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ON THE DISCRETE EXTENSION OF MARKOV'S THEOREM ON MONOTONICITY OF ZEROS*

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Abstract. Motivated by an open problem proposed by M. E. H. Ismail in his monograph "Classical and quantum orthogonal polynomials in one variable" (Cambridge University Press, 2005), we study the behavior of zeros of orthogonal polynomials associated with a positive measure on $[a, b] \subseteq \mathbb{R}$ which is modified by adding a mass at $c \in \mathbb{R} \setminus (a, b)$. We prove that the zeros of the corresponding polynomials are strictly increasing functions of c. Moreover, we establish their asymptotics when c tends to infinity or minus infinity, and it is shown that the rate of convergence is of order $1/c$.

Key words. orthogonal polynomials on the real line, Uvarov's transformation, Markov's theorem, monotonicity of zeros, asymptotic behavior, speed of convergence

AMS subject classifications. 33C45, 30C15

1. Introduction. In 1814, Gauss (Comm. Soc. Reg. Sci. Gott. Rec., vol. III, 1816) developed the quadrature rule for Legendre polynomials and Jacobi (J. Reine Angew. Math., vol. I, 1826) extended it to Jacobi polynomials twelve years later. After this, the theory of *orthogonal polynomials on the real line* (OPRL, in short) has attracted a lot of attention and has become a major theme in classical analysis in the twentieth century. From a general point of view, the pioneer works of Chebyshev, Darboux, Markov, Christoffel, and Stieltjes were fundamental. We urge the reader to consult [1, 4, 10, 14, 15, 17], where a complete account of the classical theory of OPRL can be found, for some background information.

Let $d\mu$ be a nontrivial measure on $[a, b] \subseteq \mathbb{R}$ such that

$$\int_a^b |x|^n d\mu(x) < \infty, \quad n \geq 0.$$

The application of the Gram-Schmidt process to $1, x, x^2, \ldots$ (linearly independent in the Hilbert space $L^2([a, b], d\mu)$ with norm $\|\cdot\|$) yields a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ and a sequence of positive real numbers $\{\gamma_n\}_{n\geq 0}$ such that

$$\int_a^b P_n(x)P_m(x)d\mu(x) = \gamma_n \delta_{n,m}, \quad m \geq 0,$$

where $\delta_{n,m}$ is the Kronecker delta. These polynomials are formally the OPRL. It is important to recall that the zeros of P_n, $x_{n,k}$, $1 \leq k \leq n$, are real and simple in (a, b) and that the zeros of P_n and P_{n+1} strictly interlace. Moreover, associated with $\{P_n\}_{n\geq 0}$, there are sequences of positive real numbers $\{a_n\}_{n\geq 1}$ and of real numbers $\{b_n\}_{n\geq 0}$ such that

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_nP_{n-1}(x),$$

with initial conditions $P_{-1} := 0$ and $P_0 := 1$. The converse of this result is the Favard Theorem or Spectral Theorem for OPRL.

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The behavior of the zeros plays a major role in OPRL theory. Markov [11] and Stieltjes [16] were the first who studied the monotonicity of zeros of a parameter-dependent sequence of OPRL as functions of the involved parameter. Let us consider the OPRL, $P_n(\cdot;\tau)$, associated with the parametric measure

$$\omega(x,\tau)dx + d\omega_s(x),$$

where $d\omega_s$ is singular. A general result of Markov [10, 11, 17] states that under some additional conditions on $\omega(\cdot,\tau)$, if $\partial \ln \omega(x,\tau)/\partial\tau$ is an increasing (respectively decreasing) function of x, then the zeros of $P_n(\cdot;\tau)$ are increasing (respectively decreasing) functions of τ. As the reader may note, Markov's theorem does not consider the cases of parameters in the singular part of the measure. In this sense, a natural open problem was pointed out in [9, Problem 1] and [10, Problem 24.9.1].

OPEN PROBLEM 1.1 (M. E. H. Ismail, 1989). *Extend Markov's theorem to the case when the measure is given by*

$$\omega(x,\tau)dx + d\nu(x,\tau),$$

where $\nu(\cdot,\tau)$ is a jump function or a step function.

