For this method, however, we were unable to prove fully optimal convergence. Compared to the standard two-level framework with a single smoothing step, the dependence on the degree of coarsening was reduced but not eliminated. This drawback was fixed in Section 4, where a more sophisticated smoother (and a prolongator smoother) was used and the optimal convergence bound was established for this two-level method. Our result assumed that the coarse-level problem was solved exactly. In Section 5, we extended the result of Section 4 to the case where the coarse-level problem is solved by a multigrid V-cycle without aggressive coarsening. Thus, we proved fully optimal convergence for a multilevel smoothed aggregation method with aggressive coarsening between the first two levels. This result is new and the convergence proof is completely different to that of Section 4. The most general result of Section 6 establishes a generalization into two important directions. Unlike in the preceding cases, it is fully independent of the smoothed aggregation framework and allows for aggressive coarsening between any two adjacent levels. Numerical results presented in Section 7 confirm the convergence estimates.

It is the opinion of the authors that the theoretical tools presented in this paper allow to control convergence properties of the methods with aggressive coarsening and polynomial smoothing satisfactorily. The future focus will be on a high-performance parallel implementation.

Acknowledgment. This work was sponsored by the TAČR (Technologická Agentura České Republiky) grant TA01020352, ITI (Institut Teoretické Informatiky) grant 1R0545, Department of the Navy grant N62909-11-1-7032. Part of this work has been performed while the third author was visiting the University of Nice whose support is acknowledged.

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