This last problem provides the motivation for our research. The structure of the manuscript is as follows: in Section 2, we present some preliminaries in order to fix the notation, and our extension of Markov's theorem is presented. In Section 3, our main result is proved. Finally, in Section 4, two illustrative examples associated with Jacobi and Laguerre orthogonal polynomials are shown.

2. Discrete extension of Markov's theorem. For technical reasons, in what follows we assume that either a or b can be infinity. Let us denote by $\{P_n(\cdot;\lambda,c)\}_{n\geq 0}$ the OPRL with respect to a new measure formed by adding to $d\mu$ a positive mass λ at $c \in \mathbb{R}\setminus(a,b)$, that is,

$$d\mu + \lambda\,\delta_c, \quad \lambda > 0. \tag{2.1}$$

This modification of the measure $d\mu$ is the so-called Uvarov transformation.

In the case when $d\mu$ is a classical measure, that is, one of those with respect to which Jacobi, Laguerre, and Hermite polynomials are orthogonal, rather extensive literature provides precise results on the behavior of zeros with respect to the parameter λ. General results about interlacing, convergence, and monotonicity with respect to the parameter λ can be found in [3, 8] and the references therein.

Define the polynomial $G_n(\cdot;c)$ by

$$G_n(x;c) := P_n(x) - \frac{P_n(c)}{K_{n-1}(c,c)}K_{n-1}(c,x) = (x-c)Q_{n-1}(x;c),$$

where $K_{n-1}(\cdot,\cdot)$ is the kernel polynomial given by

$$K_{n-1}(x,y) = \sum_{k=0}^{n-1}\frac{P_k(x)P_k(y)}{\|P_k\|^2}\,.$$

It is known [3, 8] that $Q_{n-1}(\cdot;c)$ is the corresponding polynomial of degree $n-1$ orthogonal with respect to the measure

$$(x-c)^2 d\mu(x).$$

Denote by $x_{n,k}(\lambda, c)$ (respectively $y_{n,k}(c)$), $1 \le k \le n$, the zeros of $P_n(\cdot;\lambda, c)$ (respectively $G_n(\cdot, c)$).

THEOREM 2.1. [3, 8] *The following statements hold:*

(i) *If* $-\infty < c \le a$, *then*

$$c = y_{n,1}(c) < x_{n,1}(\lambda, c) < x_{n,1} < \cdots < y_{n,n}(c) < x_{n,n}(\lambda, c) < x_{n,n}.$$

Moreover, $x_{n,k}(\lambda, c)$ *(for a fixed value of* k *and* $n > 0$*) is a strictly decreasing function of* λ.

(ii) *If* $b \le c < \infty$, *then*

$$x_{n,1} < x_{n,1}(\lambda, c) < y_{n,1}(c) < \cdots < x_{n,n} < x_{n,n}(\lambda, c) < y_{n,n}(c) = c.$$

Moreover, $x_{n,k}(\lambda, c)$ *(for a fixed value of* k *and* $n > 0$*) is a strictly increasing function of* λ.

Furthermore,

$$\lim_{\lambda\to\infty} x_{n,k}(\lambda, c) = y_{n,k}(c),$$

and

$$\lim_{\lambda\to\infty} \lambda(y_{n,k}(c) - x_{n,k}(\lambda, c)) = \frac{P_n(y_{n,k}(c))}{K_{n-1}(c, c)G'_n(y_{n,k}(c); c)}\,.$$

The above limit relations show that $x_{n,k}(\lambda, c)$ converges to $y_{n,k}(c)$ when λ tends to infinity with a rate of convergence of order $1/\lambda$.

After Theorem 2.1 and in connection with the Open Problem 1.1, the natural questions are: are the zeros $P_n(\cdot;\lambda, c)$ also monotonic functions with respect to c? Do these zeros converge when c tends to infinity or minus infinity? If so, what is the rate of convergence? The answers to these questions are given in the next theorem. Note that Theorem 2.1 can be proved easily. For example, one way, but not the only way, is based on the behaviour of zeros of a linear combination of polynomials with interlacing zeros [4, Ch. I, Ex. 5.4] (see also [2, Sec. 4.3]), which is closely related to the Hermite-Kakeya theorem [13, Thm. 6.3.8].

It is well known [12, Ch. 7, Lem. 15] that

$$P_n(x;\lambda, c) = P_n(x) - \frac{\lambda P_n(c)}{1 + \lambda K_{n-1}(c, c)} K_{n-1}(x, c), \tag{2.2}$$

or, after normalization,

$$\begin{aligned} p_n(x;\lambda, c) &= P_n(x) + \lambda K_{n-1}(c, c)G_n(x; c) \\ &= P_n(x) + \lambda K_{n-1}(c, c)(x - c)Q_{n-1}(x; c), \end{aligned} \tag{2.3}$$

where $p_n(x;\lambda, c) = (1+\lambda K_{n-1}(c, c))P_n(x;\lambda, c)$. By simple inspection of [4, Ch. I, Ex. 5.4], Theorem 2.1 follows from formula (2.3). However, to obtain the monotonicity of the zeros $x_{n,k}(\lambda, c)$ with respect to c instead of λ, the above approach does not work because the polynomial $Q_{n-1}(\cdot, c)$ (or $G_n(\cdot; c)$) depends on c. To the best of our knowledge, there are no preceding works in the literature on monotonicity of zeros of OPRL with respect to the point where a mass is located. Our main result reads as follows.

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THEOREM 2.2. *Let $c \in \mathbb{R} \setminus (a,b)$. Then $x_{n,k}(\lambda, c)$ (for a fixed value of k and $n > 0$) is a strictly increasing function of c. Moreover, the following statements hold:*

(i) *If $-\infty < c \leq a$, then*

$$c < x_{n,1}(\lambda,c) < x_{n,1} < x_{n-1,1} < x_{n,2}(\lambda,c) < x_{n,2} < x_{n-1,2} \cdots \\ \cdots x_{n,n-1}(\lambda,c) < x_{n,n-1} < x_{n-1,n-1} < x_{n,n}(\lambda,c) < x_{n,n}.$$

Furthermore,

$$\lim_{c\to-\infty} x_{n,k}(\lambda,c) = x_{n-1,k-1}, \quad 2 \leq k \leq n, \qquad \lim_{c\to-\infty} x_{n,1}(\lambda,c) = -\infty,$$

and

$$\lim_{c\to-\infty} c(x_{n,k}(\lambda,c) - x_{n-1,k-1}) = \frac{P_n(x_{n-1,k-1})}{P'_{n-1}(x_{n-1,k-1})}, \qquad 2 \leq k \leq n.$$

(ii) *If $b \leq c < \infty$, then*

$$x_{n,1} < x_{n,1}(\lambda,c) < x_{n-1,1} < x_{n,2} < x_{n,2}(\lambda,c) < x_{n-1,2} < \cdots \\ \cdots < x_{n,n-1} < x_{n,n-1}(\lambda,c) < x_{n-1,n-1} < x_{n,n} < x_{n,n}(\lambda,c) < c.$$

Furthermore,

$$\lim_{c\to\infty} x_{n,k}(\lambda,c) = x_{n-1,k}, \quad 1 \leq k \leq n-1, \qquad \lim_{c\to\infty} x_{n,n}(\lambda,c) = \infty,$$

and

$$\lim_{c\to\infty} c(x_{n-1,k} - x_{n,k}(\lambda,c)) = \frac{P_n(x_{n-1,k})}{P'_{n-1}(x_{n-1,k})}, \quad 1 \leq k \leq n-1.$$

Note that by combining Markov's theorem, Theorem 2.1, and Theorem 2.2, we can give a first answer to Ismail's open problem. To be specific, we give an answer to a very particular case of the open problem. It was also brought to our attention by one of the referees that after the initial submission of the present work, the first part of the above theorem was proved in a more elegant way in [5]. In this article, the author approximates the Dirac delta by the normal distribution and applies the classical Markov's theorem. Instead, our approach is based on the well-developed theory of so-called spectral transformations of orthogonal polynomials. In any case, we opted for our approach for two main reasons. First, we are interested in obtaining more information than just the monotonicity of zeros, as we stated in Theorem 2.2. Therefore, we need the explicit expression of the perturbed polynomials in terms of known polynomials. This need is natural, as can be seen in other papers [3, 6, 7, 8]. Also, all the formulas involved in our proof are very well known, and thus, the proof is just a clever combination of elementary facts. Secondly, we are interested in the possibility of a numerical implementation of our ideas bearing in mind future practical applications.

3. Proof of the main result and related questions. As the reader can note, we only need to prove Theorem 2.2 for one case. In other words, if the theorem is true for $b \leq c < \infty$, then it will be true for $-\infty < c \leq a$. Let us assume that $b \leq c < \infty$ and consider the following modification of the measure (2.1):

$$d\mu + \lambda\, \delta_{c+\epsilon}, \quad \epsilon > 0.$$

We need to prove that

$$x_{n,k}(\lambda,c) < x_{n,k}(\lambda, c+\epsilon), \quad 1 \leq k \leq n.$$

3.1. Proof of Theorem 2.2. Let us consider the following modification of the measure $d\mu$ known as the Christoffel transformation:

$$(x-c)d\mu(x). \tag{3.1}$$

It is easy to verify [4, Ch. 1, Eq. 7.3] that the OPRL associated with (3.1), $\{P_n(\cdot;c)\}_{n\geq 0}$, are given by

$$P_n(x;c) = \frac{\|P_n\|^2}{P_n(c)} K_n(x,c) = \frac{1}{x-c}\left(P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)} P_n(x)\right). \tag{3.2}$$

Combining (2.2) with (3.2), we deduce that

$$\frac{\|P_{n-1}\|^2}{\lambda P_n^2(c)}(x-c)p_n(x;\lambda,c) = P_{n-1}(x) - y_c(x)P_n(x), \tag{3.3}$$

where

$$y_c(x) = m(\lambda,c)(x-c) + \frac{P_{n-1}(c)}{P_n(c)} \quad \text{and} \quad m(\lambda,c) = -\frac{1+\lambda K_{n-1}(c,c)}{\lambda P_n^2(c)}\|P_{n-1}\|^2 < 0.$$

If $p_n(x;\lambda,c)=0$, then (3.3) implies that

$$\frac{P_{n-1}(x)}{P_n(x)} = y_c(x).$$

Clearly, the zeros of $p_n(\cdot;\lambda,c)$ are the intersection points of P_{n-1}/P_n and y_c in (a,c), which implies the interlacing properties stated in Theorem 2.2. We can also deduce this easily from formula (3.3). Of course, the following decomposition into partial fractions holds [17, Thm. 3.3.5]:

$$\frac{P_{n-1}(x)}{P_n(x)} = \sum_{k=1}^{n} \frac{l_k}{x-x_{n,k}}, \quad l_k > 0.$$

Having arrived at this point, note that P_{n-1}/P_n has vertical asymptotes at $x_{n,k}$, $1\leq k\leq n$, and it is a monotonically decreasing function in the open intervals $(-\infty, x_{n,1})$, $(x_{n,k-1}, x_{n,k})$, $2\leq k\leq n-1$, and $(x_{n,n},\infty)$; see Figure 3.1 where P_{n-1}/P_n (continuous line) is presented for $n=5$. Moreover, the lines y_c (small–dashed line) and $y_{c+\epsilon}$ (large–dashed line) have negative slopes. On account of the previous ideas, all we need to prove is that

$$x_{n,n}(\lambda,c) < x_{n,n}(\lambda,c+\epsilon), \tag{3.4}$$

and the result is true for the remaining zeros. Since the zeros of the OPRL lie in the convex hull of the support of the orthogonality measure, if $P_n(x;\lambda,c+\epsilon)=0$ for some value of $x\in[c,c+\epsilon)$, then (3.4) holds. Let us now prove the result for the case that $P_n([c,c+\epsilon);\lambda,c+\epsilon)\neq 0$.

We can write

$$P_n(\cdot;\lambda,c+\epsilon) = P_n(\cdot;\lambda,c) + \sum_{k=0}^{n-1}\alpha_{n,k}P_k(\cdot;\lambda,c),$$

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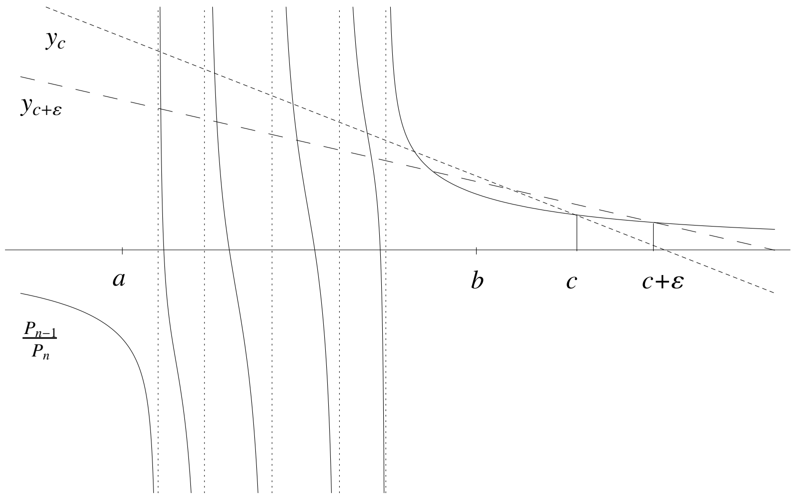

FIGURE 3.1. *Graphs of* $\frac{P_{n-1}}{P_n}$, y_c, *and* $y_{c+\epsilon}$ *for* $n = 5$.

where the coefficients $\alpha_{n,k}$ are determined by

$$\alpha_{n,k}\|P_k(\cdot;\lambda,c)\|^2_{\lambda,c} = \int_a^b P_n(x;\lambda,c+\epsilon)P_k(x;\lambda,c)d\mu(x) + \lambda P_n(c;\lambda,c+\epsilon)P_k(c;\lambda,c). \tag{3.5}$$

Here $\|\cdot\|_{\lambda,c}$ is the L^2–norm associated with the measure (2.1).

Taking into account that $P_k([c,c+\epsilon);\lambda,c+\epsilon) \neq 0$, for $1 \leq k \leq n-1$, we get

$$P_n(c+\epsilon;\lambda,c+\epsilon)P_k(c+\epsilon;\lambda,c) > P_n(c;\lambda,c+\epsilon)P_k(c;\lambda,c). \tag{3.6}$$

Using (3.6) and the orthogonality conditions, (3.5) reduces to

$$\alpha_{n,k}\|P_k(\cdot;\lambda,c)\|^2_{\lambda,c} = -\lambda(\lambda_k-1)P_n(c;\lambda,c+\epsilon)P_k(c;\lambda,c)$$

for some $\lambda_k > 1$, which implies that

$$P_n(\cdot;\lambda,c+\epsilon) = P_n(\cdot;\lambda,c) - \lambda P_n(c;\lambda,c+\epsilon)\sum_{k=0}^{n-1}(\lambda_k-1)\frac{P_k(c;\lambda,c)}{\|P_k(\cdot;\lambda,c)\|^2_{\lambda,c}}P_k(\cdot;\lambda,c). \tag{3.7}$$

Hence, (3.7) yields

$$P_n(x_{n,n}(\lambda,c);\lambda,c+\epsilon) < 0,$$

and consequently, $x_{n,n}(\lambda,c+\epsilon) \in (x_{n,n}(\lambda,c),c)$.

On the other hand, the fact $\lim_{c\to\infty} m(\lambda,c) = 0$ implies the asymptotic behaviour of the zeros when c tends to infinity. It can also be observed directly from the first part of the theorem or after a simple inspection of Figure 3.1.

Finally, in order to establish the rate of convergence stated in the theorem, we apply the Mean Value Theorem to P_{n-1} on the closed intervals $[x_{n,k}(\lambda,c),x_{n-1,k}]$, $1 \leq k \leq n-1$. In mathematical terms, we have

$$\frac{P_{n-1}(x_{n-1,k}) - P_{n-1}(x_{n,k}(\lambda,c))}{x_{n-1,k} - x_{n,k}(\lambda,c)} = P'_{n-1}(\zeta_k(c)),$$

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where $\zeta_k(c) \in (x_{n,k}(\lambda,c), x_{n-1,k})$. So multiplying the last equation by c and using (3.3), we get

$$\lim_{c\to\infty} c(x_{n-1,k} - x_{n,k}(\lambda,c)) = \lim_{c\to\infty} -c\frac{P_{n-1}(x_{n,k}(\lambda,c))}{P'_{n-1}(\zeta_k(c))}$$
$$= \lim_{c\to\infty} -c\, y_c(x_{n,k}(\lambda,c))\frac{P_n(x_{n,k}(\lambda,c))}{P'_{n-1}(\zeta_k(c))} = \frac{P_n(x_{n-1,k})}{P'_{n-1}(x_{n-1,k})}.$$

This finishes the proof. ∎

3.2. Alternative proof of Theorem 2.1. As we have mentioned in the introduction, the part concerning the monotonicity in Theorem 2.2 cannot be proved using the ideas contained in [3, 8]. On the other hand, using the approach developed in our manuscript, we can easily prove the monotonicity behavior stated in Theorem 2.1. Let us only consider the case $c \le a$. According to (2.3), $p_n(\cdot;\lambda,c) = 0$ if and only if

$$r_c(x) := -\frac{P_n(x)}{K_{n-1}(c,c)G_n(x;c)} = \lambda.$$

The zeros of $p_n(\cdot;\lambda,c)$ are the intersection points of r_c and λ in (c,b). Note that r_c has vertical asymptotes at $c = y_{n,1}(c) < y_{n,2}(c) < \cdots < y_{n,n}(c)$. Moreover, r_c is a monotonically decreasing function in the open intervals $(-\infty, y_{n,1}(c))$, $(y_{n,k}(c), y_{n,k+1}(c))$, $1 \le k \le n-1$, and $(y_{n,n}(c), \infty)$; see Figure 3.2 where r_c (continuous line) is presented for $n = 3$. By the same arguments as in the proof of Theorem 2.2, Theorem 2.1 follows. In this case, the result is straightforward, and there is no need to consider (3.4). This shows that our approach unifies the study of monotonicity when the parameter is in the discrete part of the measure. By comparing Figure 3.1 and Figure 3.2, the differences between the cases considered in Theorem 2.1 and Theorem 2.2 are evident.

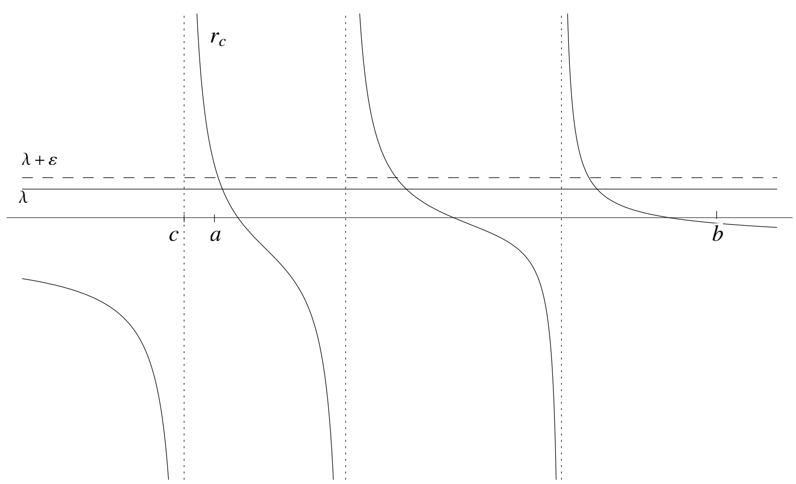

FIGURE 3.2. *Graphs of r_c, λ, and $\lambda+\epsilon$ for $n = 3$.*

4. Two examples associated with classical OPRL. To illustrate the results obtained in Theorem 2.2 for classical polynomials, we furnish figures with the aid of the functions $\mathbf{JacobiP}[\mathbf{n},\alpha,\beta,\mathbf{x}]$ and $\mathbf{LaguerreL}[\mathbf{n},\alpha,\mathbf{x}]$ implemented in Wolfram Mathematica® 9.0.

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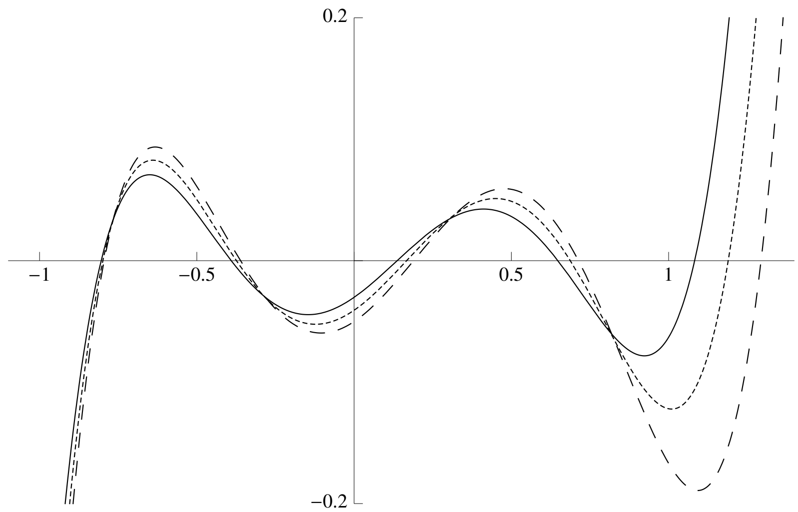

FIGURE 4.1. *Monotonicity of zeros in the Jacobi case for different values of c.*

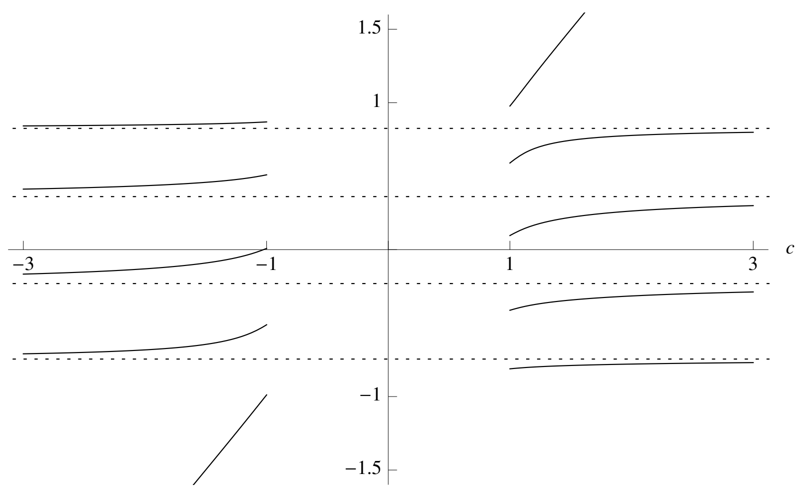

FIGURE 4.2. *Convergence of zeros in the Jacobi case when c tends to $-\infty$ or ∞.*

EXAMPLE 4.1 (Jacobi–type polynomials). The Jacobi polynomials, $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$, are orthogonal on $(-1,1)$ with respect to the weight function

$$(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1. \tag{4.1}$$

Consider the polynomials $P_5^{(0.5,1)}(\cdot;0.2,c)$ associated with a modification of (4.1) by adding a mass $\lambda = 0.2$ at $c \in \mathbb{R}\setminus(-1,1)$. Figure 4.1 displays the corresponding polynomials for different values of c, namely $c = 1.1$ (continuous line), $c = 1.2$ (small–dashed line), and $c = 1.3$ (large–dashed line). According to Figure 4.1, the zeros of these polynomials are strictly increasing functions with respect to the parameter c. Figure 4.2 illustrates the convergence of four zeros of $P_5^{(0.5,1)}(\cdot;0.2,c)$ (continuous line) to the zeros of $P_4^{(0.5,1)}$ (small–dashed line). Observe that the extreme zeros behave in accordance with our result.

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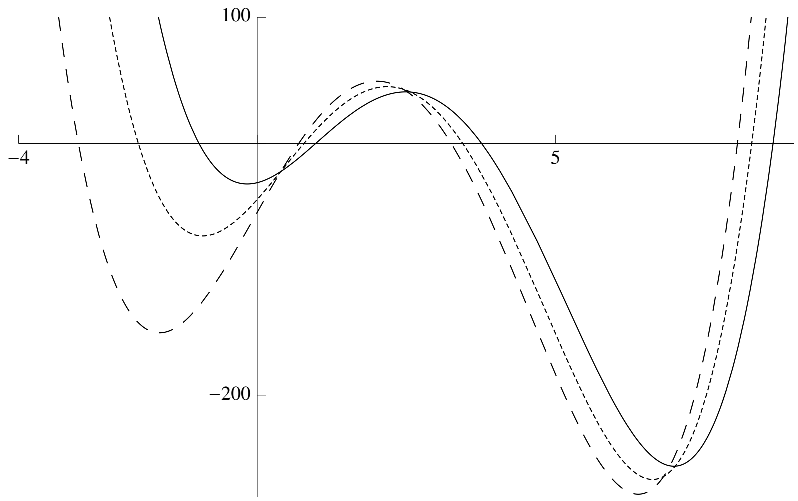

FIGURE 4.3. *Monotonicity of zeros in the Laguerre case for different values of c.*

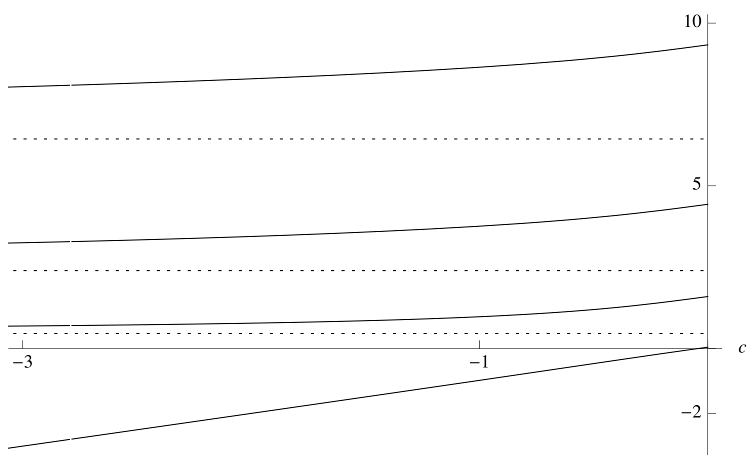

FIGURE 4.4. *Convergence of zeros in the Laguerre case when c tends to $-\infty$.*

EXAMPLE 4.2 (Laguerre–type polynomials). The Laguerre polynomials, $\{L_n^{(\alpha)}\}_{n\geq 0}$, are orthogonal on $(0,\infty)$ with respect to the weight function

$$x^{\alpha}e^{-x}, \quad \alpha > -1. \tag{4.2}$$

Consider the polynomials $L_4^{(0.1)}(\cdot;2,c)$ associated with a modification of (4.2) by adding a mass $\lambda = 2$ at $c \in \mathbb{R}\setminus(0,\infty)$. Figure 4.3 displays the corresponding polynomials for different values of c, namely $c = -1$ (continuous line), $c = -2$ (small–dashed line), and $c = -3$ (large–dashed line). Figure 4.4 illustrates the convergence of three zeros of $L_4^{(0.1)}(\cdot;2,c)$ (continuous line) to the zeros of $L_3^{(0.1)}$ (small–dashed line). As in Example 4.1, all the zeros behave in accordance with our result.

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